

3.1 The Definition of the Derivative

In the first [section](#) of the Limits chapter we saw that the computation of the slope of a tangent line, the instantaneous rate of change of a function, and the instantaneous velocity of an object at $x = a$ all required us to compute the following limit.

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

We also saw that with a small change of notation this limit could also be written as,

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (3.1)$$

This is such an important limit and it arises in so many places that we give it a name. We call it a **derivative**. Here is the official definition of the derivative.

Definition of the Derivative

The **derivative of $f(x)$ with respect to x** is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \quad (3.2)$$

Note that we replaced all the a 's in [Equation 3.1](#) with x 's to acknowledge the fact that the derivative is really a function as well. We often “read” $f'(x)$ as “ f prime of x ”.

Let's compute a couple of derivatives using the definition.

Example 1

Find the derivative of the following function using the definition of the derivative.

$$f(x) = 2x^2 - 16x + 35$$

Solution

So, all we really need to do is to plug this function into the definition of the derivative, [Equation 3.2](#), and do some algebra. While, admittedly, the algebra will get somewhat unpleasant at times, but it's just algebra so don't get excited about the fact that we're now computing derivatives.

First plug the function into the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h} \end{aligned}$$

Be careful and make sure that you properly deal with parenthesis when doing the subtracting.

Now, we know from the previous chapter that we can't just plug in $h = 0$ since this will give us a division by zero error. So, we are going to have to do some work. In this case that means multiplying everything out and distributing the minus sign through on the second term. Doing this gives,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h} \\ &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 - 16h}{h} \end{aligned}$$

Notice that every term in the numerator that didn't have an h in it canceled out and we can now factor an h out of the numerator which will cancel against the h in the denominator. After that we can compute the limit.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{h(4x + 2h - 16)}{h} \\ &= \lim_{h \rightarrow 0} 4x + 2h - 16 \\ &= 4x - 16 \end{aligned}$$

So, the derivative is,

$$f'(x) = 4x - 16$$

Example 2

Find the derivative of the following function using the definition of the derivative.

$$g(t) = \frac{t}{t+1}$$

Solution

This one is going to be a little messier as far as the algebra goes. However, outside of that it

will work in exactly the same manner as the previous examples. First, we plug the function into the definition of the derivative,

$$\begin{aligned} g'(t) &= \lim_{h \rightarrow 0} \frac{g(t+h) - g(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{t+h}{t+h+1} - \frac{t}{t+1} \right) \end{aligned}$$

Note that we changed all the letters in the definition to match up with the given function. Also note that we wrote the fraction a much more compact manner to help us with the work.

As with the first problem we can't just plug in $h = 0$. So, we will need to simplify things a little. In this case we will need to combine the two terms in the numerator into a single rational expression as follows.

$$\begin{aligned} g'(t) &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{(t+h)(t+1) - t(t+h+1)}{(t+h+1)(t+1)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{t^2 + t + th + h - (t^2 + th + t)}{(t+h+1)(t+1)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{h}{(t+h+1)(t+1)} \right) \end{aligned}$$

Before finishing this let's note a couple of things. First, we didn't multiply out the denominator. Multiplying out the denominator will just overly complicate things so let's keep it simple. Next, as with the first example, after the simplification we only have terms with h 's in them left in the numerator and so we can now cancel an h out.

So, upon canceling the h we can evaluate the limit and get the derivative.

$$\begin{aligned} g'(t) &= \lim_{h \rightarrow 0} \frac{1}{(t+h+1)(t+1)} \\ &= \frac{1}{(t+1)(t+1)} \\ &= \frac{1}{(t+1)^2} \end{aligned}$$

The derivative is then,

$$g'(t) = \frac{1}{(t+1)^2}$$

Example 3

Find the derivative of the following function using the definition of the derivative.

$$R(z) = \sqrt{5z - 8}$$

Solution

First plug into the definition of the derivative as we've done with the previous two examples.

$$\begin{aligned} R'(z) &= \lim_{h \rightarrow 0} \frac{R(z+h) - R(z)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{5(z+h) - 8} - \sqrt{5z - 8}}{h} \end{aligned}$$

In this problem we're going to have to rationalize the numerator. You do remember [rationalization](#) from an Algebra class right? In an Algebra class you probably only rationalized the denominator, but you can also rationalize numerators. Remember that in rationalizing the numerator (in this case) we multiply both the numerator and denominator by the numerator except we change the sign between the two terms. Here's the rationalizing work for this problem,

$$\begin{aligned} R'(z) &= \lim_{h \rightarrow 0} \frac{(\sqrt{5(z+h) - 8} - \sqrt{5z - 8}) (\sqrt{5(z+h) - 8} + \sqrt{5z - 8})}{h (\sqrt{5(z+h) - 8} + \sqrt{5z - 8})} \\ &= \lim_{h \rightarrow 0} \frac{5z + 5h - 8 - (5z - 8)}{h (\sqrt{5(z+h) - 8} + \sqrt{5z - 8})} \\ &= \lim_{h \rightarrow 0} \frac{5h}{h (\sqrt{5(z+h) - 8} + \sqrt{5z - 8})} \end{aligned}$$

Again, after the simplification we have only h 's left in the numerator. So, cancel the h and evaluate the limit.

$$\begin{aligned} R'(z) &= \lim_{h \rightarrow 0} \frac{5}{\sqrt{5(z+h) - 8} + \sqrt{5z - 8}} \\ &= \frac{5}{\sqrt{5z - 8} + \sqrt{5z - 8}} \\ &= \frac{5}{2\sqrt{5z - 8}} \end{aligned}$$

And so we get a derivative of,

$$R'(z) = \frac{5}{2\sqrt{5z - 8}}$$

Let's work one more example. This one will be a little different, but it's got a point that needs to be made.

Example 4

Determine $f'(0)$ for $f(x) = |x|$.

Solution

Since this problem is asking for the derivative at a specific point we'll go ahead and use that in our work. It will make our life easier and that's always a good thing.

So, plug into the definition and simplify.

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h|}{h} \end{aligned}$$

We saw a situation like this back when we were looking at [limits at infinity](#). As in that section we can't just cancel the h 's. We will have to look at the two one sided limits and recall that

$$|h| = \begin{cases} h & \text{if } h \geq 0 \\ -h & \text{if } h < 0 \end{cases}$$

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{|h|}{h} &= \lim_{h \rightarrow 0^-} \frac{-h}{h} && \text{because } h < 0 \text{ in a left-hand limit.} \\ &= \lim_{h \rightarrow 0^-} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{|h|}{h} &= \lim_{h \rightarrow 0^+} \frac{h}{h} && \text{because } h > 0 \text{ in a right-hand limit.} \\ &= \lim_{h \rightarrow 0^+} 1 \\ &= 1 \end{aligned}$$

The two one-sided limits are different and so

$$\lim_{h \rightarrow 0} \frac{|h|}{h}$$

doesn't exist. However, this is the limit that gives us the derivative that we're after.

If the limit doesn't exist then the derivative doesn't exist either.

In this example we have finally seen a function for which the derivative doesn't exist at a point. This is a fact of life that we've got to be aware of. Derivatives will not always exist. Note as well that this doesn't say anything about whether or not the derivative exists anywhere else. In fact, the derivative of the absolute value function exists at every point except the one we just looked at, $x = 0$.

The preceding discussion leads to the following definition.

Definition

A function $f(x)$ is called **differentiable** at $x = a$ if $f'(a)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval.

The next theorem shows us a very nice relationship between functions that are continuous and those that are differentiable.

Theorem

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$.

See the [Proof of Various Derivative Formulas](#) section of the Extras appendix to see the proof of this theorem.

Note that this theorem does not work in reverse. Consider $f(x) = |x|$ and take a look at,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} |x| = 0 = f(0)$$

So, $f(x) = |x|$ is continuous at $x = 0$ but we've just shown above in Example 4 that $f(x) = |x|$ is not differentiable at $x = 0$.

Alternate Notation

Next, we need to discuss some alternate notation for the derivative. The typical derivative notation is the "prime" notation. However, there is another notation that is used on occasion so let's cover that.

Given a function $y = f(x)$ all of the following are equivalent and represent the derivative of $f(x)$ with respect to x .

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = \frac{d}{dx}(y)$$

Because we also need to evaluate derivatives on occasion we also need a notation for evaluating derivatives when using the fractional notation. So, if we want to evaluate the derivative at $x = a$ all of the following are equivalent.

$$f'(a) = y'|_{x=a} = \left. \frac{df}{dx} \right|_{x=a} = \left. \frac{dy}{dx} \right|_{x=a}$$

Note as well that on occasion we will drop the (x) part on the function to simplify the notation somewhat. In these cases the following are equivalent.

$$f'(x) = f'$$

As a final note in this section we'll acknowledge that computing most derivatives directly from the definition is a fairly complex (and sometimes painful) process filled with opportunities to make mistakes. In a couple of sections we'll start developing formulas and/or properties that will help us to take the derivative of many of the common functions so we won't need to resort to the definition of the derivative too often.

This does not mean however that it isn't important to know the definition of the derivative! It is an important definition that we should always know and keep in the back of our minds. It is just something that we're not going to be working with all that much.

3.2 Interpretation of the Derivative

Before moving on to the section where we learn how to compute derivatives by avoiding the limits we were evaluating in the previous section we need to take a quick look at some of the interpretations of the derivative. All of these interpretations arise from recalling how our definition of the derivative came about. The definition came about by noticing that all the problems that we worked in the first [section](#) in the Limits chapter required us to evaluate the same limit.

