1 Review

Technically a student coming into a Calculus class is supposed to know both Algebra and Trigonometry. Unfortunately, the reality is often much different. Most students enter a Calculus class woefully unprepared for both the algebra and the trig that is in a Calculus class. This is very unfortunate since good algebra skills are absolutely vital to successfully completing any Calculus course and if your Calculus course includes trig (as this one does) good trig skills are also important in many sections.

The above statement is not meant to denigrate your favorite Algebra or Trig instructor. It is simply an acknowledgment of the fact that many of these courses, especially Algebra courses, are aimed at a more general audience and so do not always put the time into topics that are vital to a Calculus course and/or the level of difficulty is kept lower than might be best for students heading on towards Calculus.

Far too often the biggest impediment to students being successful in a Calculus course is they do not have sufficient skills in the underlying algebra and trig that will be in many of the calculus problems we'll be looking at. These students end up struggling with the algebra and trig in the problems rather than working to understand the calculus topics which in turn negatively impacts their grade in a Calculus course. The intent of this chapter, therefore, is to do a very cursory review of some algebra and trig skills that are vital to a calculus course that many students just didn't learn as well as they should have from their Algebra and Trig courses.

This chapter does not include all the algebra and trig skills that are needed to be successful in a Calculus course. It only includes those topics that most students are particularly deficient in. For instance, factoring is also vital to completing a standard calculus class but is not included here as it is assumed that if you are taking a Calculus course then you do know how to factor. Likewise, it is assumed that if you are taking a Calculus course then you know how to solve linear and quadratic equations so those topics are not covered here either. For a more in depth review of Algebra topics you should check out the full set of Algebra notes at http://tutorial.math.lamar.edu.

Note that even though these topics are very important to a Calculus class we rarely cover all of them in the actual class itself. We simply don't have the time to do that. We will cover certain portions of this chapter in class, but for the most part we leave it to the students to read this chapter on their own to make sure they are ready for these topics as they arise in class.

1.1 Functions

In this section we're going to make sure that you're familiar with functions and function notation. Both will appear in almost every section in a Calculus class so you will need to be able to deal with them.

First, what exactly is a function? The simplest definition is an equation will be a function if, for any x in the domain of the equation (the domain is all the x's that can be plugged into the equation), the equation will yield exactly one value of y when we evaluate the equation at a specific x.

This is usually easier to understand with an example.

Example 1

Determine if each of the following are functions.

(a) $y = x^2 + 1$

(b) $y^2 = x + 1$

Solution

(a) $y = x^2 + 1$

This first one is a function. Given an x, there is only one way to square it and then add 1 to the result. So, no matter what value of x you put into the equation, there is only one possible value of y when we evaluate the equation at that value of x.

(b) $y^2 = x + 1$

The only difference between this equation and the first is that we moved the exponent off the x and onto the y. This small change is all that is required, in this case, to change the equation from a function to something that isn't a function.

To see that this isn't a function is fairly simple. Choose a value of x, say $x = 3$ and plug this into the equation.

$$y^2 = 3 + 1 = 4$$

Now, there are two possible values of y that we could use here. We could use $y = 2$ or $y = -2$. Since there are two possible values of y that we get from a single x this equation isn't a function.

Note that this only needs to be the case for a single value of x to make an equation not be a function. For instance, we could have used $x = -1$ and in this case, we would get a single y ($y = 0$). However, because of what happens at $x = 3$ this equation will not be a function.

Next, we need to take a quick look at function notation. Function notation is nothing more than a fancy way of writing the y in a function that will allow us to simplify notation and some of our work a little.

Let's take a look at the following function.

$$y = 2x^2 - 5x + 3$$

Using function notation, we can write this as any of the following.

$$\begin{aligned} f(x) &= 2x^2 - 5x + 3 & g(x) &= 2x^2 - 5x + 3 \\ h(x) &= 2x^2 - 5x + 3 & R(x) &= 2x^2 - 5x + 3 \\ w(x) &= 2x^2 - 5x + 3 & y(x) &= 2x^2 - 5x + 3 \\ & & \vdots & \end{aligned}$$

Recall that this is NOT a letter times x, this is just a fancy way of writing y.

So, why is this useful? Well let's take the function above and let's get the value of the function at $x = -3$. Using function notation we represent the value of the function at $x = -3$ as $f(-3)$. Function notation gives us a nice compact way of representing function values.

Now, how do we actually evaluate the function? That's really simple. Everywhere we see an x on the right side we will substitute whatever is in the parenthesis on the left side. For our function this gives,

$$\begin{aligned} f(-3) &= 2(-3)^2 - 5(-3) + 3 \\ &= 2(9) + 15 + 3 \\ &= 36 \end{aligned}$$

Let's take a look at some more function evaluation.

Example 2

Given $f(x) = -x^2 + 6x - 11$ find each of the following.

(a) $f(2)$

(b) $f(-10)$

(c) $f(t)$

(d) $f(t-3)$

(e) $f(x-3)$

(f) $f(4x-1)$

Solution

(a) $f(2)$

$$f(2) = -(2)^2 + 6(2) - 11 = -3$$

(b) $f(-10)$

$$f(-10) = -(-10)^2 + 6(-10) - 11 = -100 - 60 - 11 = -171$$

Be careful when squaring negative numbers!

(c) $f(t)$

$$f(t) = -t^2 + 6t - 11$$

Remember that we substitute for the x's WHATEVER is in the parenthesis on the left. Often this will be something other than a number. So, in this case we put t's in for all the x's on the left.

(d) $f(t-3)$

$$f(t-3) = -(t-3)^2 + 6(t-3) - 11 = -t^2 + 12t - 38$$

Often instead of evaluating functions at numbers or single letters we will have some fairly complex evaluations so make sure that you can do these kinds of evaluations.

(e) $f(x-3)$

$$f(x-3) = -(x-3)^2 + 6(x-3) - 11 = -x^2 + 12x - 38$$

The only difference between this one and the previous one is that we changed the t to an x. Other than that, there is absolutely no difference between the two! Don't get excited if an x appears inside the parenthesis on the left.

(f) $f(4x-1)$

$$f(4x-1) = -(4x-1)^2 + 6(4x-1) - 11 = -16x^2 + 32x - 18$$

This one is not much different from the previous part. All we did was change the equation that we were plugging into the function.

All throughout a calculus course we will be finding roots of functions. A root of a function is nothing more than a number for which the function is zero. In other words, finding the roots of a function, $g(x)$, is equivalent to solving

$$g(x) = 0$$

Example 3

Determine all the roots of $f(t) = 9t^3 - 18t^2 + 6t$

Solution

So, we will need to solve,

$$9t^3 - 18t^2 + 6t = 0$$

First, we should factor the equation as much as possible. Doing this gives,

$$3t\left(3t^2 - 6t + 2\right) = 0$$

Next recall that if a product of two things are zero then one (or both) of them had to be zero. This means that,

$$3t = 0 \qquad \text{OR} \qquad 3t^2 - 6t + 2 = 0$$

From the first it's clear that one of the roots must then be $t = 0$. To get the remaining roots we will need to use the quadratic formula on the second equation. Doing this gives,

$$\begin{aligned} t &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} \\ &= \frac{6 \pm \sqrt{12}}{6} \\ &= \frac{6 \pm \sqrt{(4)(3)}}{6} \\ &= \frac{6 \pm 2\sqrt{3}}{6} \\ &= \frac{3 \pm \sqrt{3}}{3} \\ &= 1 \pm \frac{1}{3}\sqrt{3} = 1 \pm \frac{1}{\sqrt{3}} \end{aligned}$$

In order to remind you how to simplify radicals we gave several forms of the answer.

To complete the problem, here is a complete list of all the roots of this function.

$$t = 0, \ t = \frac{3 + \sqrt{3}}{3}, \ t = \frac{3 - \sqrt{3}}{3}$$

Note we didn't use the final form for the roots from the quadratic. This is usually where we'll stop with the simplification for these kinds of roots. Also note that, for the sake of the practice, we broke up the compact form for the two roots of the quadratic. You will need to be able to do this so make sure that you can.

This example had a couple of points other than finding roots of functions.

The first was to remind you of the quadratic formula. This won't be the last time that you'll need it in this class.

The second was to get you used to seeing "messy" answers. In fact, the answers in the above example are not really all that messy. However, most students come out of an Algebra class very used to seeing only integers and the occasional "nice" fraction as answers.

So, here is fair warning. In this class I often will intentionally make the answers look "messy" just to get you out of the habit of always expecting "nice" answers. In "real life" (whatever that is) the answer is rarely a simple integer such as two. In most problems the answer will be a decimal that came about from a messy fraction and/or an answer that involved radicals.

One of the more important ideas about functions is that of the **domain** and **range** of a function. In simplest terms the domain of a function is the set of all values that can be plugged into a function and have the function exist and have a real number for a value. So, for the domain we need to avoid division by zero, square roots of negative numbers, logarithms of zero and logarithms of negative numbers (if not familiar with logarithms we'll take a look at them a little later), *etc*. The range of a function is simply the set of all possible values that a function can take.

Let's find the domain and range of a few functions.

Example 4

Find the domain and range of each of the following functions.

(a) $f(x) = 5x - 3$

(b) $g(t) = \sqrt{4 - 7t}$

(c) $h(x) = -2x^2 + 12x + 5$

(d) $f(z) = |z - 6| - 3$

(e) $g(x) = 8$

Solution

(a) $f(x) = 5x - 3$

We know that this is a line and that it's not a horizontal line (because the slope is 5 and not zero...). This means that this function can take on any value and so the range is all real numbers. Using "mathematical" notation this is,

$$\text{Range : } \quad (-\infty, \infty)$$

This is more generally a polynomial and we know that we can plug any value into a polynomial and so the domain in this case is also all real numbers or,

$$\text{Domain : } -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

(b) $g(t) = \sqrt{4 - 7t}$

This is a square root and we know that square roots are always positive or zero. We know then that the range will be,

$$\text{Range : } [0, \infty)$$

For the domain we have a little bit of work to do, but not much. We need to make sure that we don't take square roots of any negative numbers, so we need to require that,

$$\begin{aligned} 4 - 7t &\geq 0 \\ 4 &\geq 7t \\ \frac{4}{7} &\geq t \qquad \Rightarrow \qquad t \leq \frac{4}{7} \end{aligned}$$

The domain is then,

$$\text{Domain : } t \leq \frac{4}{7} \quad \text{or} \quad \left(-\infty, \frac{4}{7}\right]$$

(c) $h(x) = -2x^2 + 12x + 5$

Here we have a quadratic, which is a polynomial, so we again know that the domain is all real numbers or,

$$\text{Domain : } -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

In this case the range requires a little bit of work. From an Algebra class we know that the graph of this will be a parabola that opens down (because the coefficient of the x^2 is negative) and so the vertex will be the highest point on the graph. If we know the vertex we can then get the range. The vertex is then,

$$x = -\frac{12}{2(-2)} = 3 \qquad y = h(3) = -2(3)^2 + 12(3) + 5 = 23 \qquad \Rightarrow \qquad (3, 23)$$

So, as discussed, we know that this will be the highest point on the graph or the largest value of the function and the parabola will take all values less than this, so the range is then,

$$\text{Range : } (-\infty, 23]$$

(d) $f(z) = |z-6| - 3$

This function contains an absolute value and we know that absolute value will be either positive or zero. In this case the absolute value will be zero if $z = 6$ and so the absolute value portion of this function will always be greater than or equal to zero. We are subtracting 3 from the absolute value portion and so we then know that the range will be,

$$\text{Range : } [-3, \infty)$$

We can plug any value into an absolute value and so the domain is once again all real numbers or,

$$\text{Domain : } -\infty < z < \infty \quad \text{or} \quad (-\infty, \infty)$$

(e) $g(x) = 8$

This function may seem a little tricky at first but is actually the easiest one in this set of examples. This is a constant function and so any value of x that we plug into the function will yield a value of 8. This means that the range is a single value or,

$$\text{Range : } 8$$

The domain is all real numbers,

$$\text{Domain : } -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

In general, determining the range of a function can be somewhat difficult. As long as we restrict ourselves down to "simple" functions, some of which we looked at in the previous example, finding the range is not too bad, but for most functions it can be a difficult process.

