

2.1 Tangent Lines and Rates of Change

In this section we are going to take a look at two fairly important problems in the study of calculus. There are two reasons for looking at these problems now.

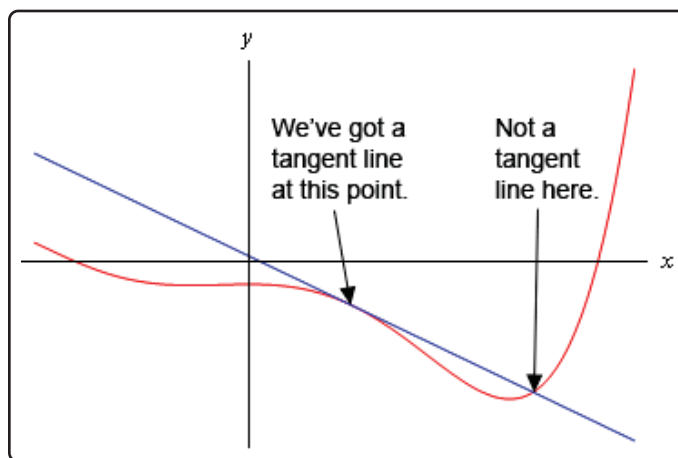
First, both of these problems will lead us into the study of limits, which is the topic of this chapter after all. Looking at these problems here will allow us to start to understand just what a limit is and what it can tell us about a function.

Secondly, the rate of change problem that we're going to be looking at is one of the most important concepts that we'll encounter in the second chapter of this course. In fact, it's probably one of the most important concepts that we'll encounter in the whole course. So, looking at it now will get us to start thinking about it from the very beginning.

Tangent Lines

The first problem that we're going to take a look at is the tangent line problem. Before getting into this problem it would probably be best to define a tangent line.

A tangent line to the function $f(x)$ at the point $x = a$ is a line that just touches the graph of the function at the point in question and is "parallel" (in some way) to the graph at that point. Take a look at the graph below.



In this graph the line is a tangent line at the indicated point because it just touches the graph at that point and is also "parallel" to the graph at that point. Likewise, at the second point shown, the line does just touch the graph at that point, but it is not "parallel" to the graph at that point and so it's not a tangent line to the graph at that point.

At the second point shown (the point where the line isn't a tangent line) we will sometimes call the line a **secant line**.

We've used the word parallel a couple of times now and we should probably be a little careful with it. In general, we will think of a line and a graph as being parallel at a point if they are both moving in the same direction at that point. So, in the first point above the graph and the line are moving

in the same direction and so we will say they are parallel at that point. At the second point, on the other hand, the line and the graph are not moving in the same direction so they aren't parallel at that point.

Okay, now that we've gotten the definition of a tangent line out of the way let's move on to the tangent line problem. That's probably best done with an example.

Example 1

Find the tangent line to $f(x) = 15 - 2x^2$ at $x = 1$.

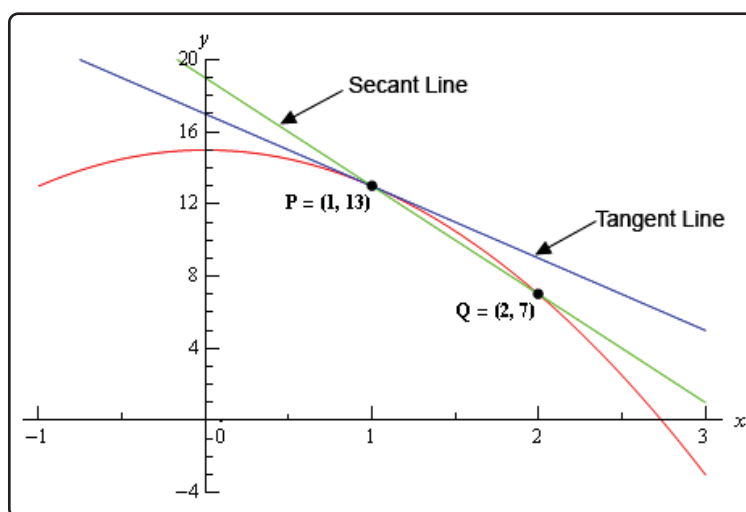
Solution

We know from algebra that to find the equation of a line we need either two points on the line or a single point on the line and the slope of the line. Since we know that we are after a tangent line we do have a point that is on the line. The tangent line and the graph of the function must touch at $x = 1$ so the point $(1, f(1)) = (1, 13)$ must be on the line.