Rate of Change

The first interpretation of a derivative is rate of change. This was not the first problem that we looked at in the Limits chapter, but it is the most important interpretation of the derivative. If $f(x)$ represents a quantity at any x then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x = a$.

Example 1

Suppose that the amount of water in a holding tank at t minutes is given by $V(t) = 2t^2 - 16t + 35$. Determine each of the following.

- (a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?
- (b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?
- (c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?
- (d) Is the volume of water in the tank ever not changing? If so, when?

Solution

In the solution to this example we will use both notations for the derivative just to get you familiar with the different notations.

We are going to need the rate of change of the volume to answer these questions. This means that we will need the derivative of this function since that will give us a formula for the rate of change at any time t . Now, notice that the function giving the volume of water in the tank is the same function that we saw in Example 1 in the last [section](#) except the letters have changed. The change in letters between the function in this example versus the function in the example from the last section won't affect the work and so we can just use the answer from that example with an appropriate change in letters.

The derivative is.

$$V'(t) = 4t - 16 \quad \text{OR} \quad \frac{dV}{dt} = 4t - 16$$

Recall from our work in the first limits section that we determined that if the rate of change was positive then the quantity was increasing and if the rate of change was negative then the quantity was decreasing.

We can now work the problem.

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?

In this case all that we need is the rate of change of the volume at $t = 1$ or,

$$V'(1) = -12 \quad \text{OR} \quad \left. \frac{dV}{dt} \right|_{t=1} = -12$$

So, at $t = 1$ the rate of change is negative and so the volume must be decreasing at this time.

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?

Again, we will need the rate of change at $t = 5$.

$$V'(5) = 4 \quad \text{OR} \quad \left. \frac{dV}{dt} \right|_{t=5} = 4$$

In this case the rate of change is positive and so the volume must be increasing at $t = 5$.

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?

To answer this question all that we look at is the size of the rate of change and we don't worry about the sign of the rate of change. All that we need to know here is that the larger the number the faster the rate of change. So, in this case the volume is changing faster at $t = 1$ than at $t = 5$.

(d) Is the volume of water in the tank ever not changing? If so, when?

The volume will not be changing if it has a rate of change of zero. In order to have a rate of change of zero this means that the derivative must be zero. So, to answer this question we will then need to solve

$$V'(t) = 0 \quad \text{OR} \quad \frac{dV}{dt} = 0$$

This is easy enough to do.

$$4t - 16 = 0 \quad \Rightarrow \quad t = 4$$

So at $t = 4$ the volume isn't changing. Note that all this is saying is that for a brief instant the volume isn't changing. It doesn't say that at this point the volume will quit changing permanently.

If we go back to our answers from parts (a) and (b) we can get an idea about what is going on. At $t = 1$ the volume is decreasing and at $t = 5$ the volume is increasing. So, at some point in time the volume needs to switch from decreasing to increasing. That time is $t = 4$.

This is the time in which the volume goes from decreasing to increasing and so for the briefest instant in time the volume will quit changing as it changes from decreasing to increasing.

Note that one of the more common mistakes that students make in these kinds of problems is to try and determine increasing/decreasing from the function values rather than the derivatives. In this case if we took the function values at $t = 0$, $t = 1$ and $t = 5$ we would get,

$$V(0) = 35 \quad V(1) = 21 \quad V(5) = 5$$

Clearly as we go from $t = 0$ to $t = 1$ the volume has decreased. This might lead us to decide that AT $t = 1$ the volume is decreasing. However, we just can't say that. All we can say is that between $t = 0$ and $t = 1$ the volume has decreased at some point in time. The only way to know what is happening right at $t = 1$ is to compute $V'(1)$ and look at its sign to determine increasing/decreasing. In this case $V'(1)$ is negative and so the volume really is decreasing at $t = 1$.

Now, if we'd plugged into the function rather than the derivative we would have gotten the correct answer for $t = 1$ even though our reasoning would have been wrong. It's important to not let this give you the idea that this will always be the case. It just happened to work out in the case of $t = 1$.

To see that this won't always work let's now look at $t = 5$. If we plug $t = 1$ and $t = 5$ into the volume we can see that again as we go from $t = 1$ to $t = 5$ the volume has decreased. Again, however all this says is that the volume HAS decreased somewhere between $t = 1$ and $t = 5$. It does NOT say that the volume is decreasing at $t = 5$. The only way to know what is going on right at $t = 5$ is to compute $V'(5)$ and in this case $V'(5)$ is positive and so the volume is actually increasing at $t = 5$.

So, be careful. When asked to determine if a function is increasing or decreasing at a point make sure and look at the derivative. It is the only sure way to get the correct answer. We are not looking to determine if the function has increased/decreased by the time we reach a particular point. We are looking to determine if the function is increasing/decreasing at that point in question.

Slope of Tangent Line

This is the next major interpretation of the derivative. The slope of the tangent line to $f(x)$ at $x = a$ is $f'(a)$. The tangent line then is given by,

$$y = f(a) + f'(a)(x - a)$$

Example 2

Find the tangent line to the following function at $z = 3$.

$$R(z) = \sqrt{5z - 8}$$

Solution

We first need the derivative of the function and we found that in Example 3 in the last [section](#). The derivative is,

$$R'(z) = \frac{5}{2\sqrt{5z - 8}}$$

Now all that we need is the function value and derivative (for the slope) at $z = 3$.

$$R(3) = \sqrt{7} \quad m = R'(3) = \frac{5}{2\sqrt{7}}$$

The tangent line is then,

$$y = \sqrt{7} + \frac{5}{2\sqrt{7}}(z - 3)$$

Velocity

Recall that this can be thought of as a special case of the rate of change interpretation. If the position of an object is given by $f(t)$ after t units of time the velocity of the object at $t = a$ is given by $f'(a)$.

Example 3

Suppose that the position of an object after t hours is given by,

$$g(t) = \frac{t}{t+1}$$

Answer both of the following about this object.

- (a) Is the object moving to the right or the left at $t = 10$ hours?
- (b) Does the object ever stop moving?

Solution

Once again, we need the derivative and we found that in Example 2 in the last section. The derivative is,

$$g'(t) = \frac{1}{(t+1)^2}$$

- (a) **Is the object moving to the right or the left at $t = 10$ hours?**

To determine if the object is moving to the right (velocity is positive) or left (velocity is negative) we need the derivative at $t = 10$.

$$g'(10) = \frac{1}{121}$$

So, the velocity at $t = 10$ is positive and so the object is moving to the right at $t = 10$.

- (b) **Does the object ever stop moving?**

The object will stop moving if the velocity is ever zero. However, note that the only way a rational expression will ever be zero is if the numerator is zero. Since the numerator of the derivative (and hence the speed) is a constant it can't be zero.

Therefore, the object will never stop moving.

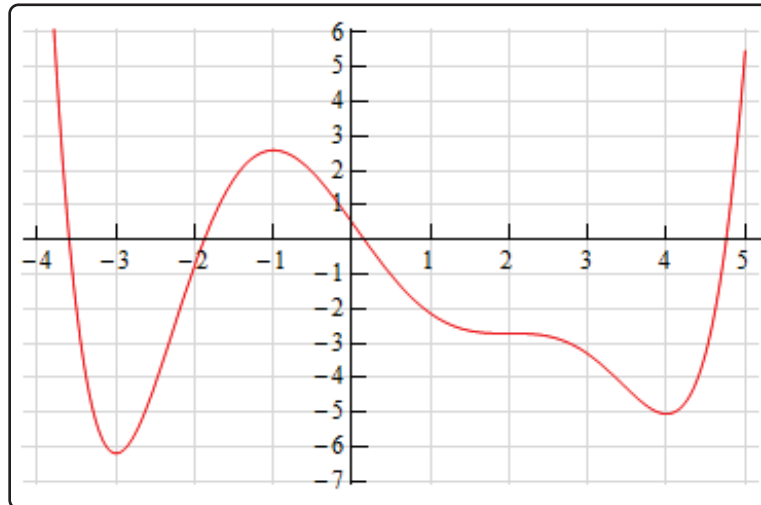
In fact, we can say a little more here. The object will always be moving to the right since the velocity is always positive.

We've seen three major interpretations of the derivative here. You will need to remember these, especially the rate of change, as they will show up continually throughout this course.

Before we leave this section let's work one more example that encompasses some of the ideas discussed here and is just a nice example to work.

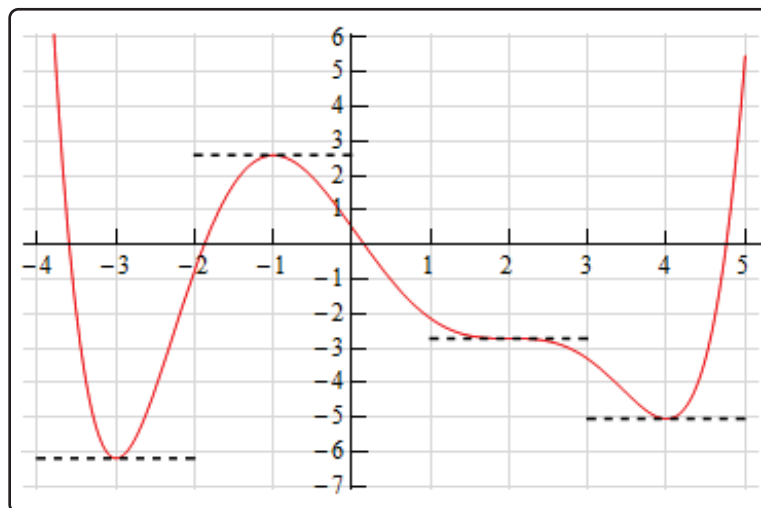
Example 4

Below is the sketch of a function $f(x)$. Sketch the graph of the derivative of this function, $f'(x)$.