Because of the difficulty in finding the range for a lot of functions we had to keep those in the previous set somewhat simple, which also meant that we couldn't really look at some of the more complicated domain examples that are liable to be important in a Calculus course. So, let's take a look at another set of functions only this time we'll just look for the domain.

Example 5

Find the domain of each of the following functions.

(a) $f(x) = \dfrac{x-4}{x^2 - 2x - 15}$

(b) $g(t) = \sqrt{6 + t - t^2}$

(c) $h(x) = \dfrac{x}{\sqrt{x^2 - 9}}$

Solution

(a) $f(x) = \dfrac{x-4}{x^2-2x-15}$

Okay, with this problem we need to avoid division by zero, so we need to determine where the denominator is zero which means solving,

$$x^2-2x-15=(x-5)(x+3)=0 \qquad \Rightarrow \qquad x=-3,\ x=5$$

So, these are the only values of x that we need to avoid and so the domain is,

$$\text{Domain : } \quad \text{All real numbers except } x=-3 \ \ \& \ x=5$$

(b) $g(t)=\sqrt{6+t-t^2}$

In this case we need to avoid square roots of negative numbers and so need to require that,

$$6+t-t^2 \ge 0 \qquad\qquad \Rightarrow \qquad\qquad t^2-t-6 \le 0$$

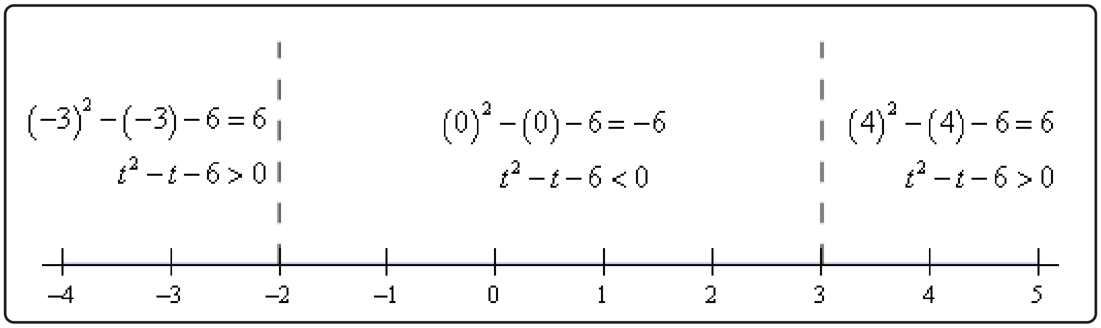
Note that we multiplied the whole inequality by -1 (and remembered to switch the direction of the inequality) to make this easier to deal with. You'll need to be able to solve inequalities like this more than a few times in a Calculus course so let's make sure you can solve these.

The first thing that we need to do is determine where the function is zero and that's not too difficult in this case.

$$t^2-t-6=(t-3)(t+2)=0$$

So, the function will be zero at $t=-2$ and $t=3$. Recall that these points will be the only place where the function *may* change sign. It's not required to change sign at these points, but these will be the only points where the function can change sign. This means that all we need to do is break up a number line into the three regions that avoid these two points and test the sign of the function at a single point in each of the regions. If the function is positive at a single point in the region it will be positive at all points in that region because it doesn't contain the any of the points where the function may change sign. We'll have a similar situation if the function is negative for the test point.

So, here is a number line showing these computations.



From this we can see that the only region in which the quadratic (in its modified form) will be negative is in the middle region. Recalling that we got to the modified region by multiplying the quadratic by a -1 this means that the quadratic under the root will only be positive in the middle region and so the domain for this function is then,

$$\text{Domain : } -2 \le t \le 3 \quad \text{or} \quad [-2, 3]$$

(c) $h(x) = \dfrac{x}{\sqrt{x^2 - 9}}$

In this case we have a mixture of the two previous parts. We have to worry about division by zero and square roots of negative numbers. We can cover both issues by requiring that,

$$x^2 - 9 > 0$$

Note that we need the inequality here to be strictly greater than zero to avoid the division by zero issues. We can either solve this by the method from the previous example or, in this case, it is easy enough to solve by inspection. The domain is this case is,

$$\text{Domain : } x < -3 \text{ \& } x > 3 \quad \text{or} \quad (-\infty, -3) \text{ \& } (3, \infty)$$

The next topic that we need to discuss here is that of **function composition**. The composition of $f(x)$ and $g(x)$ is

$$(f \circ g)(x) = f(g(x))$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note as well that order is important here. Interchanging the order will more often than not result in a different answer.

Example 6

Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$ find each of the following.

(a) $(f \circ g)(5)$

(b) $(f \circ g)(x)$

(c) $(g \circ f)(x)$

(d) $(g \circ g)(x)$

Solution

(a) $(f \circ g)(5)$

In this case we've got a number instead of an x but it works in exactly the same way.

$$\begin{aligned}(f \circ g)(5) &= f(g(5)) \\ &= f(-99) = 29512\end{aligned}$$

(b) $(f \circ g)(x)$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(1 - 20x) \\ &= 3(1 - 20x)^2 - (1 - 20x) + 10 \\ &= 3\left(1 - 40x + 400x^2\right) - 1 + 20x + 10 \\ &= 1200x^2 - 100x + 12\end{aligned}$$

Compare this answer to the next part and notice that answers are NOT the same. The order in which the functions are listed is important!

(c) $(g \circ f)(x)$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g\left(3x^2 - x + 10\right) \\ &= 1 - 20\left(3x^2 - x + 10\right) \\ &= -60x^2 + 20x - 199\end{aligned}$$

And just to make the point one more time. This answer is different from the previous part. Order is important in composition.

(d) $(g \circ g)(x)$

In this case do not get excited about the fact that it's the same function. Composition still works the same way.

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) \\ &= g(1 - 20x) \\ &= 1 - 20(1 - 20x) \\ &= 400x - 19\end{aligned}$$

Let's work one more example that will lead us into the next section.

Example 7

Given $f(x) = 3x - 2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}$ find each of the following.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

Solution

(a) $(f \circ g)(x)$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{3}x + \frac{2}{3}\right) \\ &= 3\left(\frac{1}{3}x + \frac{2}{3}\right) - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

(b) $(g \circ f)(x)$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x - 2) \\ &= \frac{1}{3}(3x - 2) + \frac{2}{3} \\ &= x - \frac{2}{3} + \frac{2}{3} \\ &= x\end{aligned}$$

In this case the two compositions were the same and in fact the answer was very simple.

$$(f \circ g)(x) = (g \circ f)(x) = x$$

This will usually not happen. However, when the two compositions are both x there is a very nice relationship between the two functions. We will take a look at that relationship in the next section.

1.2 Inverse Functions

In the last example from the previous section we looked at the two functions $f(x) = 3x - 2$ and $g(x) = \frac{x}{3} + \frac{2}{3}$ and saw that

$$(f \circ g)(x) = (g \circ f)(x) = x$$

and as noted in that section this means that there is a nice relationship between these two functions. Let's see just what that relationship is. Consider the following evaluations.

$$f(-1) = 3(-1) - 2 = -5 \qquad \Rightarrow \qquad g(-5) = \frac{-5}{3} + \frac{2}{3} = \frac{-3}{3} = -1$$

$$g(2) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \qquad \Rightarrow \qquad f\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right) - 2 = 4 - 2 = 2$$

In the first case we plugged $x = -1$ into $f(x)$ and got a value of -5. We then turned around and plugged $x = -5$ into $g(x)$ and got a value of -1, the number that we started off with.

In the second case we did something similar. Here we plugged $x = 2$ into $g(x)$ and got a value of $\frac{4}{3}$, we turned around and plugged this into $f(x)$ and got a value of 2, which is again the number that we started with.

Note that we really are doing some function composition here. The first case is really,

$$(g \circ f)(-1) = g[f(-1)] = g[-5] = -1$$

and the second case is really,

$$(f \circ g)(2) = f[g(2)] = f\left[\frac{4}{3}\right] = 2$$

Note as well that these both agree with the formula for the compositions that we found in the previous section. We get back out of the function evaluation the number that we originally plugged into the composition.

So, just what is going on here? In some way we can think of these two functions as undoing what the other did to a number. In the first case we plugged $x = -1$ into $f(x)$ and then plugged the result from this function evaluation back into $g(x)$ and in some way $g(x)$ undid what $f(x)$ had done to $x = -1$ and gave us back the original x that we started with.

Function pairs that exhibit this behavior are called **inverse functions**. Before formally defining inverse functions and the notation that we're going to use for them we need to get a definition out of the way.

A function is called **one-to-one** if no two values of x produce the same y. Mathematically this is the same as saying,

$$f(x_1) \neq f(x_2) \qquad \text{whenever} \qquad x_1 \neq x_2$$

So, a function is one-to-one if whenever we plug different values into the function we get different function values.

Sometimes it is easier to understand this definition if we see a function that isn't one-to-one. Let's take a look at a function that isn't one-to-one. The function $f(x) = x^2$ is not one-to-one because both $f(-2) = 4$ and $f(2) = 4$. In other words, there are two different values of x that produce the same value of y. Note that we can turn $f(x) = x^2$ into a one-to-one function if we restrict ourselves to $0 \le x < \infty$. This can sometimes be done with functions.

Showing that a function is one-to-one is often tedious and/or difficult. For the most part we are going to assume that the functions that we're going to be dealing with in this course are either one-to-one or we have restricted the domain of the function to get it to be a one-to-one function.

Now, let's formally define just what inverse functions are. Given two one-to-one functions $f(x)$ and $g(x)$ if

$$(f \circ g)(x) = x \qquad \text{AND} \qquad (g \circ f)(x) = x$$

then we say that $f(x)$ and $g(x)$ are **inverses** of each other. More specifically we will say that $g(x)$ is the **inverse** of $f(x)$ and denote it by

$$g(x) = f^{-1}(x)$$

Likewise, we could also say that $f(x)$ is the **inverse** of $g(x)$ and denote it by

$$f(x) = g^{-1}(x)$$

The notation that we use really depends upon the problem. In most cases either is acceptable.

For the two functions that we started off this section with we could write either of the following two sets of notation.

$$f(x) = 3x - 2 \qquad\qquad f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

$$g(x) = \frac{x}{3} + \frac{2}{3} \qquad\qquad g^{-1}(x) = 3x - 2$$

Now, be careful with the notation for inverses. The "-1" is NOT an exponent despite the fact that it sure does look like one! When dealing with inverse functions we've got to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

This is one of the more common mistakes that students make when first studying inverse functions.