Now we reach the problem. This is all that we know about the tangent line. In order to find the tangent line we need either a second point or the slope of the tangent line. Since the only reason for needing a second point is to allow us to find the slope of the tangent line let's just concentrate on seeing if we can determine the slope of the tangent line.

At this point in time all that we're going to be able to do is to get an estimate for the slope of the tangent line, but if we do it correctly we should be able to get an estimate that is in fact the actual slope of the tangent line. We'll do this by starting with the point that we're after, let's call it $P = (1, 13)$. We will then pick another point that lies on the graph of the function, let's call that point $Q = (x, f(x))$.

For the sake of argument let's take $x = 2$ and so the second point will be $Q = (2, 7)$. Below is a graph of the function, the tangent line and the secant line that connects P and Q .

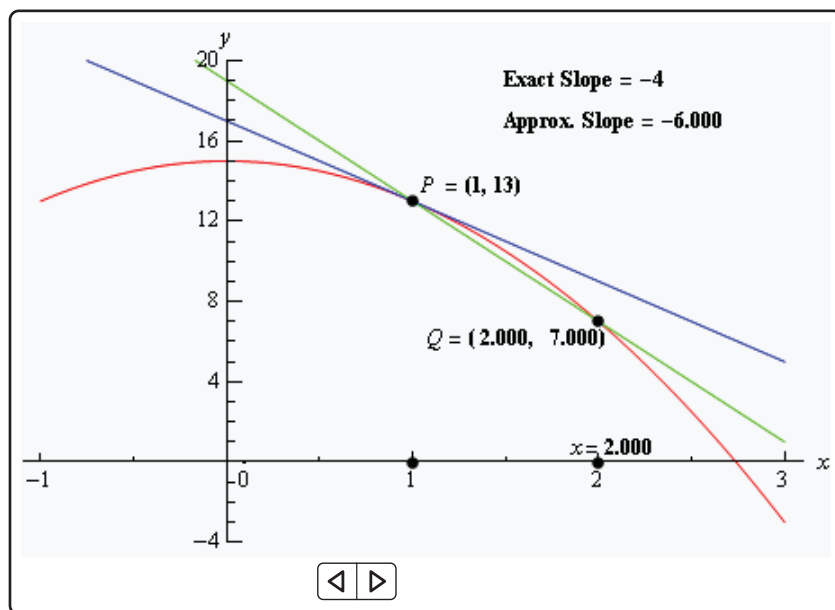


We can see from this graph that the secant and tangent lines are somewhat similar and so the slope of the secant line should be somewhat close to the actual slope of the tangent line. So, as an estimate of the slope of the tangent line we can use the slope of the secant line, let's call it m_{PQ} , which is,

$$m_{PQ} = \frac{f(2) - f(1)}{2 - 1} = \frac{7 - 13}{1} = -6$$

Now, if we weren't too interested in accuracy we could say this is good enough and use this as an estimate of the slope of the tangent line. However, we would like an estimate that is at least somewhat close the actual value. So, to get a better estimate we can take an x that is closer to $x = 1$ and redo the work above to get a new estimate on the slope. We could then take a third value of x even closer yet and get an even better estimate.

In other words, as we take Q closer and closer to P the slope of the secant line connecting Q and P should be getting closer and closer to the slope of the tangent line. If you are viewing this on the web, the image below shows this process.