**Solution**

At first glance this seems to be an all but impossible task. However, if you have some basic knowledge of the interpretations of the derivative you can get a sketch of the derivative. It will not be a perfect sketch for the most part, but you should be able to get most of the basic features of the derivative in the sketch.

Let's start off with the following sketch of the function with a couple of additions.



Notice that at $x = -3$, $x = -1$, $x = 2$ and $x = 4$ the tangent line to the function is horizontal. This means that the slope of the tangent line must be zero. Now, we know that the slope of the tangent line at a particular point is also the value of the derivative of the function at that point. Therefore, we now know that,

$$f'(-3) = 0 \quad f'(-1) = 0 \quad f'(2) = 0 \quad f'(4) = 0$$

This is a good starting point for us. It gives us a few points on the graph of the derivative. It also breaks the domain of the function up into regions where the function is increasing and decreasing. We know, from our discussions above, that if the function is increasing at a point then the derivative must be positive at that point. Likewise, we know that if the function is decreasing at a point then the derivative must be negative at that point.

We can now give the following information about the derivative.

$x < -3$	$f'(x) < 0$
$-3 < x < -1$	$f'(x) > 0$
$-1 < x < 2$	$f'(x) < 0$
$2 < x < 4$	$f'(x) < 0$
$x > 4$	$f'(x) > 0$

Remember that we are giving the signs of the derivatives here and these are solely a function of whether the function is increasing or decreasing. The sign of the function itself is completely immaterial here and will not in any way effect the sign of the derivative.

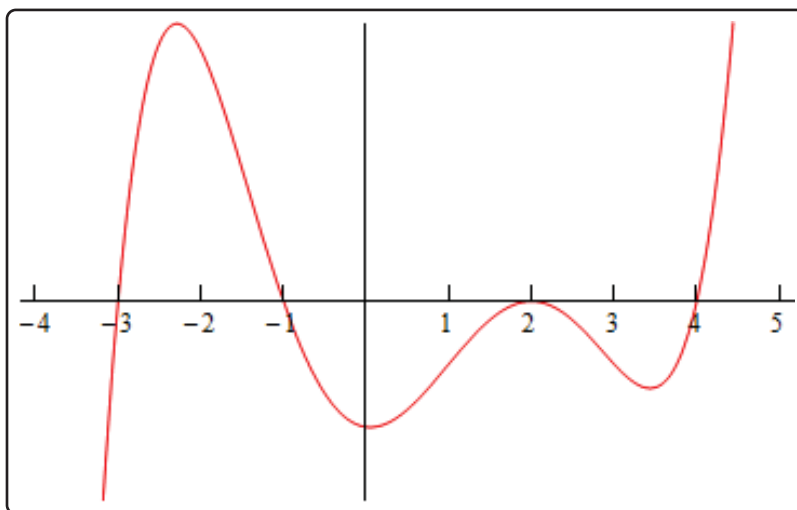
This may still seem like we don't have enough information to get a sketch, but we can get a little bit more information about the derivative from the graph of the function. In the range $x < -3$ we know that the derivative must be negative, however we can also see that the derivative needs to be increasing in this range. It is negative here until we reach $x = -3$ and at this point the derivative must be zero. The only way for the derivative to be negative to the left of $x = -3$ and zero at $x = -3$ is for the derivative to increase as we increase x towards $x = -3$.

Now, in the range $-3 < x < -1$ we know that the derivative must be zero at the endpoints and positive in between the two endpoints. Directly to the right of $x = -3$ the derivative must also be increasing (because it starts at zero and then goes positive - therefore it must be increasing). So, the derivative in this range must start out increasing and must eventually get back to zero at $x = -1$. So, at some point in this interval the derivative must start decreasing before it reaches $x = -1$. Now, we have to be careful here because this is just general behavior here at the two endpoints. We won't know where the derivative goes from increasing to decreasing and it may well change between increasing and decreasing several times before we reach $x = -1$. All we can really say is that immediately to the right of $x = -3$ the derivative will be increasing and immediately to the left of $x = -1$ the derivative will be decreasing.

Next, for the ranges $-1 < x < 2$ and $2 < x < 4$ we know the derivative will be zero at the endpoints and negative in between. Also, following the type of reasoning given above we can see in each of these ranges that the derivative will be decreasing just to the right of the left-hand endpoint and increasing just to the left of the right hand endpoint.

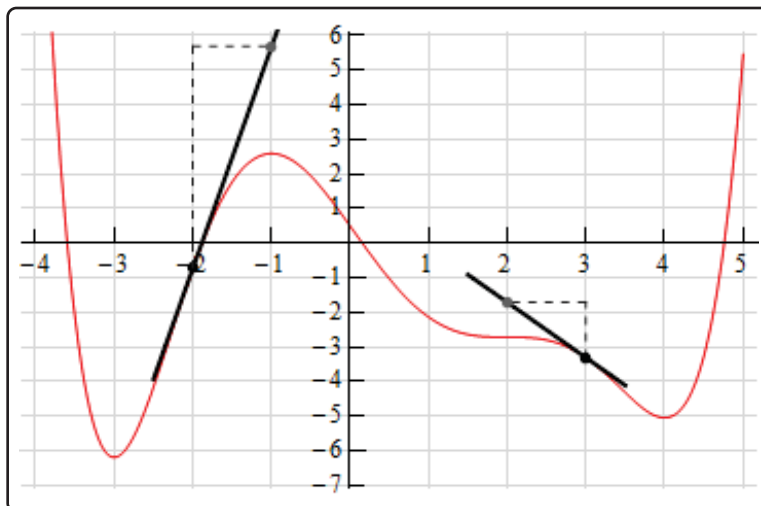
Finally, in the last region $x > 4$ we know that the derivative is zero at $x = 4$ and positive to the right of $x = 4$. Once again, following the reasoning above, the derivative must also be increasing in this range.

Putting all of this material together (and always taking the simplest choices for increasing and/or decreasing information) gives us the following sketch for the derivative.



Note that this was done with the actual derivative and so is in fact accurate. Any sketch you do will probably not look quite the same. The “humps” in each of the regions may be at different places and/or different heights for example. Also, note that we left off the vertical scale because given the information that we’ve got at this point there was no real way to know this information.

That doesn’t mean however that we can’t get some ideas of specific points on the derivative other than where we know the derivative to be zero. To see this let’s check out the following graph of the function (not the derivative, but the function).

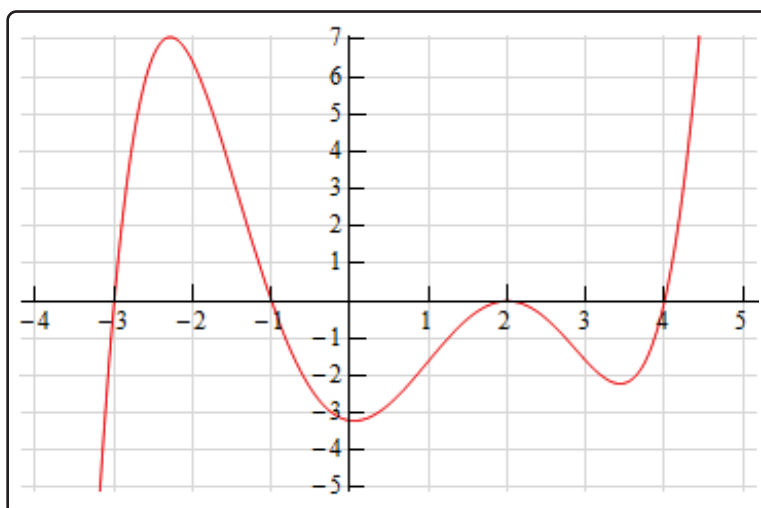


At $x = -2$ and $x = 3$ we've sketched in a couple of tangent lines. We can use the basic rise/run slope concept to estimate the value of the derivative at these points.

Let's start at $x = 3$. We've got two points on the line here. We can see that each seem to be about one-quarter of the way off the grid line. So, taking that into account and the fact that we go through one complete grid we can see that the slope of the tangent line, and hence the derivative, is approximately -1.5 .

At $x = -2$ it looks like (with some heavy estimation) that the second point is about 6.5 grids above the first point and so the slope of the tangent line here, and hence the derivative, is approximately 6.5.

Here is the sketch of the derivative with the vertical scale included and from this we can see that in fact our estimates are pretty close to reality.



Note that this idea of estimating values of derivatives can be a tricky process and does require a fair amount of (possible bad) approximations so while it can be used, you need to be careful with it.

We'll close out this section by noting that while we're not going to include an example here we could also use the graph of the derivative to give us a sketch of the function itself. In fact, in the next chapter where we discuss some applications of the derivative we will be looking using information the derivative gives us to sketch the graph of a function.

3.3 Differentiation Formulas

In the first section of this chapter we saw the [definition of the derivative](#) and we computed a couple of derivatives using the definition. As we saw in those examples there was a fair amount of work involved in computing the limits and the functions that we worked with were not terribly complicated.

For more complex functions using the definition of the derivative would be an almost impossible task. Luckily for us we won't have to use the definition terribly often. We will have to use it on occasion, however we have a large collection of formulas and properties that we can use to simplify our life considerably and will allow us to avoid using the definition whenever possible.

We will introduce most of these formulas over the course of the next several sections. We will start in this section with some of the basic properties and formulas. We will give the properties and formulas in this section in both "prime" notation and "fraction" notation.