The process for finding the inverse of a function is a fairly simple one although there are a couple of steps that can on occasion be somewhat messy. Here is the process

Finding the Inverse of a Function

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with y. This is done to make the rest of the process easier.
2. Replace every x with a y and replace every y with an x.
3. Solve the equation from Step 2 for y. This is the step where mistakes are most often made so be careful with this step.
4. Replace y with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!
5. Verify your work by checking that $\left(f \circ f^{-1}\right)(x) = x$ and $\left(f^{-1} \circ f\right)(x) = x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

That's the process. Most of the steps are not all that bad but as mentioned in the process there are a couple of steps that we really need to be careful with since it is easy to make mistakes in those steps.

In the verification step we technically really do need to check that both $\left(f \circ f^{-1}\right)(x) = x$ and $\left(f^{-1} \circ f\right)(x) = x$ are true. For all the functions that we are going to be looking at in this course if one is true then the other will also be true. However, there are functions (they are beyond the scope of this course however) for which it is possible for only one of these to be true. This is brought up because in all the problems here we will be just checking one of them. We just need to always remember that technically we should check both.

Let's work some examples.

Example 1

Given $f(x) = 3x - 2$ find $f^{-1}(x)$.

Solution

Now, we already know what the inverse to this function is as we've already done some work with it. However, it would be nice to actually start with this since we know what we should get. This will work as a nice verification of the process.

So, let's get started. We'll first replace $f(x)$ with y.

$$y = 3x - 2$$

Next, replace all x's with y and all y's with x.

$$x = 3y - 2$$

Now, solve for y.

$$\begin{aligned} x + 2 &= 3y \\ \frac{1}{3}(x+2) &= y \\ \frac{x}{3} + \frac{2}{3} &= y \end{aligned}$$

Finally replace y with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

Now, we need to verify the results. We already took care of this in the previous section, however, we really should follow the process so we'll do that here. It doesn't matter which of the two that we check we just need to check one of them. This time we'll check that $(f \circ f^{-1})(x) = x$ is true.

$$\begin{aligned} (f \circ f^{-1})(x) &= f\left[f^{-1}(x)\right] \\ &= f\left[\frac{x}{3} + \frac{2}{3}\right] \\ &= 3\left(\frac{x}{3} + \frac{2}{3}\right) - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

Example 2

Given $g(x) = \sqrt{x-3}$ find $g^{-1}(x)$.

Solution

The fact that we're using $g(x)$ instead of $f(x)$ doesn't change how the process works. Here are the first few steps.

$$y = \sqrt{x-3} \qquad \Rightarrow \qquad x = \sqrt{y-3}$$

Now, to solve for y we will need to first square both sides and then proceed as normal.

$$\begin{aligned} x &= \sqrt{y-3} \\ x^2 &= y - 3 \\ x^2 + 3 &= y \end{aligned}$$

This inverse is then,

$$g^{-1}(x) = x^2 + 3$$

Finally let's verify and this time we'll use the other one just so we can say that we've gotten both down somewhere in an example.

$$\begin{aligned}\left(g^{-1} \circ g\right)(x) &= g^{-1}\left[g(x)\right] \\ &= g^{-1}\left(\sqrt{x-3}\right) \\ &= \left(\sqrt{x-3}\right)^2 + 3 \\ &= x - 3 + 3 \\ &= x\end{aligned}$$

So, we did the work correctly and we do indeed have the inverse.

The next example can be a little messy so be careful with the work here.

Example 3

Given $h(x) = \dfrac{x+4}{2x-5}$ find $h^{-1}(x)$.

Solution

The first couple of steps are pretty much the same as the previous examples so here they are,

$$y = \frac{x+4}{2x-5} \qquad \Rightarrow \qquad x = \frac{y+4}{2y-5}$$

Now, be careful with the solution step. With this kind of problem it is very easy to make a mistake here.

$$\begin{aligned}x(2y-5) &= y + 4 \\ 2xy - 5x &= y + 4 \\ 2xy - y &= 4 + 5x \\ (2x-1)\,y &= 4 + 5x \\ y &= \frac{4+5x}{2x-1}\end{aligned}$$

So, if we've done all of our work correctly the inverse should be,

$$h^{-1}(x) = \frac{4+5x}{2x-1}$$

Finally, we'll need to do the verification. This is also a fairly messy process and it doesn't really matter which one we work with.

$$\begin{aligned}\left(h \circ h^{-1}\right)(x) &= h\left[h^{-1}(x)\right] \\ &= h\left[\frac{4+5x}{2x-1}\right] \\ &= \frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5}\end{aligned}$$

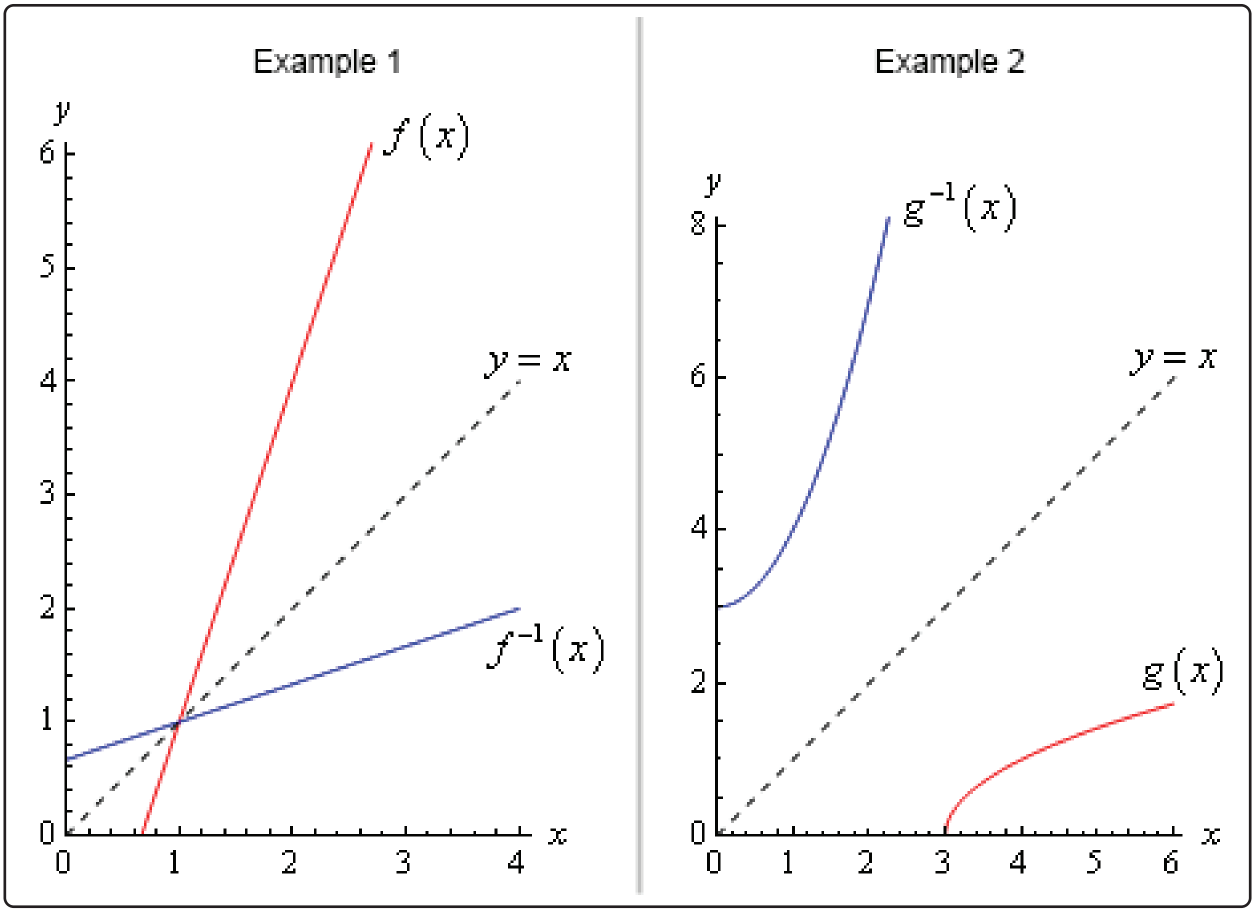
Okay, this is a mess. Let's simplify things up a little bit by multiplying the numerator and denominator by $2x-1$.

$$\begin{aligned}\left(h \circ h^{-1}\right)(x) &= \frac{2x-1}{2x-1}\,\frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5} \\ &= \frac{(2x-1)\left(\frac{4+5x}{2x-1}+4\right)}{(2x-1)\left(2\left(\frac{4+5x}{2x-1}\right)-5\right)} \\ &= \frac{4+5x+4(2x-1)}{2(4+5x)-5(2x-1)} \\ &= \frac{4+5x+8x-4}{8+10x-10x+5} \\ &= \frac{13x}{13} = x\end{aligned}$$

Wow. That was a lot of work, but it all worked out in the end. We did all of our work correctly and we do in fact have the inverse.

There is one final topic that we need to address quickly before we leave this section. There is an interesting relationship between the graph of a function and the graph of its inverse.

Here is the graph of the function and inverse from the first two examples.



In both cases we can see that the graph of the inverse is a reflection of the actual function about the line $y = x$. This will always be the case with the graphs of a function and its inverse.

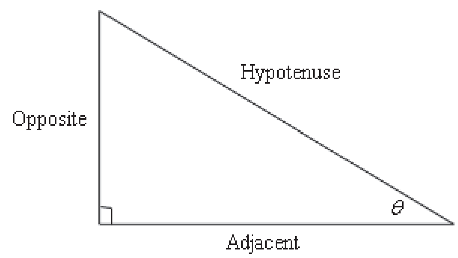
1.3 Trig Functions

The intent of this section is to remind you of some of the more important (from a Calculus standpoint...) topics from a trig class. One of the most important (but not the first) of these topics will be how to use the unit circle. We will leave the most important topic to the next section.

First let's start with the six trig functions and how they relate to each other.

$$\cos(x) \qquad \sin(x)$$
$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$
$$\sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

Recall as well that all the trig functions can be defined in terms of a right triangle.



From this right triangle we get the following definitions of the six trig functions.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

Remembering both the relationship between all six of the trig functions and their right triangle definitions will be useful in this course on occasion.

Next, we need to touch on radians. In most trig classes instructors tend to concentrate on doing everything in terms of degrees (probably because it's easier to visualize degrees). The same is true in many science classes. However, in a calculus course almost everything is done in radians. The following table gives some of the basic angles in both degrees and radians.