As you can see (animation won't work on all pdf viewers unfortunately) as we moved Q in closer and closer to P the secant lines does start to look more and more like the tangent line and so the approximate slopes (*i.e.* the slopes of the secant lines) are getting closer and closer to the exact slope. Also, do not worry about how I got the exact or approximate slopes. We'll be computing the approximate slopes shortly and we'll be able to compute the exact slope in a few sections.

In this figure we only looked at Q 's that were to the right of P , but we could have just as easily used Q 's that were to the left of P and we would have received the same results. In

fact, we should always take a look at Q 's that are on both sides of P . In this case the same thing is happening on both sides of P . However, we will eventually see that doesn't have to happen. Therefore, we should always take a look at what is happening on both sides of the point in question when doing this kind of process.

So, let's see if we can come up with the approximate slopes we showed above, and hence an estimation of the slope of the tangent line. In order to simplify the process a little let's get a formula for the slope of the line between P and Q , m_{PQ} , that will work for any x that we choose to work with. We can get a formula by finding the slope between P and Q using the "general" form of $Q = (x, f(x))$.

$$m_{PQ} = \frac{f(x) - f(1)}{x - 1} = \frac{15 - 2x^2 - 13}{x - 1} = \frac{2 - 2x^2}{x - 1}$$

Now, let's pick some values of x getting closer and closer to $x = 1$, plug in and get some slopes.

x	m_{PQ}	x	m_{PQ}
2	-6	0	-2
1.5	-5	0.5	-3
1.1	-4.2	0.9	-3.8
1.01	-4.02	0.99	-3.98
1.001	-4.002	0.999	-3.998
1.0001	-4.0002	0.9999	-3.9998

So, if we take x 's to the right of 1 and move them in very close to 1 it appears the slope of the secant lines is approaching -4 . Likewise, if we take x 's to the left of 1 and move them in very close to 1 the slope of the secant lines again appears to be approaching -4 .

Based on this evidence it seems that the slopes of the secant lines are approaching -4 as we move in towards $x = 1$, so we will estimate that the slope of the tangent line is also -4 . As noted above, this is the correct value and we will be able to prove this eventually.

Now, the equation of the line that goes through $(a, f(a))$ is given by

$$y = f(a) + m(x - a)$$

Therefore, the equation of the tangent line to $f(x) = 15 - 2x^2$ at $x = 1$ is

$$y = 13 - 4(x - 1) = -4x + 17$$

There are a couple of important points to note about our work above. First, we looked at points that were on both sides of $x = 1$. In this kind of process it is important to never assume that what is happening on one side of a point will also be happening on the other side as well. We should

always look at what is happening on both sides of the point. In this example we could sketch a graph and from that guess that what is happening on one side will also be happening on the other, but we will usually not have the graphs in front of us or be able to easily get them.

Next, notice that when we say we're going to move in close to the point in question we do mean that we're going to move in very close and we also used more than just a couple of points. We should never try to determine a trend based on a couple of points that aren't really all that close to the point in question.

The next thing to notice is really a warning more than anything. The values of m_{PQ} in this example were fairly "nice" and it was pretty clear what value they were approaching after a couple of computations. In most cases this will not be the case. Most values will be far "messier" and you'll often need quite a few computations to be able to get an estimate. You should always use at least four points, on each side to get the estimate. Two points is never sufficient to get a good estimate and three points will also often not be sufficient to get a good estimate. Generally, you keeping picking points closer and closer to the point you are looking at until the change in the value between two successive points is getting very small.

Last, we were after something that was happening at $x = 1$ and we couldn't actually plug $x = 1$ into our formula for the slope. Despite this limitation we were able to determine some information about what was happening at $x = 1$ simply by looking at what was happening around $x = 1$. This is more important than you might at first realize and we will be discussing this point in detail in later sections.