Properties

$$1. (f(x) \pm g(x))' = f'(x) \pm g'(x) \quad \text{OR} \quad \frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}$$

In other words, to differentiate a sum or difference all we need to do is differentiate the individual terms and then put them back together with the appropriate signs. Note as well that this property is not limited to two functions.

See the [Proof of Various Derivative Formulas](#) section of the Extras appendix to see the proof of this property. It's a very simple proof using the definition of the derivative.

$$2. (cf(x))' = cf'(x) \quad \text{OR} \quad \frac{d}{dx}(cf(x)) = c \frac{df}{dx}, c \text{ is any number}$$

In other words, we can "factor" a multiplicative constant out of a derivative if we need to. See the [Proof of Various Derivative Formulas](#) section of the Extras appendix to see the proof of this property.

Note that we have not included formulas for the derivative of products or quotients of two functions here. The derivative of a product or quotient of two functions is not the product or quotient of the derivatives of the individual pieces. We will take a look at these in the next section.

Next, let's take a quick look at a couple of basic "computation" formulas that will allow us to actually compute some derivatives.

Formulas

1. If $f(x) = c$ then $f'(x) = 0$ OR $\frac{d}{dx}(c) = 0$

The derivative of a constant is zero. See the [Proof of Various Derivative Formulas](#) section of the Extras appendix to see the proof of this formula.

2. If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ OR $\frac{d}{dx}(x^n) = nx^{n-1}$, n is any number.

This formula is sometimes called the **power rule**. All we are doing here is bringing the original exponent down in front and multiplying and then subtracting one from the original exponent.

Note as well that in order to use this formula n must be a number, it can't be a variable. Also note that the base, the x , must be a variable, it can't be a number. It will be tempting in some later sections to misuse the Power Rule when we run in some functions where the exponent isn't a number and/or the base isn't a variable.

See the [Proof of Various Derivative Formulas](#) section of the Extras appendix to see the proof of this formula. There are actually three different proofs in this section. The first two restrict the formula to n being an integer because at this point that is all that we can do at this point. The third proof is for the general rule but does suppose that you've read most of this chapter.

These are the only properties and formulas that we'll give in this section. Let's compute some derivatives using these properties.

Example 1

Differentiate each of the following functions.

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

(b) $g(t) = 2t^6 + 7t^{-6}$

(c) $y = 8z^3 - \frac{1}{3z^5} + z - 23$

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \frac{2}{\sqrt{x^2}}$

(e) $h(x) = x^\pi - x^{\sqrt{2}}$

Solution

$$(a) f(x) = 15x^{100} - 3x^{12} + 5x - 46$$

In this case we have the sum and difference of four terms and so we will differentiate each of the terms using the first property from above and then put them back together with the proper sign. Also, for each term with a multiplicative constant remember that all we need to do is “factor” the constant out (using the second property) and then do the derivative.

$$\begin{aligned} f'(x) &= 15(100)x^{99} - 3(12)x^{11} + 5(1)x^0 - 0 \\ &= 1500x^{99} - 36x^{11} + 5 \end{aligned}$$

Notice that in the third term the exponent was a one and so upon subtracting 1 from the original exponent we get a new exponent of zero. Now recall that $x^0 = 1$. Don't forget to do any basic arithmetic that needs to be done such as any multiplication and/or division in the coefficients.

$$(b) g(t) = 2t^6 + 7t^{-6}$$

The point of this problem is to make sure that you deal with negative exponents correctly. Here is the derivative.

$$\begin{aligned} g'(t) &= 2(6)t^5 + 7(-6)t^{-7} \\ &= 12t^5 - 42t^{-7} \end{aligned}$$

Make sure that you correctly deal with the exponents in these cases, especially the negative exponents. It is an easy mistake to “go the other way” when subtracting one off from a negative exponent and get $-6t^{-5}$ instead of the correct $-6t^{-7}$.

$$(c) y = 8z^3 - \frac{1}{3z^5} + z - 23$$

Now in this function the second term is not correctly set up for us to use the power rule. The power rule requires that the term be a variable to a power only and the term must be in the numerator. So, prior to differentiating we first need to rewrite the second term into a form that we can deal with.

$$y = 8z^3 - \frac{1}{3}z^{-5} + z - 23$$

Note that we left the 3 in the denominator and only moved the variable up to the numerator. Remember that the only thing that gets an exponent is the term that is immediately to the left of the exponent. If we'd wanted the three to come up as well we'd have written,

$$\frac{1}{(3z)^5}$$

so be careful with this! It's a very common mistake to bring the 3 up into the numerator as well at this stage.

Now that we've gotten the function rewritten into a proper form that allows us to use the Power Rule we can differentiate the function. Here is the derivative for this part.

$$y' = 24z^2 + \frac{5}{3}z^{-6} + 1$$

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \frac{2}{\sqrt[5]{x^2}}$

All of the terms in this function have roots in them. In order to use the power rule we need to first convert all the roots to fractional exponents. Again, remember that the Power Rule requires us to have a variable to a number and that it must be in the numerator of the term. Here is the function written in "proper" form.

$$\begin{aligned} T(x) &= x^{\frac{1}{2}} + 9(x^7)^{\frac{1}{3}} - \frac{2}{(x^2)^{\frac{1}{5}}} \\ &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - \frac{2}{x^{\frac{2}{5}}} \\ &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - 2x^{-\frac{2}{5}} \end{aligned}$$

In the last two terms we combined the exponents. You should always do this with this kind of term. In a later section we will learn of a technique that would allow us to differentiate this term without combining exponents, however it will take significantly more work to do. Also, don't forget to move the term in the denominator of the third term up to the numerator. We can now differentiate the function.

$$\begin{aligned} T'(x) &= \frac{1}{2}x^{-\frac{1}{2}} + 9\left(\frac{7}{3}\right)x^{\frac{4}{3}} - 2\left(-\frac{2}{5}\right)x^{-\frac{7}{5}} \\ &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{63}{3}x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}} \end{aligned}$$

Make sure that you can deal with fractional exponents. You will see a lot of them in this class.

(e) $h(x) = x^\pi - x^{\sqrt{2}}$

In all of the previous examples the exponents have been nice integers or fractions. That is usually what we'll see in this class. However, the exponent only needs to be a number so don't get excited about problems like this one. They work exactly the same.

$$h'(x) = \pi x^{\pi-1} - \sqrt{2}x^{\sqrt{2}-1}$$

The answer is a little messy and we won't reduce the exponents down to decimals. However, this problem is not terribly difficult it just looks that way initially.

There is a general rule about derivatives in this class that you will need to get into the habit of using. When you see radicals you should always first convert the radical to a fractional exponent and then simplify exponents as much as possible. Following this rule will save you a lot of grief in the future.

Back when we first put down the properties we noted that we hadn't included a property for products and quotients. That doesn't mean that we can't differentiate any product or quotient at this point. There are some that we can do.

Example 2

Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2} (2x - x^2)$

(b) $h(t) = \frac{2t^5 + t^2 - 5}{t^2}$

Solution

(a) $y = \sqrt[3]{x^2} (2x - x^2)$

In this function we can't just differentiate the first term, differentiate the second term and then multiply the two back together. That just won't work. We will discuss this in detail in the next section so if you're not sure you believe that hold on for a bit and we'll be looking at that soon as well as showing you an example of why it won't work.

It is still possible to do this derivative however. All that we need to do is convert the radical to fractional exponents (as we should anyway) and then multiply this through the parenthesis.

$$y = x^{\frac{2}{3}} (2x - x^2) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$$

Now we can differentiate the function.

$$y' = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

(b) $h(t) = \frac{2t^5 + t^2 - 5}{t^2}$

As with the first part we can't just differentiate the numerator and the denominator and then put it back together as a fraction. Again, if you're not sure you believe this hold on until the next section and we'll take a more detailed look at this.

We can simplify this rational expression however as follows.

$$h(t) = \frac{2t^5}{t^2} + \frac{t^2}{t^2} - \frac{5}{t^2} = 2t^3 + 1 - 5t^{-2}$$

This is a function that we can differentiate.

$$h'(t) = 6t^2 + 10t^{-3}$$

So, as we saw in this example there are a few products and quotients that we can differentiate. If we can first do some simplification the functions will sometimes simplify into a form that can be differentiated using the properties and formulas in this section.

Before moving on to the next section let's work a couple of examples to remind us once again of some of the interpretations of the derivative.

Example 3

Is $f(x) = 2x^3 + \frac{300}{x^3} + 4$ increasing, decreasing or not changing at $x = -2$?

Solution

We know that the rate of change of a function is given by the functions derivative so all we need to do is it rewrite the function (to deal with the second term) and then take the derivative.

$$f(x) = 2x^3 + 300x^{-3} + 4 \quad \Rightarrow \quad f'(x) = 6x^2 - 900x^{-4} = 6x^2 - \frac{900}{x^4}$$

Note that we rewrote the last term in the derivative back as a fraction. This is not something we've done to this point and is only being done here to help with the evaluation in the next step. It's often easier to do the evaluation with positive exponents.

So, upon evaluating the derivative we get

$$f'(-2) = 6(4) - \frac{900}{16} = -\frac{129}{4} = -32.25$$

So, at $x = -2$ the derivative is negative and so the function is decreasing at $x = -2$.

Example 4

Find the equation of the tangent line to $f(x) = 4x - 8\sqrt{x}$ at $x = 16$.