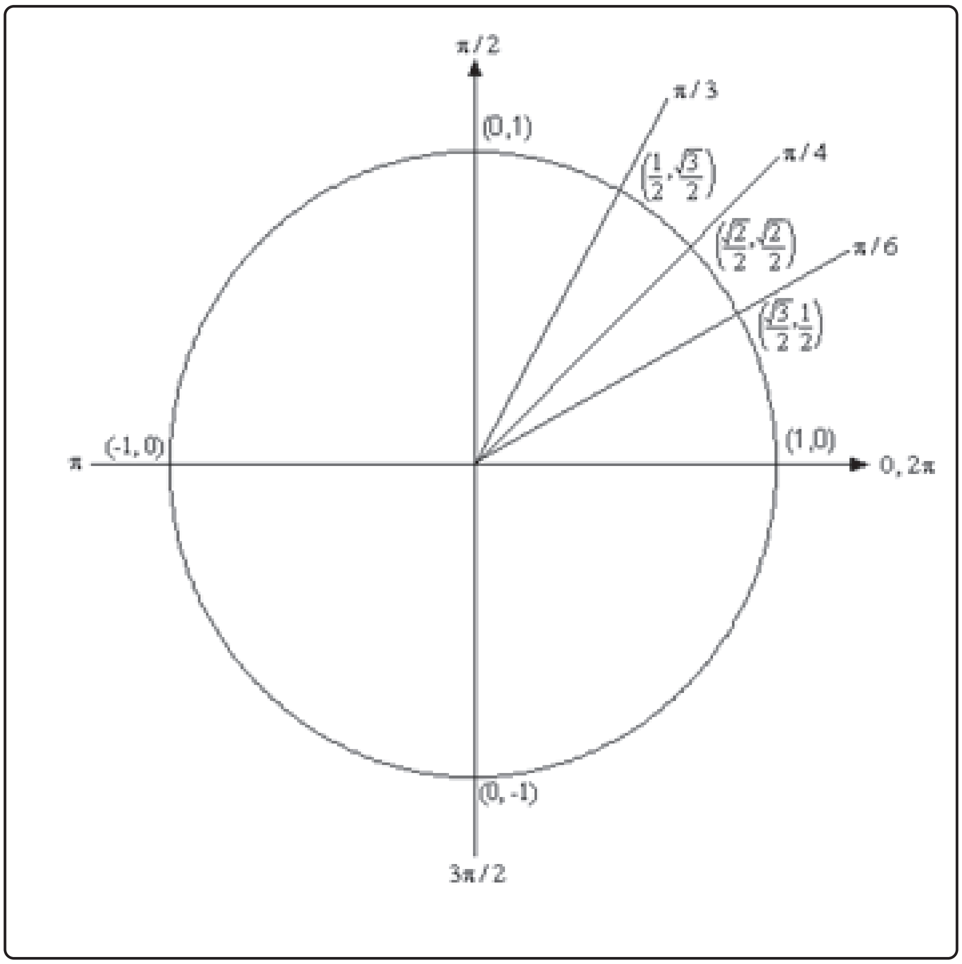
Degree	0	30	45	60	90	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Know this table! We may not see these specific angles all that much when we get into the Calculus portion of these notes, but knowing these can help us to visualize each angle. Now, one more time just make sure this is clear.

Be forewarned, everything in most calculus classes will be done in radians!

Let's next take a look at one of the most overlooked ideas from a trig class. The unit circle is one of the more useful tools to come out of a trig class. Unfortunately, most people don't learn it as well as they should in their trig class.

Below is unit circle with just the first quadrant filled in with the "standard" angles. The way the unit circle works is to draw a line from the center of the circle outwards corresponding to a given angle. Then look at the coordinates of the point where the line and the circle intersect. The first coordinate, *i.e.* the x-coordinate, is the cosine of that angle and the second coordinate, *i.e.* the y-coordinate, is the sine of that angle. We've put some of the angles along with the coordinates of their intersections on the unit circle.



So, from the unit circle above we can see that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

Also, remember how the signs of angles work. If you rotate in a counter clockwise direction the angle is positive and if you rotate in a clockwise direction the angle is negative.

Recall as well that one complete revolution is 2π, so the positive x-axis can correspond to either an angle of 0 or 2π (or 4π, or 6π, or -2π, or -4π, *etc.* depending on the direction of rotation). Likewise, the angle $\frac{\pi}{6}$ (to pick an angle completely at random) can also be any of the following angles:

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate once around counter clockwise)}$$

$$\frac{\pi}{6} + 4\pi = \frac{25\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate around twice counter clockwise)}$$

$$\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate once around clockwise)}$$

$$\frac{\pi}{6} - 4\pi = -\frac{23\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate around twice clockwise)}$$

etc.

In fact, $\frac{\pi}{6}$ can be any of the following angles $\frac{\pi}{6} + 2\pi\, n\,, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$ In this case n is the number of complete revolutions you make around the unit circle starting at $\frac{\pi}{6}$. Positive values of n correspond to counter clockwise rotations and negative values of n correspond to clockwise rotations.

So, why did we only put in the first quadrant? The answer is simple. If you know the first quadrant then you can get all the other quadrants from the first with a small application of geometry. You'll see how this is done in the following set of examples.

Example 1

Evaluate each of the following.

(a) sin $\left(\frac{2\pi}{3}\right)$ and sin $\left(-\frac{2\pi}{3}\right)$

(b) cos $\left(\frac{7\pi}{6}\right)$ and cos $\left(-\frac{7\pi}{6}\right)$

(c) tan $\left(-\frac{\pi}{4}\right)$ and tan $\left(\frac{7\pi}{4}\right)$

(d) sec $\left(\frac{25\pi}{6}\right)$

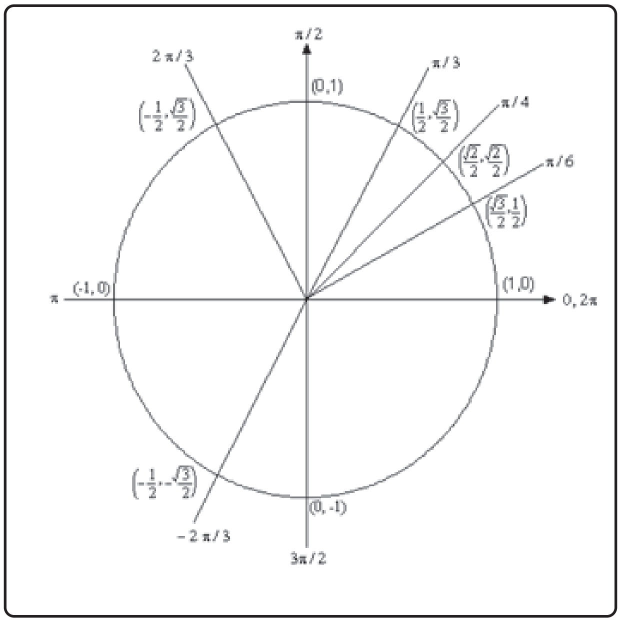
Solution

(a) sin $\left(\frac{2\pi}{3}\right)$ and sin $\left(-\frac{2\pi}{3}\right)$

The first evaluation in this part uses the angle $\frac{2\pi}{3}$. That's not on our unit circle above, however notice that $\frac{2\pi}{3} = \pi - \frac{\pi}{3}$. So $\frac{2\pi}{3}$ is found by rotating up $\frac{\pi}{3}$ from the negative x-axis. This means that the line for $\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the second quadrant. The coordinates for $\frac{2\pi}{3}$ will be the coordinates for $\frac{\pi}{3}$ except the x coordinate will be negative.

Likewise, for $-\frac{2\pi}{3}$ we can notice that $-\frac{2\pi}{3} = -\pi + \frac{\pi}{3}$, so this angle can be found by rotating down $\frac{\pi}{3}$ from the negative x-axis. This means that the line for $-\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the third quadrant and the coordinates will be the same as the coordinates for $\frac{\pi}{3}$ except both will be negative.

Both of these angles, along with the coordinates of the intersection points, are shown on the following unit circle.



From this unit circle we can see that $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

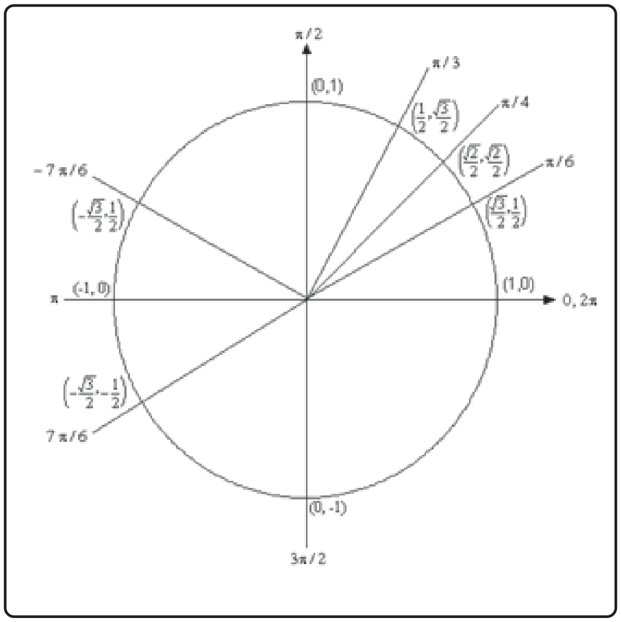
This leads to a nice fact about the sine function. The sine function is called an **odd** function and so for ANY angle we have

$$\sin(-\theta) = -\sin(\theta)$$

(b) $\cos\left(\frac{7\pi}{6}\right)$ and $\cos\left(-\frac{7\pi}{6}\right)$

For this example, notice that $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ so this means we would rotate down $\frac{\pi}{6}$ from the negative x-axis to get to this angle. Also $-\frac{7\pi}{6} = -\pi - \frac{\pi}{6}$ so this means we would rotate up $\frac{\pi}{6}$ from the negative x-axis to get to this angle. So, as with the last part, both of these angles will be mirror images of $\frac{\pi}{6}$ in the third and second quadrants respectively and we can use this to determine the coordinates for both of these new angles.

Both of these angles are shown on the following unit circle along with the coordinates for the intersection points.

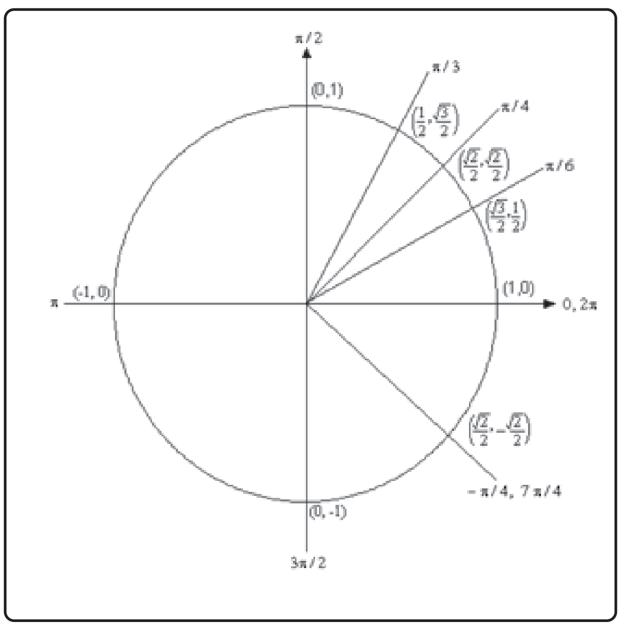


From this unit circle we can see that cos $\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and cos $\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. In this case the cosine function is called an **even** function and so for ANY angle we have

$$\cos(-\theta) = \cos(\theta)$$

(c) tan $\left(-\frac{\pi}{4}\right)$ and tan $\left(\frac{7\pi}{4}\right)$

Here we should note that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ so $\frac{7\pi}{4}$ and $-\frac{\pi}{4}$ are in fact the same angle! Also note that this angle will be the mirror image of $\frac{\pi}{4}$ in the fourth quadrant. The unit circle for this angle is



Now, if we remember that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ we can use the unit circle to find the values of the tangent function. So,

$$\tan\left(\frac{7\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right) = \frac{\sin(-\pi/4)}{\cos(-\pi/4)} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1$$

On a side note, notice that $\tan\left(\frac{\pi}{4}\right) = 1$ and we can see that the tangent function is also called an **odd** function and so for ANY angle we will have

$$\tan(-\theta) = -\tan(\theta)$$

(d) $\sec\left(\frac{25\pi}{6}\right)$

Here we need to notice that $\frac{25\pi}{6} = 4\pi + \frac{\pi}{6}$. In other words, we've started at $\frac{\pi}{6}$ and rotated around twice to end back up at the same point on the unit circle. This means that

$$\sec\left(\frac{25\pi}{6}\right) = \sec\left(4\pi + \frac{\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right)$$

Now, let's also not get excited about the secant here. Just recall that

$$\sec(x) = \frac{1}{\cos(x)}$$

and so all we need to do here is evaluate a cosine! Therefore,

$$\sec\left(\frac{25\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$

So, in the last example we saw how the unit circle can be used to determine the value of the trig functions at any of the "common" angles. It's important to notice that all of these examples used the fact that if you know the first quadrant of the unit circle and can relate all the other angles to "mirror images" of one of the first quadrant angles you don't really need to know whole unit circle. If you'd like to see a complete unit circle I've got one on my Trig Cheat Sheet that is available at https://tutorial.math.lamar.edu.