Before moving on let's do a quick review of just what we did in the above example. We wanted the tangent line to $f(x)$ at a point $x = a$. First, we know that the point $P = (a, f(a))$ will be on the tangent line. Next, we'll take a second point that is on the graph of the function, call it $Q = (x, f(x))$ and compute the slope of the line connecting P and Q as follows,

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

We then take values of x that get closer and closer to $x = a$ (making sure to look at x 's on both sides of $x = a$ and use this list of values to estimate the slope of the tangent line, m .

The tangent line will then be,

$$y = f(a) + m(x - a)$$

Rates of Change

The next problem that we need to look at is the rate of change problem. As mentioned earlier, this will turn out to be one of the most important concepts that we will look at throughout this course.

Here we are going to consider a function, $f(x)$, that represents some quantity that varies as x varies. For instance, maybe $f(x)$ represents the amount of water in a holding tank after x minutes. Or maybe $f(x)$ is the distance traveled by a car after x hours. In both of these example we used x

to represent time. Of course x doesn't have to represent time, but it makes for examples that are easy to visualize.

What we want to do here is determine just how fast $f(x)$ is changing at some point, say $x = a$. This is called the **instantaneous rate of change** or sometimes just **rate of change** of $f(x)$ at $x = a$.

As with the tangent line problem all that we're going to be able to do at this point is to estimate the rate of change. So, let's continue with the examples above and think of $f(x)$ as something that is changing in time and x being the time measurement. Again, x doesn't have to represent time but it will make the explanation a little easier. While we can't compute the instantaneous rate of change at this point we can find the average rate of change.

To compute the average rate of change of $f(x)$ at $x = a$ all we need to do is to choose another point, say x , and then the average rate of change will be,

$$\begin{aligned} A.R.C. &= \frac{\text{change in } f(x)}{\text{change in } x} \\ &= \frac{f(x) - f(a)}{x - a} \end{aligned}$$

Then to estimate the instantaneous rate of change at $x = a$ all we need to do is to choose values of x getting closer and closer to $x = a$ (don't forget to choose them on both sides of $x = a$) and compute values of $A.R.C.$ We can then estimate the instantaneous rate of change from that.

Let's take a look at an example.

Example 2

Suppose that the amount of air in a balloon after t hours is given by

$$V(t) = t^3 - 6t^2 + 35$$

Estimate the instantaneous rate of change of the volume after 5 hours.

Solution

Okay. The first thing that we need to do is get a formula for the average rate of change of the volume. In this case this is,

$$A.R.C. = \frac{V(t) - V(5)}{t - 5} = \frac{t^3 - 6t^2 + 35 - 10}{t - 5} = \frac{t^3 - 6t^2 + 25}{t - 5}$$

To estimate the instantaneous rate of change of the volume at $t = 5$ we just need to pick values of t that are getting closer and closer to $t = 5$. Here is a table of values of t and the average rate of change for those values.

t	$A.R.C.$	t	$A.R.C.$
6	25.0	4	7.0
5.5	19.75	4.5	10.75
5.1	15.91	4.9	14.11
5.01	15.0901	4.99	14.9101
5.001	15.009001	4.999	14.991001
5.0001	15.00090001	4.9999	14.99910001

So, from this table it looks like the average rate of change is approaching 15 and so we can estimate that the instantaneous rate of change is 15 at this point.

So, just what does this tell us about the volume at $t = 5$? Let's put some units on the answer from above. This might help us to see what is happening to the volume at this point. Let's suppose that the units on the volume were in cm^3 . The units on the rate of change (both average and instantaneous) are then cm^3/hr .

We have estimated that at $t = 5$ the volume is changing at a rate of $15 \text{ cm}^3/\text{hr}$. This means that at $t = 5$ the volume is changing in such a way that, if the rate were constant, then an hour later there would be 15 cm^3 more air in the balloon than there was at $t = 5$.