Solution

We know that the equation of a tangent line is given by,

$$y = f(a) + f'(a)(x - a)$$

So, we will need the derivative of the function (don't forget to get rid of the radical).

$$f(x) = 4x - 8x^{\frac{1}{2}} \quad \Rightarrow \quad f'(x) = 4 - 4x^{-\frac{1}{2}} = 4 - \frac{4}{x^{\frac{1}{2}}}$$

Again, notice that we eliminated the negative exponent in the derivative solely for the sake of the evaluation. All we need to do then is evaluate the function and the derivative at the point in question, $x = 16$.

$$f(16) = 64 - 8(4) = 32 \quad f'(16) = 4 - \frac{4}{4} = 3$$

The tangent line is then,

$$y = 32 + 3(x - 16) = 3x - 16$$

Example 5

The position of an object at any time t (in hours) is given by,

$$s(t) = 2t^3 - 21t^2 + 60t - 10$$

Determine when the object is moving to the right and when the object is moving to the left.

Solution

The only way that we'll know for sure which direction the object is moving is to have the velocity in hand. Recall that if the velocity is positive the object is moving off to the right and if the velocity is negative then the object is moving to the left.

We need the derivative in order to get the velocity of the object. The derivative, and hence

the velocity, is,

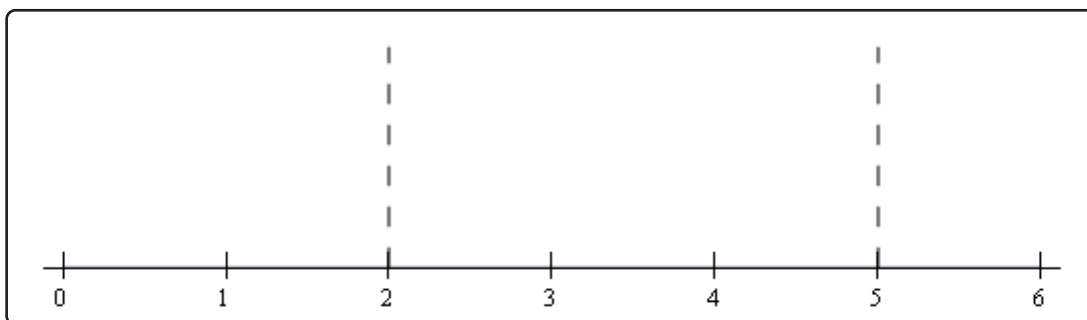
$$s'(t) = 6t^2 - 42t + 60 = 6(t^2 - 7t + 10) = 6(t - 2)(t - 5)$$

The reason for factoring the derivative will be apparent shortly.

Now, we need to determine where the derivative is positive and where the derivative is negative. There are several ways to do this. The method that we tend to prefer is the following.

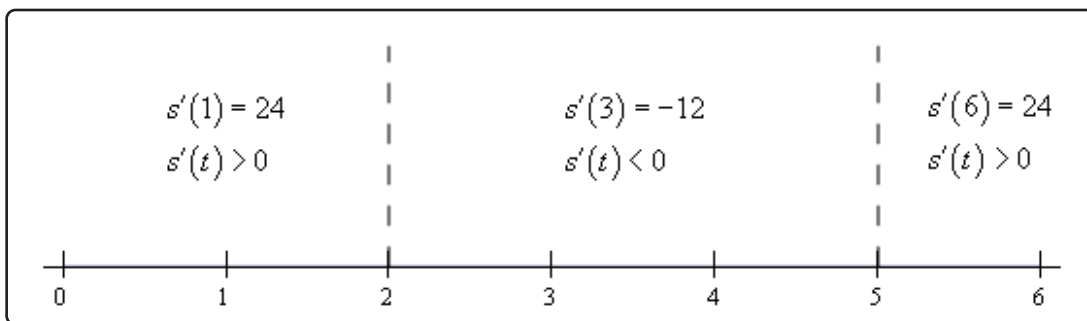
Since polynomials are continuous we know from the [Intermediate Value Theorem](#) that if the polynomial ever changes sign then it must have first gone through zero. So, if we knew where the derivative was zero we would know the only points where the derivative *might* change sign.

We can see from the factored form of the derivative that the derivative will be zero at $t = 2$ and $t = 5$. Let's graph these points on a number line.



Now, we can see that these two points divide the number line into three distinct regions. In each of these regions we **know** that the derivative will be the same sign. Recall the derivative can only change sign at the two points that are used to divide the number line up into the regions.

Therefore, all that we need to do is to check the derivative at a test point in each region and the derivative in that region will have the same sign as the test point. Here is the number line with the test points and results shown.



Here are the intervals in which the derivative is positive and negative.

$$\begin{aligned}\text{positive : } & -\infty < t < 2 \quad \& \quad 5 < t < \infty \\ \text{negative : } & 2 < t < 5\end{aligned}$$

We included negative t 's here because we could even though they may not make much sense for this problem. Once we know this we also can answer the question. The object is moving to the right and left in the following intervals.

$$\begin{aligned}\text{moving to the right : } & -\infty < t < 2 \quad \& \quad 5 < t < \infty \\ \text{moving to the left : } & 2 < t < 5\end{aligned}$$

Make sure that you can do the kind of work that we just did in this example. You will be asked numerous times over the course of the next two chapters to determine where functions are positive and/or negative. If you need some review or want to practice these kinds of problems you should check out the [Solving Inequalities](#) section of the [Algebra/Trig Review](#).

3.4 Product and Quotient Rule

In the previous section we noted that we had to be careful when differentiating products or quotients. It's now time to look at products and quotients and see why.

First let's take a look at why we have to be careful with products and quotients. Suppose that we have the two functions $f(x) = x^3$ and $g(x) = x^6$. Let's start by computing the derivative of the product of these two functions. This is easy enough to do directly.

$$(fg)' = (x^3x^6)' = (x^9)' = 9x^8$$

Remember that on occasion we will drop the (x) part on the functions to simplify notation somewhat. We've done that in the work above.

Now, let's try the following.

$$f'(x)g'(x) = (3x^2)(6x^5) = 18x^7$$

So, we can very quickly see that.

$$(fg)' \neq f'g'$$

In other words, the derivative of a product is not the product of the derivatives.

Using the same functions we can do the same thing for quotients.

$$\left(\frac{f}{g}\right)' = \left(\frac{x^3}{x^6}\right)' = \left(\frac{1}{x^3}\right)' = (x^{-3})' = -3x^{-4} = -\frac{3}{x^4}$$

$$\frac{f'(x)}{g'(x)} = \frac{3x^2}{6x^5} = \frac{1}{2x^3}$$

So, again we can see that,

$$\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$$

To differentiate products and quotients we have the **Product Rule** and the **Quotient Rule**.

Product Rule

If the two functions $f(x)$ and $g(x)$ are differentiable (*i.e.* the derivative exist) then the product is differentiable and,

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

The proof of the Product Rule is shown in the [Proof of Various Derivative Formulas](#) section of the Extras appendix.

Quotient Rule

If the two functions $f(x)$ and $g(x)$ are differentiable (*i.e.* the derivative exist) then the quotient is differentiable and,

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Note that the numerator of the quotient rule is very similar to the product rule so be careful to not mix the two up!

The proof of the Quotient Rule is shown in the [Proof of Various Derivative Formulas](#) section of the Extras appendix.

Let's do a couple of examples of the product rule.

Example 1

Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2} (2x - x^2)$

(b) $f(x) = (6x^3 - x)(10 - 20x)$

Solution

At this point there really aren't a lot of reasons to use the product rule. As we noted in the previous section all we would need to do for either of these is to just multiply out the product and then differentiate.

With that said we will use the product rule on these so we can see an example or two. As we add more functions to our repertoire and as the functions become more complicated the product rule will become more useful and in many cases required.

(a) $y = \sqrt[3]{x^2} (2x - x^2)$

Note that we took the derivative of this function in the previous [section](#) and didn't use the product rule at that point. We should however get the same result here as we did then.

Now let's do the problem here. There's not really a lot to do here other than use the product rule. However, before doing that we should convert the radical to a fractional exponent as always.

$$y = x^{\frac{2}{3}} (2x - x^2)$$

Now let's take the derivative. So, we take the derivative of the first function times the second then add on to that the first function times the derivative of the second function.

$$y' = \frac{2}{3}x^{-\frac{1}{3}}(2x - x^2) + x^{\frac{2}{3}}(2 - 2x)$$

This is NOT what we got in the previous section for this derivative. However, with some simplification we can arrive at the same answer.

$$y' = \frac{4}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{5}{3}} + 2x^{\frac{2}{3}} - 2x^{\frac{5}{3}} = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

This is what we got for an answer in the previous section so that is a good check of the product rule.

(b) $f(x) = (6x^3 - x)(10 - 20x)$

This one is actually easier than the previous one. Let's just run it through the product rule.

$$\begin{aligned} f'(x) &= (18x^2 - 1)(10 - 20x) + (6x^3 - x)(-20) \\ &= -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

Since it was easy to do we went ahead and simplified the results a little.

Let's now work an example or two with the quotient rule. In this case, unlike the product rule examples, a couple of these functions will require the quotient rule in order to get the derivative. For the last two however, we can avoid the quotient rule if we'd like to as we'll see.

Example 2

Differentiate each of the following functions.