Another important idea from the last example is that when it comes to evaluating trig functions all that you really need to know is how to evaluate sine and cosine. The other four trig functions are defined in terms of these two so if you know how to evaluate sine and cosine you can also evaluate the remaining four trig functions.

We've not covered many of the topics from a trig class in this section, but we did cover some of the more important ones from a calculus standpoint. There are many important trig formulas that you will use occasionally in a calculus class. Most notably are the half-angle and double-angle formulas. If you need reminded of what these are, you might want to download my Trig Cheat Sheet as most of the important facts and formulas from a trig class are listed there.

1.4 Solving Trig Equations

In this section we will take a look at solving trig equations. This is something that you will be asked to do on a fairly regular basis in many classes.

Let's just jump into the examples and see how to solve trig equations.

Example 1

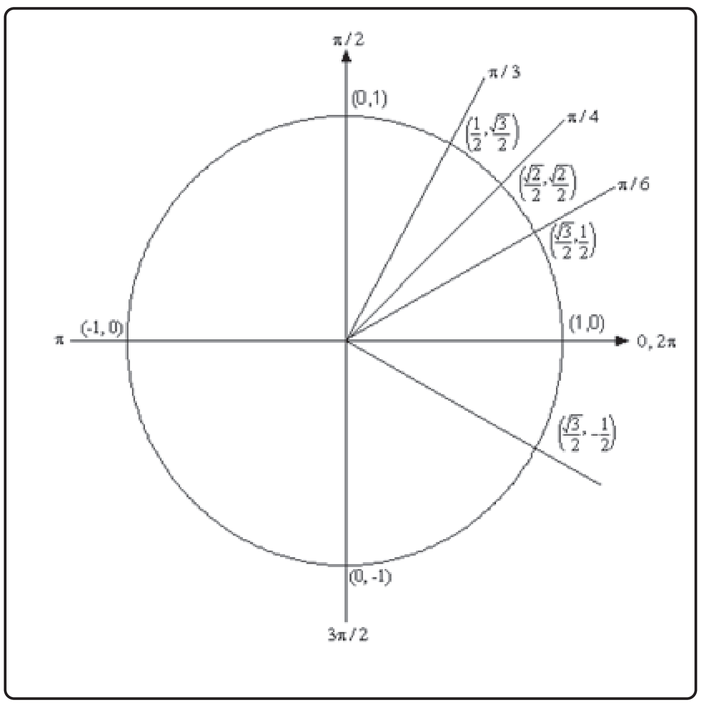
Solve $2\cos(t) = \sqrt{3}$.

Solution

There's really not a whole lot to do in solving this kind of trig equation. We first need to get the trig function on one side by itself. To do this all we need to do is divide both sides by 2.

$$\begin{aligned} 2\cos(t) &= \sqrt{3} \\ \cos(t) &= \frac{\sqrt{3}}{2} \end{aligned}$$

We are looking for all the values of t for which cosine will have the value of $\frac{\sqrt{3}}{2}$. So, let's take a look at the following unit circle.



From quick inspection we can see that $t = \frac{\pi}{6}$ is a solution. However, as we have shown on the unit circle there is another angle which will also be a solution. We need to determine what this angle is. When we look for these angles we typically want *positive* angles that lie between 0 and 2π. This angle will not be the only possibility of course, but we typically look for angles that meet these conditions.

To find this angle for this problem all we need to do is use a little geometry. The angle in the first quadrant makes an angle of $\frac{\pi}{6}$ with the positive x-axis, then so must the angle in the fourth quadrant. So, we have two options. We could use $-\frac{\pi}{6}$, but again, it's more common to use positive angles. To get a positive angle all we need to do is use the fact that the angle is $\frac{\pi}{6}$ with the positive x-axis (as noted above) and a positive angle will be $t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

One way to remember how to get the positive form of the second angle is to think of making one full revolution from the positive x-axis (*i.e.* 2π) and then backing off (*i.e.* subtracting) $\frac{\pi}{6}$.

We aren't done with this problem. As the discussion about finding the second angle has shown there are many ways to write any given angle on the unit circle. Sometimes it will be $-\frac{\pi}{6}$ that we want for the solution and sometimes we will want both (or neither) of the listed angles. Therefore, since there isn't anything in this problem (contrast this with the next problem) to tell us which is the correct solution we will need to list ALL possible solutions.

This is very easy to do. Recall from the previous section and you'll see there that we used

$$\frac{\pi}{6} + 2\pi\, n\,, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

to represent all the possible angles that can end at the same location on the unit circle, *i.e.* angles that end at $\frac{\pi}{6}$. Remember that all this says is that we start at $\frac{\pi}{6}$ then rotate around in the counter-clockwise direction (n is positive) or clockwise direction (n is negative) for n complete rotations. The same thing can be done for the second solution.

So, all together the complete solution to this problem is

$$\frac{\pi}{6} + 2\pi\, n\,, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$
$$\frac{11\pi}{6} + 2\pi\, n\,, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

As a final thought, notice that we can get $-\frac{\pi}{6}$ by using $n = -1$ in the second solution.

Now, in a calculus class this is not a typical trig equation that we'll be asked to solve. A more typical example is the next one.

Example 2

Solve $2\cos(t) = \sqrt{3}$ on $[-2\pi, 2\pi]$.

Solution

In a calculus class we are often more interested in only the solutions to a trig equation that fall in a certain interval. The first step in this kind of problem is to find all possible solutions. We did this in the previous example.

$$\begin{aligned} &\frac{\pi}{6} + 2\pi\, n\,, \quad n = 0, \pm1, \pm2, \pm3, \ldots \\ &\frac{11\pi}{6} + 2\pi\, n\,, \quad n = 0, \pm1, \pm2, \pm3, \ldots \end{aligned}$$

Now, to find the solutions in the interval all we need to do is start picking values of n, plugging them in and getting the solutions that will fall into the interval that we've been given.

$n = 0$.

$$\begin{aligned} \frac{\pi}{6} + 2\pi\,(0) &= \frac{\pi}{6} < 2\pi \\ \frac{11\pi}{6} + 2\pi\,(0) &= \frac{11\pi}{6} < 2\pi \end{aligned}$$

Now, notice that if we take any positive value of n we will be adding on positive multiples of 2π onto a positive quantity and this will take us past the upper bound of our interval so we don't need to take any positive value of n.

However, just because we aren't going to take any positive value of n doesn't mean that we shouldn't also look at negative values of n.

$n = -1$.

$$\begin{aligned} \frac{\pi}{6} + 2\pi\,(-1) &= -\frac{11\pi}{6} > -2\pi \\ \frac{11\pi}{6} + 2\pi\,(-1) &= -\frac{\pi}{6} > -2\pi \end{aligned}$$

These are both greater than -2π and so are solutions, but if we subtract another 2π off (*i.e* use $n = -2$) we will once again be outside of the interval so we've found all the possible solutions that lie inside the interval $[-2\pi, 2\pi]$.

So, the solutions are : $\frac{\pi}{6}$, $\frac{11\pi}{6}$, $-\frac{\pi}{6}$, $-\frac{11\pi}{6}$.

So, let's see if you've got all this down.

Example 3

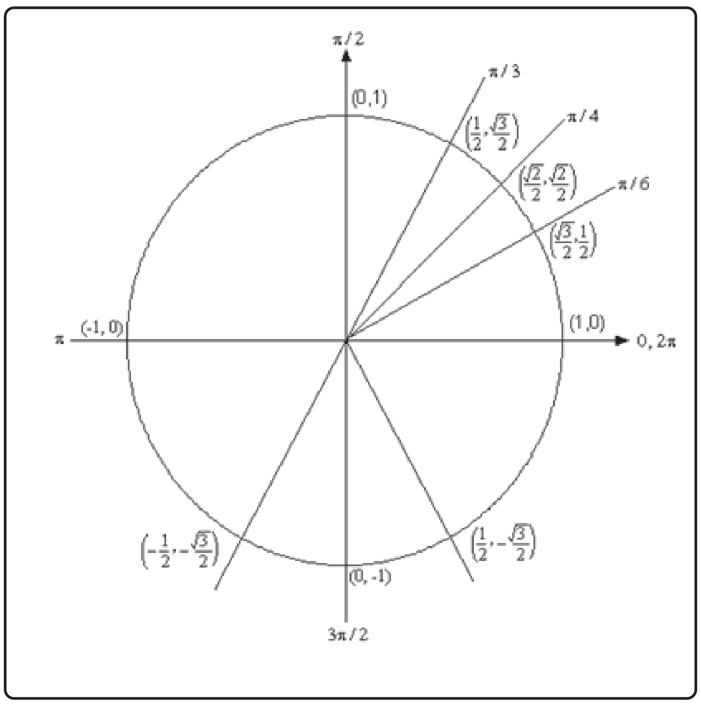
Solve $2\sin(5x) = -\sqrt{3}$ on $[-\pi, 2\pi]$.

Solution

This problem is very similar to the other problems in this section with a very important difference. We'll start this problem in exactly the same way as we did in the first example. So, first get the sine on one side by itself.

$$2\sin(5x) = -\sqrt{3}$$
$$\sin(5x) = \frac{-\sqrt{3}}{2}$$

We are looking for angles that will give $-\frac{\sqrt{3}}{2}$ out of the sine function. Let's again go to our trusty unit circle.



Now, there are no angles in the first quadrant for which sine has a value of $-\frac{\sqrt{3}}{2}$. However, there are two angles in the lower half of the unit circle for which sine will have a value of $-\frac{\sqrt{3}}{2}$. So, what are these angles?

Notice that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$. Given this we now know that the angle in the third quadrant will be $\frac{\pi}{3}$ below the ***negative*** x-axis or $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$. An easy way to remember this is to notice that we'll rotate half a revolution from the positive x-axis to get to the negative x-axis then add on $\frac{\pi}{3}$ to reach the angle we are looking for.

Likewise, the angle in the fourth quadrant will $\frac{\pi}{3}$ below the ***positive*** x-axis. So, we could use $-\frac{\pi}{3}$ or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$. Remember that we're typically looking for positive angles between 0 and 2π so we'll use the positive angle. An easy way to remember how to the positive angle here is to rotate one full revolution from the positive x-axis (*i.e.* 2π) and then backing off (*i.e.* subtracting) $\frac{\pi}{3}$.