We do need to be careful here however. In reality there probably won't be 15 cm^3 more air in the balloon after an hour. The rate at which the volume is changing is generally not constant so we can't make any real determination as to what the volume will be in another hour. What we can say is that the volume is increasing, since the instantaneous rate of change is positive, and if we had rates of change for other values of t we could compare the numbers and see if the rate of change is faster or slower at the other points.

For instance, at $t = 4$ the instantaneous rate of change is $0 \text{ cm}^3/\text{hr}$ and at $t = 3$ the instantaneous rate of change is $-9 \text{ cm}^3/\text{hr}$. We'll leave it to you to check these rates of change. In fact, that would be a good exercise to see if you can build a table of values that will support our claims on these rates of change.

Anyway, back to the example. At $t = 4$ the rate of change is zero and so at this point in time the volume is not changing at all. That doesn't mean that it will not change in the future. It just means that exactly at $t = 4$ the volume isn't changing. Likewise, at $t = 3$ the volume is decreasing since the rate of change at that point is negative. We can also say that, regardless of the increasing/decreasing aspects of the rate of change, the volume of the balloon is changing faster at $t = 5$ than it is at $t = 3$ since 15 is larger than 9 .

We will be talking a lot more about rates of change when we get into the next chapter.

Velocity Problem

Let's briefly look at the velocity problem. Many calculus books will treat this as its own problem. We however, like to think of this as a special case of the rate of change problem. In the velocity problem we are given a position function of an object, $f(t)$, that gives the position of an object at time t . Then to compute the instantaneous velocity of the object we just need to recall that the velocity is nothing more than the rate at which the position is changing.

In other words, to estimate the instantaneous velocity we would first compute the average velocity,

$$\begin{aligned} A.V. &= \frac{\text{change in position}}{\text{time traveled}} \\ &= \frac{f(t) - f(a)}{t - a} \end{aligned}$$

and then take values of t closer and closer to $t = a$ and use these values to estimate the instantaneous velocity.

Change of Notation

There is one last thing that we need to do in this section before we move on. The main point of this section was to introduce us to a couple of key concepts and ideas that we will see throughout the first portion of this course as well as get us started down the path towards limits.

Before we move into limits officially let's go back and do a little work that will relate both (or all three if you include velocity as a separate problem) problems to a more general concept.

First, notice that whether we wanted the tangent line, instantaneous rate of change, or instantaneous velocity each of these came down to using exactly the same formula. Namely,

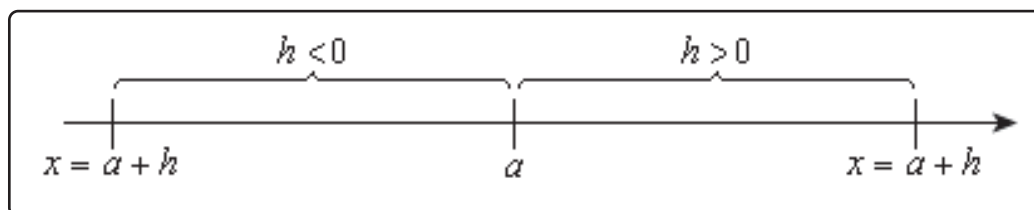
$$\frac{f(x) - f(a)}{x - a} \quad (2.1)$$

This should suggest that all three of these problems are then really the same problem. In fact this is the case as we will see in the next chapter. We are really working the same problem in each of these cases the only difference is the interpretation of the results.

In preparation for the next section where we will discuss this in much more detail we need to do a quick change of notation. It's easier to do here since we've already invested a fair amount of time into these problems.

In all of these problems we wanted to determine what was happening at $x = a$. To do this we chose another value of x and plugged into [Equation 2.1](#). For what we were doing here that is probably most intuitive way of doing it. However, when we start looking at these problems as a single problem [Equation 2.1](#) will not be the best formula to work with.

What we'll do instead is to first determine how far from $x = a$ we want to move and then define our new point based on that decision. So, if we want to move a distance of h from $x = a$ the new point would be $x = a + h$. This is shown in the sketch below.