(a) $W(z) = \frac{3z + 9}{2 - z}$

(b) $h(x) = \frac{4\sqrt{x}}{x^2 - 2}$

(c) $f(x) = \frac{4}{x^6}$

(d) $y = \frac{w^6}{5}$

Solution

$$(a) W(z) = \frac{3z + 9}{2 - z}$$

There isn't a lot to do here other than to use the quotient rule. Here is the work for this function.

$$\begin{aligned} W'(z) &= \frac{3(2 - z) - (3z + 9)(-1)}{(2 - z)^2} \\ &= \frac{15}{(2 - z)^2} \end{aligned}$$

$$(b) h(x) = \frac{4\sqrt{x}}{x^2 - 2}$$

Again, not much to do here other than use the quotient rule. Don't forget to convert the square root into a fractional exponent.

$$\begin{aligned} h'(x) &= \frac{4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}(x^2 - 2) - 4x^{\frac{1}{2}}(2x)}{(x^2 - 2)^2} \\ &= \frac{2x^{\frac{3}{2}} - 4x^{-\frac{1}{2}} - 8x^{\frac{3}{2}}}{(x^2 - 2)^2} \\ &= \frac{-6x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}}{(x^2 - 2)^2} \end{aligned}$$

$$(c) f(x) = \frac{4}{x^6}$$

It seems strange to have this one here rather than being the first part of this example given that it definitely appears to be easier than any of the previous two. In fact, it is easier. There is a point to doing it here rather than first. In this case there are two ways to do compute this derivative. There is an easy way and a hard way and in this case the hard way is the quotient rule. That's the point of this example.

Let's do the quotient rule and see what we get.

$$f'(x) = \frac{(0)(x^6) - 4(6x^5)}{(x^6)^2} = \frac{-24x^5}{x^{12}} = -\frac{24}{x^7}$$

Now, that was the "hard" way. So, what was so hard about it? Well actually it wasn't that hard, there is just an easier way to do it that's all. However, having said that, a common mistake here is to do the derivative of the numerator (a constant) incorrectly. For some reason many people will give the derivative of the numerator in these kinds of problems as a 1 instead of 0! Also, there is some simplification that needs to be done in these kinds of problems if you do the quotient rule.

The easy way is to do what we did in the previous section.

$$f(x) = 4x^{-6} \quad f'(x) = -24x^{-7} = -\frac{24}{x^7}$$

Either way will work, but I'd rather take the easier route if I had the choice.

(d) $y = \frac{w^6}{5}$

This problem also seems a little out of place. However, it is here again to make a point. Do not confuse this with a quotient rule problem. While you can do the quotient rule on this function there is no reason to use the quotient rule on this. Simply rewrite the function as

$$y = \frac{1}{5}w^6$$

and differentiate as always.

$$y' = \frac{6}{5}w^5$$

Finally, let's not forget about our applications of derivatives.

Example 3

Suppose that the amount of air in a balloon at any time t is given by

$$V(t) = \frac{6\sqrt[3]{t}}{4t+1}$$

Determine if the balloon is being filled with air or being drained of air at $t = 8$.

Solution

If the balloon is being filled with air then the volume is increasing and if it's being drained of air then the volume will be decreasing. In other words, we need to get the derivative so that we can determine the rate of change of the volume at $t = 8$.

This will require the quotient rule.

$$\begin{aligned} V'(t) &= \frac{2t^{-\frac{2}{3}}(4t+1) - 6t^{\frac{1}{3}}(4)}{(4t+1)^2} \\ &= \frac{-16t^{\frac{1}{3}} + 2t^{-\frac{2}{3}}}{(4t+1)^2} \\ &= \frac{-16t^{\frac{1}{3}} + \frac{2}{t^{\frac{2}{3}}}}{(4t+1)^2} \end{aligned}$$

Note that we simplified the numerator more than usual here. This was only done to make the derivative easier to evaluate.

The rate of change of the volume at $t = 8$ is then,

$$\begin{aligned} V'(8) &= \frac{-16(2) + \frac{2}{4}}{(33)^2} & (8)^{\frac{1}{3}} &= 2 & (8)^{\frac{2}{3}} &= \left((8)^{\frac{1}{3}}\right)^2 = (2)^2 = 4 \\ &= -\frac{63}{2178} = -\frac{7}{242} \end{aligned}$$

So, the rate of change of the volume at $t = 8$ is negative and so the volume must be decreasing. Therefore, air is being drained out of the balloon at $t = 8$.

As a final topic let's note that the product rule can be extended to more than two functions, for instance.

$$\begin{aligned} (fgh)' &= f'gh + fg'h + fgh' \\ (fghw)' &= f'ghw + fg'hw + fgh'w + fghw' \end{aligned}$$

Deriving these products of more than two functions is actually pretty simple. For example, let's take a look at the three function product rule.

First, we don't think of it as a product of three functions but instead of the product rule of the two functions fg and h which we can then use the two function product rule on. Doing this gives,

$$(fgh)' = ([fg]h)' = [fg]'h + [fg]h'$$

Note that we put brackets on the fg part to make it clear we are thinking of that term as a single function. Now all we need to do is use the two function product rule on the $[fg]'$ term and then do a little simplification.

$$(fgh)' = [f'g + fg']h + [fg]h' = f'gh + fg'h + fgh'$$

Any product rule with more functions can be derived in a similar fashion.

With this section and the previous section we are now able to differentiate powers of x as well as sums, differences, products and quotients of these kinds of functions. However, there are many more functions out there in the world that are not in this form. The next few sections give many of these functions as well as give their derivatives.

3.5 Derivatives of Trig Functions

With this section we're going to start looking at the derivatives of functions other than polynomials or roots of polynomials. We'll start this process off by taking a look at the derivatives of the six trig functions. Two of the derivatives will be derived. The remaining four are left to you and will follow similar proofs for the two given here.

Before we actually get into the derivatives of the trig functions we need to give a couple of limits that will show up in the derivation of two of the derivatives.

Fact

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \qquad \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$$

See the [Proof of Trig Limits](#) section of the Extras appendix to see the proof of these two limits.

Before proceeding a quick note. Students often ask why we always use radians in a Calculus class. This is the reason why! The proof of the formula involving sine above requires the angles to be in radians. If the angles are in degrees the limit involving sine is not 1 and so the formulas we will derive below would also change. The formulas below would pick up an extra constant that would just get in the way of our work and so we use radians to avoid that. So, remember to always use radians in a Calculus class!

Before we start differentiating trig functions let's work a quick set of limit problems that this fact now allows us to do.

Example 1

Evaluate each of the following limits.

(a) $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{6\theta}$

(b) $\lim_{x \rightarrow 0} \frac{\sin(6x)}{x}$

(c) $\lim_{x \rightarrow 0} \frac{x}{\sin(7x)}$

(d) $\lim_{t \rightarrow 0} \frac{\sin(3t)}{\sin(8t)}$

(e) $\lim_{x \rightarrow 4} \frac{\sin(x-4)}{x-4}$

$$(f) \lim_{z \rightarrow 0} \frac{\cos(2z) - 1}{z}$$

Solution

$$(a) \lim_{\theta \rightarrow 0} \frac{\sin \theta}{6\theta}$$

There really isn't a whole lot to this limit. In fact, it's only here to contrast with the next example so you can see the difference in how these work. In this case since there is only a 6 in the denominator we'll just factor this out and then use the fact.

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{6\theta} = \frac{1}{6} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{1}{6} (1) = \frac{1}{6}$$

$$(b) \lim_{x \rightarrow 0} \frac{\sin(6x)}{x}$$

Now, in this case we can't factor the 6 out of the sine so we're stuck with it there and we'll need to figure out a way to deal with it. To do this problem we need to notice that in the fact the argument of the sine is the same as the denominator (*i.e.* both θ 's). So we need to get both of the argument of the sine and the denominator to be the same. We can do this by multiplying the numerator and the denominator by 6 as follows.

$$\lim_{x \rightarrow 0} \frac{\sin(6x)}{x} = \lim_{x \rightarrow 0} \frac{6 \sin(6x)}{6x} = 6 \lim_{x \rightarrow 0} \frac{\sin(6x)}{6x}$$

Note that we factored the 6 in the numerator out of the limit. At this point, while it may not look like it, we can use the fact above to finish the limit.

To see that we can use the fact on this limit let's do a **change of variables**. A change of variables is really just a renaming of portions of the problem to make something look more like something we know how to deal with. They can't always be done, but sometimes, such as this case, they can simplify the problem. The change of variables here is to let $\theta = 6x$ and then notice that as $x \rightarrow 0$ we also have $\theta \rightarrow 6(0) = 0$. When doing a change of variables in a limit we need to change all the x 's into θ 's and that includes the one in the limit.

Doing the change of variables on this limit gives,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(6x)}{x} &= 6 \lim_{x \rightarrow 0} \frac{\sin(6x)}{6x} && \text{let } \theta = 6x \\ &= 6 \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \\ &= 6(1) \\ &= 6 \end{aligned}$$

And there we are. Note that we didn't really need to do a change of variables here. All we really need to notice is that the argument of the sine is the same as the denominator and then we can use the fact. A change of variables, in this case, is really only needed to make it clear that the fact does work.

$$(c) \lim_{x \rightarrow 0} \frac{x}{\sin(7x)}$$

In this case we appear to have a small problem in that the function we're taking the limit of here is upside down compared to that in the fact. This is not the problem it appears to be once we notice that,

$$\frac{x}{\sin(7x)} = \frac{1}{\frac{\sin(7x)}{x}}$$

and then all we need to do is recall a nice property of limits that allows us to do ,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x}{\sin(7x)} &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin(7x)}{x}} \\ &= \frac{\lim_{x \rightarrow 0} 1}{\lim_{x \rightarrow 0} \frac{\sin(7x)}{x}} \\ &= \frac{1}{\lim_{x \rightarrow 0} \frac{\sin(7x)}{x}} \end{aligned}$$