Now we come to the very important difference between this problem and the previous problems in this section. The solution is **NOT**

$$x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

This is not the set of solutions because we are NOT looking for values of x for which $\sin(x) = -\frac{\sqrt{3}}{2}$, but instead we are looking for values of x for which $\sin(5x) = -\frac{\sqrt{3}}{2}$. Note the difference in the arguments of the sine function! One is x and the other is $5x$. This makes all the difference in the world in finding the solution! Therefore, the set of solutions is

$$5x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$5x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Well, actually, that's not quite the solution. We are looking for values of x so divide everything by 5 to get.

$$x = \frac{4\pi}{15} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{\pi}{3} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Notice that we also divided the $2\pi n$ by 5 as well! This is important! If we don't do that you **WILL** miss solutions. For instance, take $n = 1$.

$$x = \frac{4\pi}{15} + \frac{2\pi}{5} = \frac{10\pi}{15} = \frac{2\pi}{3} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{2\pi}{3}\right)\right) = \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3} + \frac{2\pi}{5} = \frac{11\pi}{15} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{11\pi}{15}\right)\right) = \sin\left(\frac{11\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

We'll leave it to you to verify our work showing they are solutions. However, it makes the point. If you didn't divide the $2\pi n$ by 5 you would have missed these solutions!

Okay, now that we've gotten all possible solutions it's time to find the solutions on the given interval. We'll do this as we did in the previous problem. Pick values of n and get the solutions.

$n = 0$.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(0\right)}{5} = \frac{4\pi}{15} < 2\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi\left(0\right)}{5} = \frac{\pi}{3} < 2\pi$$

$n = 1$.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(1\right)}{5} = \frac{2\pi}{3} < 2\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi\left(1\right)}{5} = \frac{11\pi}{15} < 2\pi$$

$n = 2$.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(2\right)}{5} = \frac{16\pi}{15} < 2\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi\left(2\right)}{5} = \frac{17\pi}{15} < 2\pi$$

$n = 3$.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(3\right)}{5} = \frac{22\pi}{15} < 2\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi\left(3\right)}{5} = \frac{23\pi}{15} < 2\pi$$

$n = 4$.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(4\right)}{5} = \frac{28\pi}{15} < 2\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi\left(4\right)}{5} = \frac{29\pi}{15} < 2\pi$$

$n = 5$.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(5\right)}{5} = \frac{34\pi}{15} > 2\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi\left(5\right)}{5} = \frac{35\pi}{15} > 2\pi$$

Okay, so we finally got past the right endpoint of our interval so we don't need any more positive n. Now let's take a look at the negative n and see what we've got.

$n = -1$.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(-1\right)}{5} = -\frac{2\pi}{15} > -\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi\left(-1\right)}{5} = -\frac{\pi}{15} > -\pi$$

$n = -2$.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(-2\right)}{5} = -\frac{8\pi}{15} > -\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi\left(-2\right)}{5} = -\frac{7\pi}{15} > -\pi$$

$n = -3.$

$$x = \frac{4\pi}{15} + \frac{2\pi(-3)}{5} = -\frac{14\pi}{15} > -\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi(-3)}{5} = -\frac{13\pi}{15} > -\pi$$

$n = -4.$

$$x = \frac{4\pi}{15} + \frac{2\pi(-4)}{5} = -\frac{4\pi}{3} < -\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi(-4)}{5} = -\frac{19\pi}{15} < -\pi$$

And we're now past the left endpoint of the interval. Sometimes, there will be many solutions as there were in this example. Putting all of this together gives the following set of solutions that lie in the given interval.

$$\frac{4\pi}{15}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{11\pi}{15}, \frac{16\pi}{15}, \frac{17\pi}{15}, \frac{22\pi}{15}, \frac{23\pi}{15}, \frac{28\pi}{15}, \frac{29\pi}{15}$$
$$-\frac{\pi}{15}, -\frac{2\pi}{15}, -\frac{7\pi}{15}, -\frac{8\pi}{15}, -\frac{13\pi}{15}, -\frac{14\pi}{15}$$

Let's work another example.

Example 4

Solve $\sin(2x) = -\cos(2x)$ on $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$.

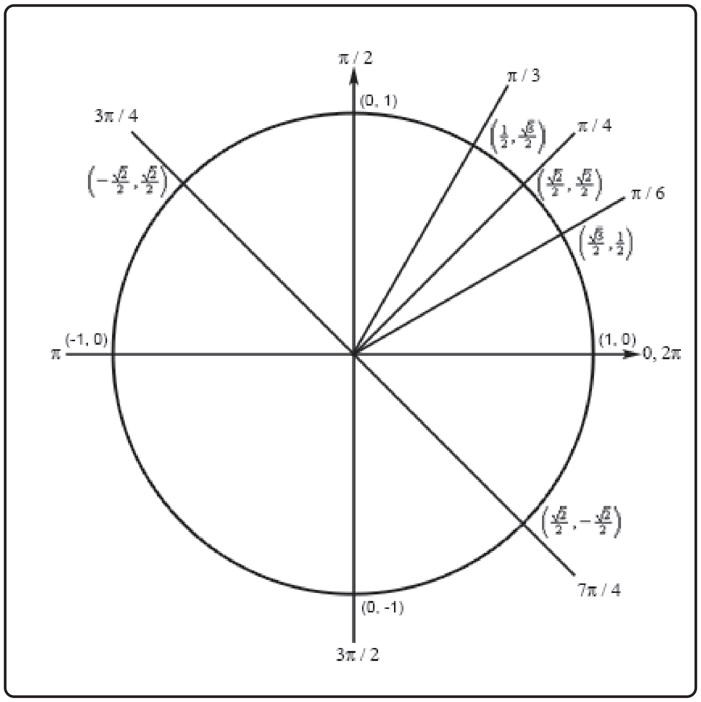
Solution

This problem is a little different from the previous ones. First, we need to do some rearranging and simplification.

$$\sin(2x) = -\cos(2x)$$
$$\frac{\sin(2x)}{\cos(2x)} = -1$$
$$\tan(2x) = -1$$

So, solving $\sin(2x) = -\cos(2x)$ is the same as solving $\tan(2x) = -1$. Hopefully, you'll recall that the smallest positive angle where tangent it -1 is $\frac{3\pi}{4}$ and this angle is in the 2^{nd} quadrant.

There is also a second angle for which tangent will be -1 and we can use the unit circle to illustrate this second angle. Let's take a look at the following unit circle.



As shown in this unit circle if we add π to our first angle we get $\frac{3\pi}{4} + \pi = \frac{7\pi}{4}$ and we get an angle that is in the fourth quadrant and has the same coordinates except for opposite signs. This means that tangent will also have a value of -1 here and so is a second angle.

This will always be true when solving tangent equations. Once we have one angle that will solve the equation a second angle will always be π plus the first angle.

All possible angles are then,

$$2x = \frac{3\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$2x = \frac{7\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Or, upon dividing by the 2 we get all possible solutions.

$$x = \frac{3\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{7\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Now, let's determine the solutions that lie in the given interval.

$n = 0$.

$$x = \frac{3\pi}{8} + \pi(0) = \frac{3\pi}{8} < \frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(0) = \frac{7\pi}{8} < \frac{3\pi}{2}$$

$n = 1$.

$$x = \frac{3\pi}{8} + \pi(1) = \frac{11\pi}{8} < \frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(1) = \frac{15\pi}{8} > \frac{3\pi}{2}$$

Unlike the previous example only one of these will be in the interval. This will happen occasionally so don't always expect both answers from a particular n to work. Also, we should now check $n = 2$ for the first to see if it will be in or out of the interval. I'll leave it to you to check that it's out of the interval.

Now, let's check the negative n.

$n = -1$.

$$x = \frac{3\pi}{8} + \pi(-1) = -\frac{5\pi}{8} > -\frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(-1) = -\frac{\pi}{8} > -\frac{3\pi}{2}$$

$n = -2$.

$$x = \frac{3\pi}{8} + \pi(-2) = -\frac{13\pi}{8} < -\frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(-2) = -\frac{9\pi}{8} > -\frac{3\pi}{2}$$

Again, only one will work here. I'll leave it to you to verify that $n = -3$ will give two answers that are both out of the interval.

The complete list of solutions is then,

$$-\frac{9\pi}{8}, -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}$$

Before moving on we need to address one issue about the previous example. The solution method used there is not the "standard" solution method. Because the second angle is just π plus the first and if we added π onto the second angle we'd be back at the line representing the first angle the more standard solution method is to just add πn onto the first angle to get,

$$2x = \frac{3\pi}{4} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Then dividing by 2 to get the full set of solutions,

$$x = \frac{3\pi}{8} + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

This set of solutions is identical to the set of solutions we got in the example (we'll leave it to you to plug in some n's and verify that). So, why did we not use the method in the previous example? Simple. The method in the previous example more closely mirrors the solution method for cosine and sine (*i.e.* they both, generally, give two sets of angles) and so for students that aren't comfortable with solving trig equations this gives a "consistent" solution method.

Let's work one more example so that we can make a point that needs to be understood when solving some trig equations.

Example 5

Solve $\cos(3x) = 2$.

Solution

This example is designed to remind you of certain properties about sine and cosine. Recall that $-1 \le \cos(\theta) \le 1$ and $-1 \le \sin(\theta) \le 1$. Therefore, since cosine will never be greater that 1 it definitely can't be 2. So **THERE ARE NO SOLUTIONS** to this equation!

It is important to remember that not all trig equations will have solutions.

In this section we solved some simple trig equations. There are more complicated trig equations that we can solve so don't leave this section with the feeling that there is nothing harder out there in the world to solve. In fact, we'll see at least one of the more complicated problems in the next section. Also, every one of these problems came down to solutions involving one of the "common" or "standard" angles. Most trig equations won't come down to one of those and will in fact need a calculator to solve. The next section is devoted to this kind of problem.

1.10 Common Graphs

The purpose of this section is to make sure that you're familiar with the graphs of many of the basic functions that you're liable to run across in a calculus class.

Example 1

Graph $y = -\frac{2}{5}x + 3$.

Solution

This is a line in the slope intercept form

$$y = mx + b$$

In this case the line has a y intercept of $(0, b)$ and a slope of m. Recall that slope can be thought of as

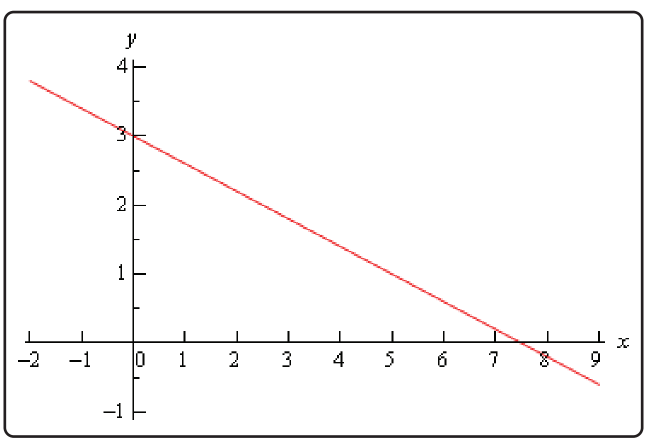
$$m = \frac{\text{rise}}{\text{run}}$$

Note that if the slope is negative we tend to think of the rise as a fall.

The slope allows us to get a second point on the line. Once we have any point on the line and the slope we move right by *run* and up/down by *rise* depending on the sign. This will be a second point on the line.

In this case we know $(0, 3)$ is a point on the line and the slope is $-\frac{2}{5}$. So starting at $(0, 3)$ we'll move 5 to the right (*i.e.* $0 \to 5$) and down 2 (*i.e.* $3 \to 1$) to get $(5, 1)$ as a second point on the line. Once we've got two points on a line all we need to do is plot the two points and connect them with a line.

Here's the sketch for this line.