As we saw in our work above it is important to take values of x that are both sides of $x = a$. This way of choosing new value of x will do this for us as we can see in the sketch above. If $h > 0$ we will get value of x that are to the right of $x = a$ and if $h < 0$ we will get values of x that are to the left of $x = a$ and both are given by $x = a + h$.

Now, with this new way of getting a second value of x [Equation 2.1](#) will become,

$$\frac{f(x) - f(a)}{x - a} = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}$$

Now, this is for a specific value of x , *i.e.* $x = a$ and we'll rarely be looking at these at specific values of x . So, we take the final step in the above equation and replace the a with x to get,

$$\frac{f(x + h) - f(x)}{h}$$

This gives us a formula for a general value of x and on the surface it might seem that this is going to be an overly complicated way of dealing with this stuff. However, as we will see it will often be easier to deal with this form than it will be to deal with the original form, [Equation 2.1](#).

2.2 The Limit

In the previous [section](#) we looked at a couple of problems and in both problems we had a function (slope in the tangent problem case and average rate of change in the rate of change problem) and we wanted to know how that function was behaving at some point $x = a$. At this stage of the game we no longer care where the functions came from and we no longer care if we're going to see them down the road again or not. All that we need to know or worry about is that we've got these functions and we want to know something about them.

To answer the questions in the last section we choose values of x that got closer and closer to $x = a$ and we plugged these into the function. We also made sure that we looked at values of x that were on both the left and the right of $x = a$. Once we did this we looked at our table of function values and saw what the function values were approaching as x got closer and closer to $x = a$ and used this to guess the value that we were after.

This process is called **taking a limit** and we have some notation for this. The limit notation for the two problems from the last section is,

$$\lim_{x \rightarrow 1} \frac{2 - 2x^2}{x - 1} = -4 \qquad \lim_{t \rightarrow 5} \frac{t^3 - 6t^2 + 25}{t - 5} = 15$$

In this notation we will note that we always give the function that we're working with and we also give the value of x (or t) that we are moving in towards.

In this section we are going to take an intuitive approach to limits and try to get a feel for what they are and what they can tell us about a function. With that goal in mind we are not going to get into how we actually compute limits yet. We will instead rely on what we did in the previous section as well as another approach to guess the value of the limits.

Both approaches that we are going to use in this section are designed to help us understand just what limits are. In general, we don't typically use the methods in this section to compute limits and in many cases can be very difficult to use to even estimate the value of a limit and/or will give the wrong value on occasion. We will look at actually computing limits in a couple of sections.

Let's first start off with the following "definition" of a limit.

Definition

We say that the limit of $f(x)$ is L as x approaches a and write this as

$$\lim_{x \rightarrow a} f(x) = L$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently close to a , from both sides, without actually letting x be a .

This is not the exact, precise definition of a limit. If you would like to see the more precise and mathematical definition of a limit you should check out the [The Definition of a Limit](#) section at the

end of this chapter. The definition given above is more of a “working” definition. This definition helps us to get an idea of just what limits are and what they can tell us about functions.

So just what does this definition mean? Well let’s suppose that we know that the limit does in fact exist. According to our “working” definition we can then decide how close to L that we’d like to make $f(x)$. For sake of argument let’s suppose that we want to make $f(x)$ no more than 0.001 away from L . This means that we want one of the following

$$\begin{array}{ll} f(x) - L < 0.001 & \text{if } f(x) \text{ is larger than } L \\ L - f(x) < 0.001 & \text{if } f(x) \text{ is smaller than } L \end{array}$$

Now according to the “working” definition this means that if we get x sufficiently close to a we can make one of the above true. However, it actually says a little more. It says that somewhere out there in the world is a value of x , say X , so that for all x 's that are closer to a than X then one of the above statements will be true.