With a little rewriting we can see that we do in fact end up needing to do a limit like the one we did in the previous part. So, let's do the limit here and this time we won't bother with a change of variable to help us out. All we need to do is multiply the numerator and denominator of the fraction in the denominator by 7 to get things set up to use the fact. Here is the work for this limit.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x}{\sin(7x)} &= \frac{1}{\lim_{x \rightarrow 0} \frac{7 \sin(7x)}{7x}} \\ &= \frac{1}{7 \lim_{x \rightarrow 0} \frac{\sin(7x)}{7x}} \\ &= \frac{1}{(7)(1)} \\ &= \frac{1}{7} \end{aligned}$$

$$(d) \lim_{t \rightarrow 0} \frac{\sin(3t)}{\sin(8t)}$$

This limit looks nothing like the limit in the fact, however it can be thought of as a combination of the previous two parts by doing a little rewriting. First, we'll split the

fraction up as follows,

$$\lim_{t \rightarrow 0} \frac{\sin(3t)}{\sin(8t)} = \lim_{t \rightarrow 0} \frac{\sin(3t)}{1} \frac{1}{\sin(8t)}$$

Now, the fact wants a t in the denominator of the first and in the numerator of the second. This is easy enough to do if we multiply the whole thing by $\frac{t}{t}$ (which is just one after all and so won't change the problem) and then do a little rearranging as follows,

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sin(3t)}{\sin(8t)} &= \lim_{t \rightarrow 0} \frac{\sin(3t)}{1} \frac{1}{\sin(8t)} \frac{t}{t} \\ &= \lim_{t \rightarrow 0} \frac{\sin(3t)}{t} \frac{t}{\sin(8t)} \\ &= \left(\lim_{t \rightarrow 0} \frac{\sin(3t)}{t} \right) \left(\lim_{t \rightarrow 0} \frac{t}{\sin(8t)} \right) \end{aligned}$$

At this point we can see that this really is two limits that we've seen before. Here is the work for each of these and notice on the second limit that we're going to work it a little differently than we did in the previous part. This time we're going to notice that it doesn't really matter whether the sine is in the numerator or the denominator as long as the argument of the sine is the same as what's in the numerator the limit is still one.

Here is the work for this limit.

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sin(3t)}{\sin(8t)} &= \left(\lim_{t \rightarrow 0} \frac{3 \sin(3t)}{3t} \right) \left(\lim_{t \rightarrow 0} \frac{8t}{8 \sin(8t)} \right) \\ &= \left(3 \lim_{t \rightarrow 0} \frac{\sin(3t)}{3t} \right) \left(\frac{1}{8} \lim_{t \rightarrow 0} \frac{8t}{\sin(8t)} \right) \\ &= (3) \left(\frac{1}{8} \right) \\ &= \frac{3}{8} \end{aligned}$$

(e) $\lim_{x \rightarrow 4} \frac{\sin(x-4)}{x-4}$

This limit almost looks the same as that in the fact in the sense that the argument of the sine is the same as what is in the denominator. However, notice that, in the limit, x is going to 4 and not 0 as the fact requires. However, with a change of variables we can see that this limit is in fact set to use the fact above regardless.

So, let $\theta = x - 4$ and then notice that as $x \rightarrow 4$ we have $\theta \rightarrow 0$. Therefore, after doing the change of variable the limit becomes,

$$\lim_{x \rightarrow 4} \frac{\sin(x-4)}{x-4} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$(f) \lim_{z \rightarrow 0} \frac{\cos(2z) - 1}{z}$$

The previous parts of this example all used the sine portion of the fact. However, we could just have easily used the cosine portion so here is a quick example using the cosine portion to illustrate this. We'll not put in much explanation here as this really does work in the same manner as the sine portion.

$$\begin{aligned} \lim_{z \rightarrow 0} \frac{\cos(2z) - 1}{z} &= \lim_{z \rightarrow 0} \frac{2(\cos(2z) - 1)}{2z} \\ &= 2 \lim_{z \rightarrow 0} \frac{\cos(2z) - 1}{2z} \\ &= 2(0) \\ &= 0 \end{aligned}$$

All that is required to use the fact is that the argument of the cosine is the same as the denominator.

Okay, now that we've gotten this set of limit examples out of the way let's get back to the main point of this section, differentiating trig functions.

We'll start with finding the derivative of the sine function. To do this we will need to use the definition of the derivative. It's been a while since we've had to use this, but sometimes there just isn't anything we can do about it. Here is the definition of the derivative for the sine function.

$$\frac{d}{dx}(\sin(x)) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Since we can't just plug in $h = 0$ to evaluate the limit we will need to use the following trig formula on the first sine in the numerator.

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

Doing this gives us,

$$\begin{aligned} \frac{d}{dx}(\sin(x)) &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\ &= \lim_{h \rightarrow 0} \sin(x) \frac{\cos(h) - 1}{h} + \lim_{h \rightarrow 0} \cos(x) \frac{\sin(h)}{h} \end{aligned}$$

As you can see upon using the trig formula we can combine the first and third term and then factor a sine out of that. We can then break up the fraction into two pieces, both of which can be dealt with separately.

Now, both of the limits here are limits as h approaches zero. In the first limit we have a $\sin(x)$ and in the second limit we have a $\cos(x)$. Both of these are only functions of x only and as h moves in towards zero this has no effect on the value of x . Therefore, as far as the limits are concerned, these two functions are constants and can be factored out of their respective limits. Doing this gives,

$$\frac{d}{dx}(\sin(x)) = \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$$

At this point all we need to do is use the limits in the fact above to finish out this problem.

$$\frac{d}{dx}(\sin(x)) = \sin(x)(0) + \cos(x)(1) = \cos(x)$$

Differentiating cosine is done in a similar fashion. It will require a different trig formula, but other than that is an almost identical proof. The details will be left to you. When done with the proof you should get,

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

With these two out of the way the remaining four are fairly simple to get. All the remaining four trig functions can be defined in terms of sine and cosine and these definitions, along with appropriate derivative rules, can be used to get their derivatives.

Let's take a look at tangent. Tangent is defined as,

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Now that we have the derivatives of sine and cosine all that we need to do is use the quotient rule on this. Let's do that.

$$\begin{aligned} \frac{d}{dx}(\tan(x)) &= \frac{d}{dx} \left(\frac{\sin(x)}{\cos(x)} \right) \\ &= \frac{\cos(x) \cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \end{aligned}$$

Now, recall that $\cos^2(x) + \sin^2(x) = 1$ and if we also recall the definition of secant in terms of cosine we arrive at,

$$\begin{aligned} \frac{d}{dx}(\tan(x)) &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x) \end{aligned}$$

The remaining three trig functions are also quotients involving sine and/or cosine and so can be differentiated in a similar manner. We'll leave the details to you. Here are the derivatives of all six of the trig functions.

Derivatives of the six trig functions

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

At this point we should work some examples.

Example 2

Differentiate each of the following functions.

(a) $g(x) = 3 \sec(x) - 10 \cot(x)$

(b) $h(w) = 3w^{-4} - w^2 \tan(w)$

(c) $y = 5 \sin(x) \cos(x) + 4 \csc(x)$

(d) $P(t) = \frac{\sin(t)}{3 - 2 \cos(t)}$

Solution

(a) $g(x) = 3 \sec(x) - 10 \cot(x)$

There really isn't a whole lot to this problem. We'll just differentiate each term using the formulas from above.

$$\begin{aligned} g'(x) &= 3 \sec(x) \tan(x) - 10(-\csc^2(x)) \\ &= 3 \sec(x) \tan(x) + 10 \csc^2(x) \end{aligned}$$

(b) $h(w) = 3w^{-4} - w^2 \tan(w)$

In this part we will need to use the product rule on the second term and note that we really will need the product rule here. There is no other way to do this derivative unlike

what we saw when we first looked at the product rule. When we first looked at the product rule the only functions we knew how to differentiate were polynomials and in those cases all we really needed to do was multiply them out and we could take the derivative without the product rule. We are now getting into the point where we will be forced to do the product rule at times regardless of whether or not we want to.

We will also need to be careful with the minus sign in front of the second term and make sure that it gets dealt with properly. There are two ways to deal with this. One way is to make sure that you use a set of parentheses as follows,

$$\begin{aligned} h'(w) &= -12w^{-5} - (2w \tan(w) + w^2 \sec^2(w)) \\ &= -12w^{-5} - 2w \tan(w) - w^2 \sec^2(w) \end{aligned}$$

Because the second term is being subtracted off of the first term then the whole derivative of the second term must also be subtracted off of the derivative of the first term. The parentheses make this idea clear.

A potentially easier way to do this is to think of the minus sign as part of the first function in the product. Or, in other words the two functions in the product, using this idea, are $-w^2$ and $\tan(w)$. Doing this gives,

$$h'(w) = -12w^{-5} - 2w \tan(w) - w^2 \sec^2(w)$$

So, regardless of how you approach this problem you will get the same derivative.

(c) $y = 5 \sin(x) \cos(x) + 4 \csc(x)$

As with the previous part we'll need to use the product rule on the first term. We will also think of the 5 as part of the first function in the product to make sure we deal with it correctly. Alternatively, you could make use of a set of parentheses to make sure the 5 gets dealt with properly. Either way will work, but we'll stick with thinking of the 5 as part of the first term in the product. Here's the derivative of this function.

$$\begin{aligned} y' &= 5 \cos(x) \cos(x) + 5 \sin(x) (-\sin(x)) - 4 \csc(x) \cot(x) \\ &= 5 \cos^2(x) - 5 \sin^2(x) - 4 \csc(x) \cot(x) \end{aligned}$$

(d) $P(t) = \frac{\sin(t)}{3 - 2 \cos(t)}$

In this part we'll need to use the quotient rule to take the derivative.

$$\begin{aligned} P'(t) &= \frac{\cos(t)(3 - 2 \cos(t)) - \sin(t)(2 \sin(t))}{(3 - 2 \cos(t))^2} \\ &= \frac{3 \cos(t) - 2 \cos^2(t) - 2 \sin^2(t)}{(3 - 2 \cos(t))^2} \end{aligned}$$

Be careful with the signs when differentiating the denominator. The negative sign we get from differentiating the cosine will cancel against the negative sign that is already there.