Example 2

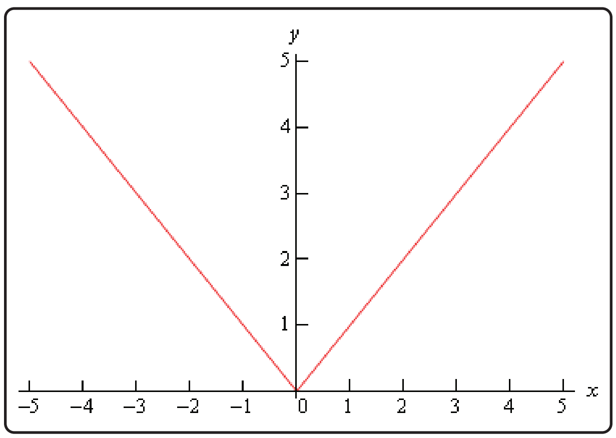
Graph $f(x) = |x|$.

Solution

There really isn't much to this problem outside of reminding ourselves of what absolute value is. Recall that the absolute value function is defined as,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph is then,



Example 3

Graph $f(x) = -x^2 + 2x + 3$.

Solution

This is a parabola in the general form.

$$f(x) = ax^2 + bx + c$$

In this form, the x-coordinate of the vertex (the highest or lowest point on the parabola) is $x = -\frac{b}{2a}$ and the y-coordinate is $y = f\left(-\frac{b}{2a}\right)$. So, for our parabola the coordinates of the vertex will be.

$$x = -\frac{2}{2(-1)} = 1$$
$$y = f(1) = -(1)^2 + 2(1) + 3 = 4$$

So, the vertex for this parabola is $(1, 4)$.

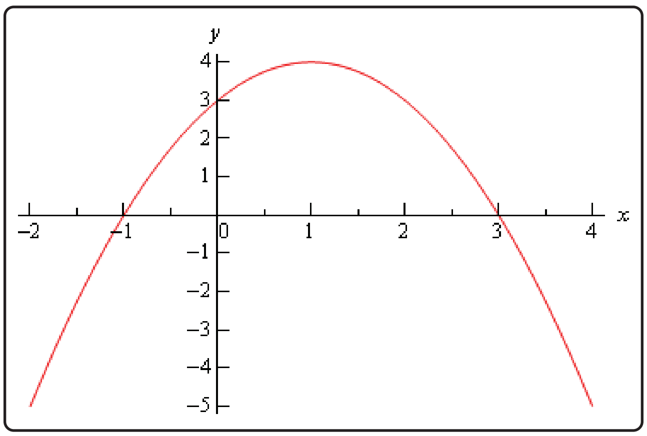
We can also determine which direction the parabola opens from the sign of a. If a is positive the parabola opens up and if a is negative the parabola opens down. In our case the parabola opens down.

Now, because the vertex is above the x-axis and the parabola opens down we know that we'll have x-intercepts (*i.e.* values of x for which we'll have $f(x) = 0$) on this graph. So, we'll solve the following.

$$\begin{aligned} -x^2 + 2x + 3 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \end{aligned}$$

So, we will have x-intercepts at $x = -1$ and $x = 3$. Notice that to make our life easier in the solution process we multiplied everything by -1 to get the coefficient of the x^2 positive. This made the factoring easier.

Here's a sketch of this parabola.



Example 4

Graph $f(y) = y^2 - 6y + 5$.

Solution

Most people come out of an Algebra class capable of dealing with functions in the form $y = f(x)$. However, many functions that you will have to deal with in a Calculus class are in the form $x = f(y)$ and can only be easily worked with in that form. So, you need to get used to working with functions in this form.

The nice thing about these kinds of function is that if you can deal with functions in the form $y = f(x)$ then you can deal with functions in the form $x = f(y)$ even if you aren't that familiar with them.

Let's first consider the equation.

$$y = x^2 - 6x + 5$$

This is a parabola that opens up and has a vertex of $(3, -4)$, as we know from our work in the previous example.

For our function we have essentially the same equation except the x and y's are switched around. In other words, we have a parabola in the form,

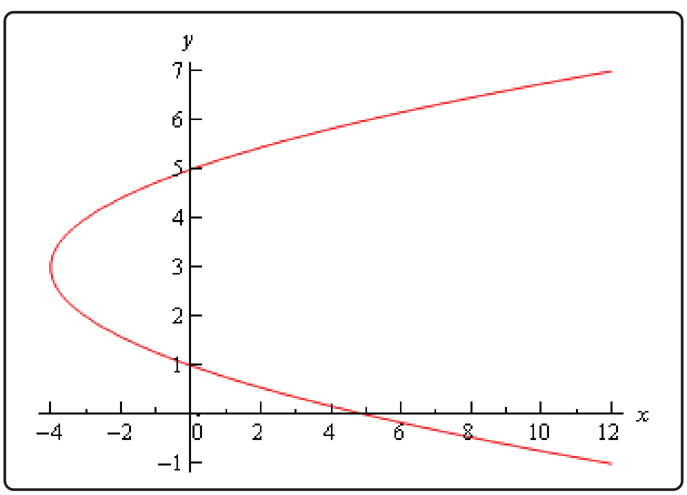
$$x = ay^2 + by + c$$

This is the general form of this kind of parabola and this will be a parabola that opens left or right depending on the sign of a. The y-coordinate of the vertex is given by $y = -\frac{b}{2a}$ and we find the x-coordinate by plugging this into the equation. So, you can see that this is very similar to the type of parabola that you're already used to dealing with.

Now, let's get back to the example. Our function is a parabola that opens to the right (a is positive) and has a vertex at $(-4, 3)$. The vertex is to the left of the y-axis and opens to the right so we'll need the y-intercepts (*i.e.* values of y for which we'll have $f(y) = 0$)). We find these just like we found x-intercepts in the previous problem.

$$y^2 - 6y + 5 = 0$$
$$(y - 5)(y - 1) = 0$$

So, our parabola will have y-intercepts at $y = 1$ and $y = 5$. Here's a sketch of the graph.



Example 5

Graph $x^2 + 2x + y^2 - 8y + 8 = 0$.

Solution

To determine just what kind of graph we've got here we need to complete the square on both the x and the y.

$$\begin{aligned} x^2 + 2x + y^2 - 8y + 8 &= 0 \\ x^2 + 2x + 1 - 1 + y^2 - 8y + 16 - 16 + 8 &= 0 \\ (x+1)^2 + (y-4)^2 &= 9 \end{aligned}$$

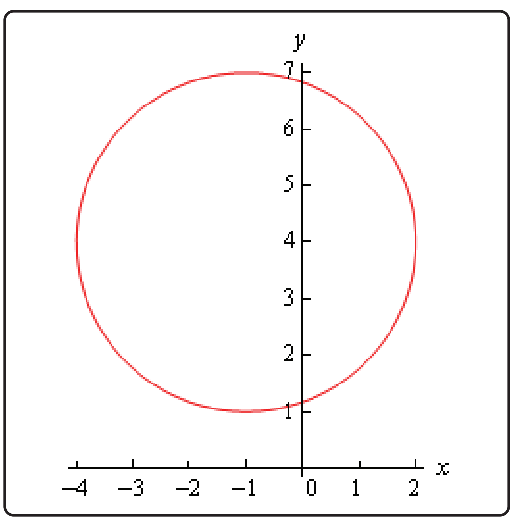
Recall that to complete the square we take the half of the coefficient of the x (or the y), square this and then add and subtract it to the equation.

Upon doing this we see that we have a circle and it's now written in standard form.

$$(x-h)^2 + (y-k)^2 = r^2$$

When circles are in this form we can easily identify the center (h, k) and radius r. Once we have these we can graph the circle simply by starting at the center and moving right, left, up and down by r to get the rightmost, leftmost, top most and bottom most points respectively.

Our circle has a center at $(-1, 4)$ and a radius of 3. Here's a sketch of this circle.



Example 6

Graph $\dfrac{(x-2)^2}{9} + 4(y+2)^2 = 1$.

Solution

This is an ellipse. The standard form of the ellipse is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

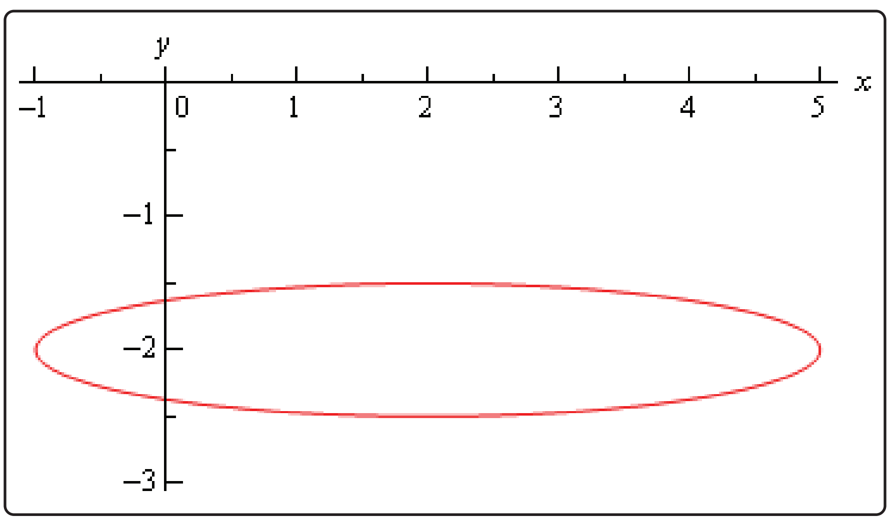
This is an ellipse with center (h, k) and the right most and left most points are a distance of a away from the center and the top most and bottom most points are a distance of b away from the center.

The ellipse for this problem has center $(2, -2)$ and has $a = 3$ and $b = \frac{1}{2}$. Note that to get the b we're really rewriting the equation as,

$$\frac{(x-2)^2}{9} + \frac{(y+2)^2}{{}^1/_4} = 1$$

to get it into standard from.

Here's a sketch of the ellipse.



Example 7

Graph $\frac{(x+1)^2}{9} - \frac{(y-2)^2}{4} = 1$.

Solution

This is a hyperbola. There are actually two standard forms for a hyperbola. Here are the basics for each form.

Form	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
Center	(h,k)	(h,k)
Opens	Opens right and left	Opens up and down
Vertices	a units right and left of center	b units up and down from center
Slope of Asymptotes	$\pm\frac{b}{a}$	$\pm\frac{b}{a}$

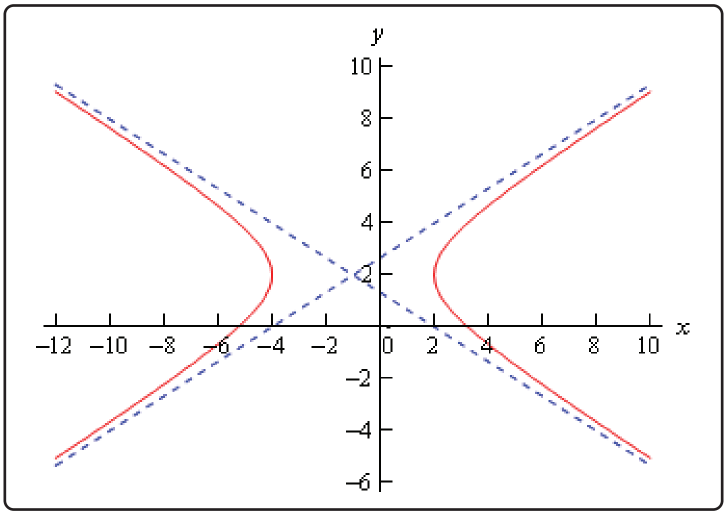
So, what does all this mean? First, notice that one of the terms is positive and the other is negative. This will determine which direction the two parts of the hyperbola open. If the x term is positive the hyperbola opens left and right. Likewise, if the y term is positive the parabola opens up and down.