This is a fairly important idea. There are many functions out there in the world that we can make as close to L for specific values of x that are close to a , but there will be other values of x closer to a that give functions values that are nowhere near close to L . In order for a limit to exist once we get $f(x)$ as close to L as we want for some x then it will need to stay in that close to L (or get closer) for all values of x that are closer to a . We’ll see an [example](#) of this later in this section.

In somewhat simpler terms the definition says that as x gets closer and closer to $x = a$ (from both sides of course...) then $f(x)$ **must** be getting closer and closer to L . Or, as we move in towards $x = a$ then $f(x)$ **must** be moving in towards L .

It is important to note once again that we must look at values of x that are on both sides of $x = a$. We should also note that we are not allowed to use $x = a$ in the definition. We will often use the information that limits give us to get some information about what is going on right at $x = a$, but the limit itself is not concerned with what is actually going on at $x = a$. The limit is only concerned with what is going on around the point $x = a$. This is an important concept about limits that we need to keep in mind.

An alternative notation that we will occasionally use in denoting limits is

$$f(x) \rightarrow L \quad \text{as} \quad x \rightarrow a$$

How do we use this definition to help us estimate limits? We do exactly what we did in the previous [section](#). We take x 's on both sides of $x = a$ that move in closer and closer to a and we plug these into our function. We then look to see if we can determine what number the function values are moving in towards and use this as our estimate.

Let’s work an example.

Example 1

Estimate the value of the following limit.

$$\lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

Solution

Notice that we did say estimate the value of the limit. Again, we are not going to directly compute limits in this section. The point of this section is to give us a better idea of how limits work and what they can tell us about the function.

So, with that in mind we are going to work this in pretty much the same way that we did in the last section. We will choose values of x that get closer and closer to $x = 2$ and plug these values into the function. Doing this gives the following table of values.

x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

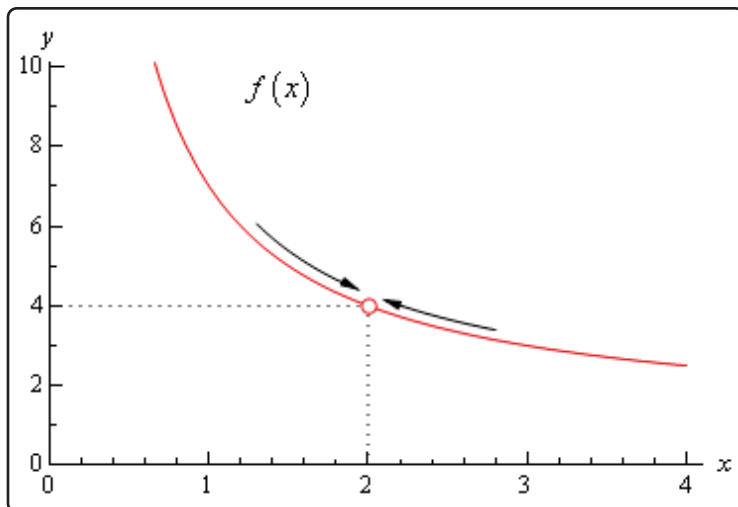
Note that we made sure and picked values of x that were on both sides of $x = 2$ and that we moved in very close to $x = 2$ to make sure that any trends that we might be seeing are in fact correct.

Also notice that we can't actually plug in $x = 2$ into the function as this would give us a division by zero error. This is not a problem since the limit doesn't care what is happening at the point in question.

From this table it appears that the function is going to 4 as x approaches 2, so

$$\lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = 4$$

Let's think a little bit more about what's going on here. Let's graph the function from the last example. The graph of the function in the range of x 's that were interested in is shown below.



First, notice that there is a rather large open dot at $x = 2$. This is there to remind us that the function (and hence the graph) doesn't exist at $x = 2$.

As we were plugging in values of x into the function we are in effect moving along the graph in towards the point as $x = 2$. This is shown in the graph by the two arrows on the graph that are moving in towards the point.