This appears to be done, but there is actually a fair amount of simplification that can yet be done. To do this we need to factor out a “-2” from the last two terms in the numerator and then make use of the fact that $\cos^2(\theta) + \sin^2(\theta) = 1$.

$$\begin{aligned} P'(t) &= \frac{3 \cos(t) - 2(\cos^2(t) + \sin^2(t))}{(3 - 2 \cos(t))^2} \\ &= \frac{3 \cos(t) - 2}{(3 - 2 \cos(t))^2} \end{aligned}$$

As a final problem here let's not forget that we still have our standard interpretations to derivatives.

Example 3

Suppose that the amount of money in a bank account is given by

$$P(t) = 500 + 100 \cos(t) - 150 \sin(t)$$

where t is in years. During the first 10 years in which the account is open when is the amount of money in the account increasing?

Solution

To determine when the amount of money is increasing we need to determine when the rate of change is positive. Since we know that the rate of change is given by the derivative that is the first thing that we need to find.

$$P'(t) = -100 \sin(t) - 150 \cos(t)$$

Now, we need to determine where in the first 10 years this will be positive. This is equivalent to asking where in the interval $[0, 10]$ is the derivative positive. Recall that both sine and cosine are continuous functions and so the derivative is also a continuous function. The [Intermediate Value Theorem](#) then tells us that the derivative can only change sign if it first goes through zero.

So, we need to solve the following equation.

$$\begin{aligned} -100 \sin(t) - 150 \cos(t) &= 0 \\ 100 \sin(t) &= -150 \cos(t) \\ \frac{\sin(t)}{\cos(t)} &= -1.5 \\ \tan(t) &= -1.5 \end{aligned}$$

The solution to this equation is,

$$\begin{aligned} t &= 2.1588 + 2\pi n, & n &= 0, \pm 1, \pm 2, \dots \\ t &= 5.3004 + 2\pi n, & n &= 0, \pm 1, \pm 2, \dots \end{aligned}$$

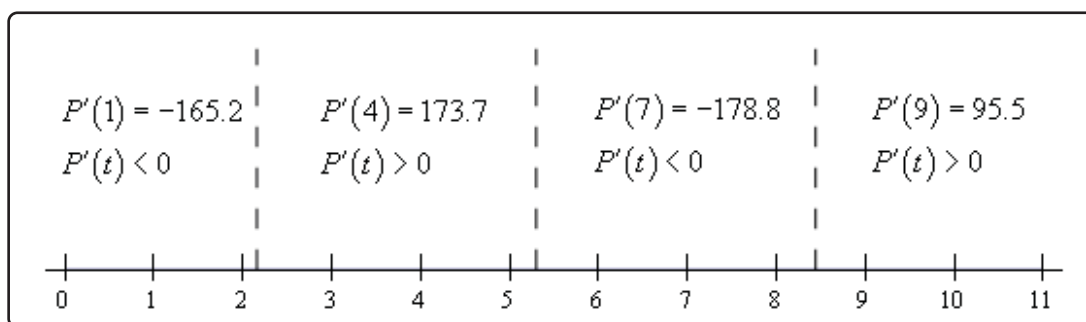
If you don't recall how to solve trig equations go back and take a look at the sections on [solving trig equations](#) in the Review chapter.

We are only interested in those solutions that fall in the range $[0, 10]$. Plugging in values of n into the solutions above we see that the values we need are,

$$\begin{aligned} t &= 2.1588 & t &= 2.1588 + 2\pi = 8.4420 \\ t &= 5.3004 \end{aligned}$$

So, much like solving polynomial inequalities all that we need to do is sketch in a number line and add in these points. These points will divide the number line into regions in which the derivative must always be the same sign. All that we need to do then is choose a test point from each region to determine the sign of the derivative in that region.

Here is the number line with all the information on it.



So, it looks like the amount of money in the bank account will be increasing during the following intervals.

$$2.1588 < t < 5.3004 \quad 8.4420 < t < 10$$

Note that we can't say anything about what is happening after $t = 10$ since we haven't done any work for t 's after that point.

In this section we saw how to differentiate trig functions. We also saw in the last example that our interpretations of the derivative are still valid so we can't forget those.

Also, it is important that we be able to solve trig equations as this is something that will arise off and on in this course. It is also important that we can do the kinds of number lines that we used in the last example to determine where a function is positive and where a function is negative. This is something that we will be doing on occasion in both this chapter and the next.

Chain Rule

Suppose that we have two functions $f(x)$ and $g(x)$ and they are both differentiable.

1. If we define $F(x) = (f \circ g)(x)$ then the derivative of $F(x)$ is,

$$F'(x) = f'(g(x)) g'(x)$$

2. If we have $y = f(u)$ and $u = g(x)$ then the derivative of y is,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Each of these forms have their uses, however we will work mostly with the first form in this class. To see the proof of the Chain Rule see the [Proof of Various Derivative Formulas](#) section of the Extras appendix.

Now, let's go back and use the Chain Rule on the function that we used when we opened this section.

Example 1

Use the Chain Rule to differentiate $R(z) = \sqrt{5z - 8}$.

Solution

We've already identified the two functions that we needed for the composition, but let's write them back down anyway and take their derivatives.

$$\begin{aligned} f(z) &= \sqrt{z} & g(z) &= 5z - 8 \\ f'(z) &= \frac{1}{2\sqrt{z}} & g'(z) &= 5 \end{aligned}$$

So, using the chain rule we get,

$$\begin{aligned} R'(z) &= f'(g(z)) g'(z) \\ &= f'(5z - 8) g'(z) \\ &= \frac{1}{2}(5z - 8)^{-\frac{1}{2}} (5) \\ &= \frac{1}{2\sqrt{5z - 8}} (5) \\ &= \frac{5}{2\sqrt{5z - 8}} \end{aligned}$$

And this is what we got using the definition of the derivative.

In general, we don't really do all the composition stuff in using the Chain Rule. That can get a little complicated and in fact obscures the fact that there is a quick and easy way of remembering the chain rule that doesn't require us to think in terms of function composition.

Let's take the function from the previous example and rewrite it slightly.

$$R(z) = \underbrace{(5z - 8)}_{\text{inside function}} \underbrace{\frac{1}{2}}_{\text{outside function}}$$

This function has an "inside function" and an "outside function". The outside function is the square root or the exponent of $\frac{1}{2}$ depending on how you want to think of it and the inside function is the stuff that we're taking the square root of or raising to the $\frac{1}{2}$, again depending on how you want to look at it.

The derivative is then,

$$R'(z) = \overbrace{\frac{1}{2}}^{\text{derivative of outside function}} \underbrace{(5z - 8)^{-\frac{1}{2}}}_{\text{inside function left alone}} \underbrace{(5)}_{\text{derivative of inside function}}$$

In general, this is how we think of the chain rule. We identify the "inside function" and the "outside function". We then differentiate the outside function leaving the inside function alone and multiply all of this by the derivative of the inside function. In its general form this is,

$$F'(x) = \underbrace{f'}_{\text{derivative of outside function}} \underbrace{(g(x))}_{\text{inside function left alone}} \underbrace{g'(x)}_{\text{derivative of inside function}}$$

We can always identify the "outside function" in the examples below by asking ourselves how we would evaluate the function. For instance in the $R(z)$ case if we were to ask ourselves what $R(2)$ is we would first evaluate the stuff under the radical and then finally take the square root of this result. The square root is the last operation that we perform in the evaluation and this is also the outside function. The outside function will always be the last operation you would perform if you were going to evaluate the function.

Let's take a look at some examples of the Chain Rule.

Example 2

Differentiate each of the following.

(a) $f(x) = \sin(3x^2 + x)$

(b) $f(t) = (2t^3 + \cos(t))^{50}$

(c) $h(w) = e^{w^4 - 3w^2 + 9}$

(d) $g(x) = \ln(x^{-4} + x^4)$

(e) $y = \sec(1 - 5x)$

(f) $P(t) = \cos^4(t) + \cos(t^4)$

Solution

(a) $f(x) = \sin(3x^2 + x)$

It looks like the outside function is the sine and the inside function is $3x^2 + x$. The derivative is then.

$$f'(x) = \underbrace{\cos}_{\substack{\text{derivative of} \\ \text{outside function}}} \underbrace{(3x^2 + x)}_{\substack{\text{leave inside} \\ \text{function alone}}} \underbrace{(6x + 1)}_{\substack{\text{times derivative} \\ \text{of inside function}}}$$

Or with a little rewriting,

$$f'(x) = (6x + 1) \cos(3x^2 + x)$$

(b) $f(t) = (2t^3 + \cos(t))^{50}$

In this case the outside function is the exponent of 50 and the inside function is all the stuff on the inside of the parenthesis. The derivative is then.

$$\begin{aligned} f'(t) &= 50(2t^3 + \cos(t))^{49} (6t^2 - \sin(t)) \\ &= 50(6t^2 - \sin(t))(2t^3 + \cos(t))^{49} \end{aligned}$$

(c) $h(w) = e^{w^4 - 3w^2 + 9}$

Identifying the outside function in the previous two was fairly simple since it really was the “outside” function in some sense. In this case we need to be a little careful. Recall