Both have the same "center". Note that hyperbolas don't really have a center in the sense that circles and ellipses have centers. The center is the starting point in graphing a hyperbola. It tells us how to get to the vertices and how to get the asymptotes set up.

The asymptotes of a hyperbola are two lines that intersect at the center and have the slopes listed above. As you move farther out from the center the graph will get closer and closer to the asymptotes.

For the equation listed here the hyperbola will open left and right. Its center is $(-1,2)$. The two vertices are $(-4,2)$ and $(2,2)$. The asymptotes will have slopes $\pm\frac{2}{3}$.

Here is a sketch of this hyperbola. Note that the asymptotes are denoted by the two dashed lines.



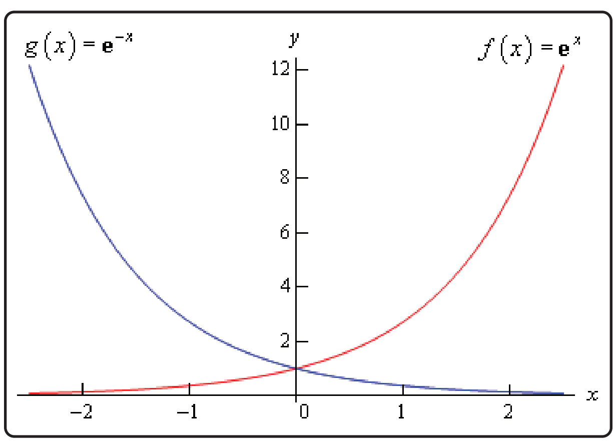
Example 8

Graph $f(x) = \mathbf{e}^x$ and $g(x) = \mathbf{e}^{-x}$.

Solution

There really isn't a lot to this problem other than making sure that both of these exponentials are graphed somewhere.

These will both show up with some regularity in later sections and their behavior as x goes to both plus and minus infinity will be needed and from this graph we can clearly see this behavior.

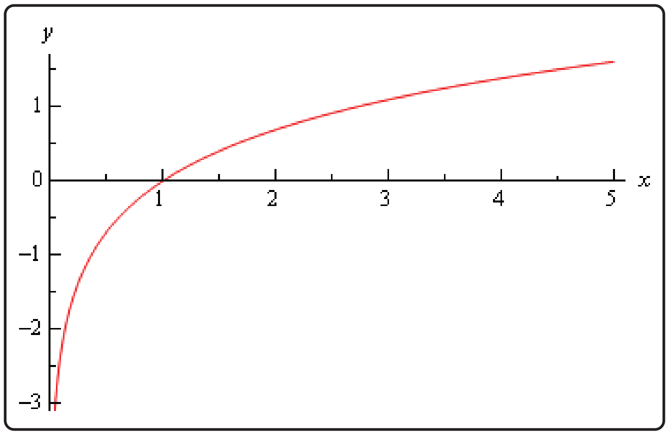


Example 9

Graph $f(x) = \ln(x)$.

Solution

This has already been graphed once in this review, but this puts it here with all the other "important" graphs.

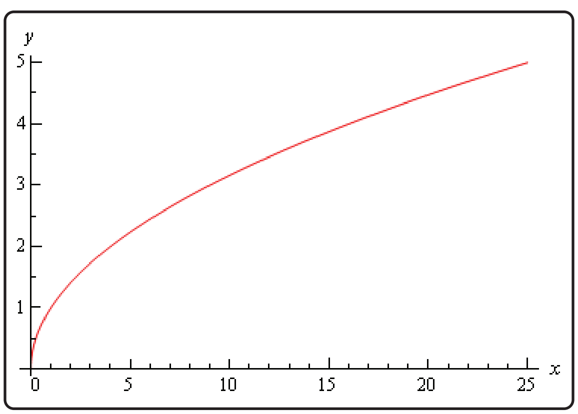


Example 10

Graph $y = \sqrt{x}$.

Solution

This one is fairly simple, we just need to make sure that we can graph it when need be.



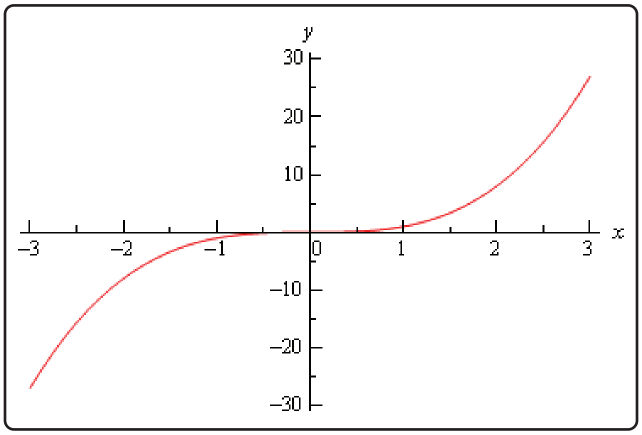
Remember that the domain of the square root function is $x \geq 0$.

Example 11

Graph $y = x^3$.

Solution

Again, there really isn't much to this other than to make sure it's been graphed somewhere so we can say we've done it.

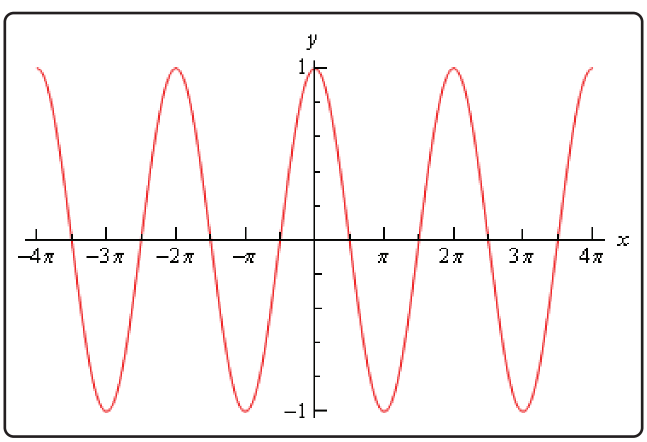


Example 12

Graph $y = \cos(x)$.

Solution

There really isn't a whole lot to this one. Here's the graph for $-4\pi \le x \le 4\pi$.



Let's also note here that we can put all values of x into cosine (which won't be the case for most of the trig functions) and so the domain is all real numbers. Also note that

$$-1 \le \cos(x) \le 1$$

It is important to notice that cosine will never be larger than 1 or smaller than -1. This will be useful on occasion in a calculus class. In general we can say that

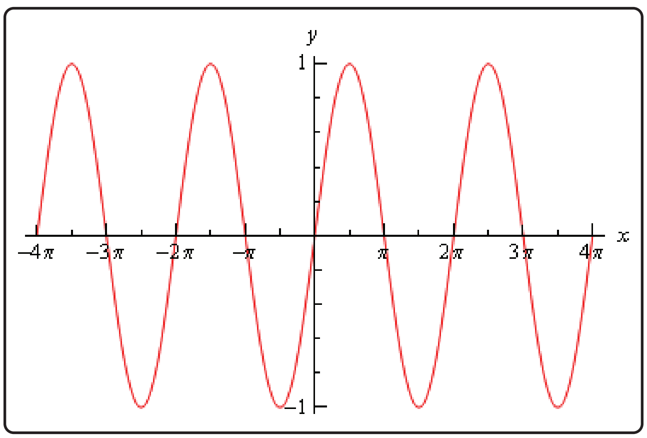
$$-R \le R\cos(\omega x) \le R$$

Example 13

Graph $y = \sin(x)$.

Solution

As with the previous problem there really isn't a lot to do other than graph it. Here is the graph for $-4\pi \le x \le 4\pi$.



From this graph we can see that sine has the same range that cosine does. In general

$$-R \le R\sin(\omega x) \le R$$

As with cosine, sine itself will never be larger than 1 and never smaller than -1. Also the domain of sine is all real numbers.

Example 14

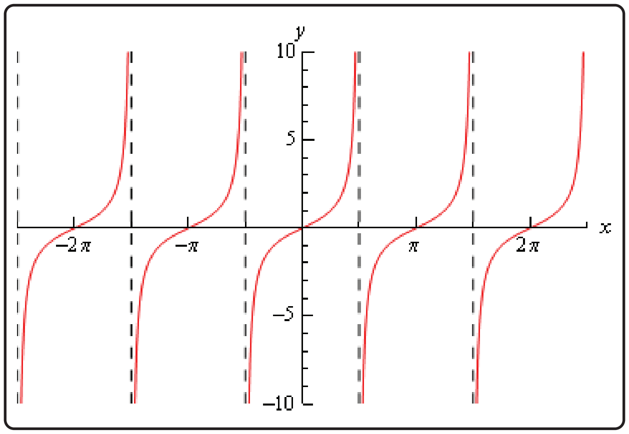
Graph $y = \tan(x)$.

Solution

In the case of tangent we have to be careful when plugging x's in since tangent doesn't exist wherever cosine is zero (remember that $\tan x = \frac{\sin x}{\cos x}$). Tangent will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of tangent on the range $-\frac{5\pi}{2} < x < \frac{5\pi}{2}$.



Example 15

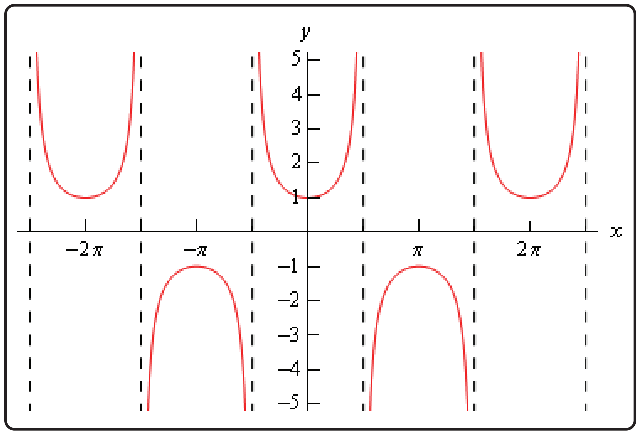
Graph $y = \sec(x)$.

Solution

As with tangent we will have to avoid x's for which cosine is zero (remember that $\sec x = \frac{1}{\cos x}$). Secant will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of secant on the range $-\frac{5\pi}{2} < x < \frac{5\pi}{2}$.



Notice that the graph is always greater than 1 or less than -1. This should not be terribly surprising. Recall that

$$-1 \leq \cos(x) \leq 1$$

So, one divided by something less than one will be greater than 1. Also, $^1/_{\pm 1} = \pm 1$ and so we get the following ranges for secant.

$$\sec(\omega\, x) \geq 1 \qquad \text{and} \qquad \sec(\omega\, x) \leq -1$$

Note that we did not graph cotangent or cosecant here. However, they are similar to the graphs of tangent and secant and you should be able to do quick sketches of them given the work above if needed.

Finally, note that we did not cover any of the basic transformations that are often used in graphing functions here. The practice problems for this section have quite a few problems designed to help you remember them. If you know the basic transformations it often makes graphing a much simpler process so if you are not comfortable with them you should work through the practice problems for this section.