When we are computing limits the question that we are really asking is what y value is our graph approaching as we move in towards $x = a$ on our graph. We are **NOT** asking what y value the graph takes at the point in question. In other words, we are asking what the graph is doing **around** the point $x = a$. In our case we can see that as x moves in towards 2 (from both sides) the function is approaching $y = 4$ even though the function itself doesn't even exist at $x = 2$. Therefore, we can say that the limit is in fact 4.

So, what have we learned about limits? Limits are asking what the function is doing **around** $x = a$ and are **not** concerned with what the function is actually doing at $x = a$. This is a good thing as many of the functions that we'll be looking at won't even exist at $x = a$ as we saw in our last example.

Let's work another example to drive this point home.

Example 2

Estimate the value of the following limit.

$$\lim_{x \rightarrow 2} g(x) \quad \text{where,} \quad g(x) = \begin{cases} \frac{x^2 + 4x - 12}{x^2 - 2x} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$$

Solution

The first thing to note here is that this is exactly the same function as the first example with the exception that we've now given it a value for $x = 2$. So, let's first note that

$$g(2) = 6$$

As far as estimating the value of this limit goes, nothing has changed in comparison to the first example. We could build up a table of values as we did in the first example or we could take a quick look at the graph of the function. Either method will give us the value of the limit.

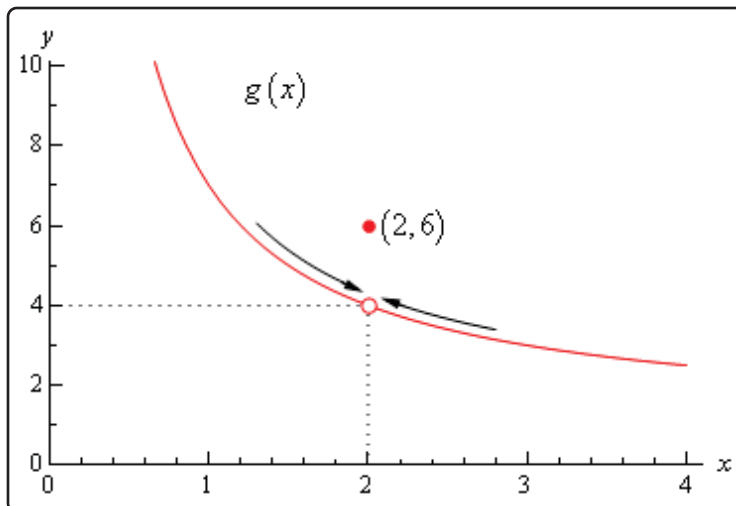
Let's first take a look at a table of values and see what that tells us. Notice that the presence of the value for the function at $x = 2$ will not change our choices for x . We only choose values of x that are getting closer to $x = 2$ but we never take $x = 2$. In other words, the table of values that we used in the first example will be exactly the same table that we'll use here. So, since we've already got it down once there is no reason to redo it here.

From this table it is again clear that the limit is,

$$\lim_{x \rightarrow 2} g(x) = 4$$

The limit is **NOT** 6! Remember from the discussion after the first example that limits do not care what the function is actually doing at the point in question. Limits are only concerned with what is going on **around** the point. Since the only thing about the function that we actually changed was its behavior at $x = 2$ this will not change the limit.

Let's also take a quick look at this function's graph to see if this says the same thing.



Again, we can see that as we move in towards $x = 2$ on our graph the function is still approaching a y value of 4. Remember that we are only asking what the function is doing **around** $x = 2$ and we don't care what the function is actually doing at $x = 2$. The graph then also supports the conclusion that the limit is,

$$\lim_{x \rightarrow 2} g(x) = 4$$

Let's make the point one more time just to make sure we've got it. Limits are **not** concerned with what is going on at $x = a$. Limits are only concerned with what is going on **around** $x = a$. We keep saying this, but it is a very important concept about limits that we must always keep in mind. So, we will take every opportunity to remind ourselves of this idea.

Since limits aren't concerned with what is actually happening at $x = a$ we will, on occasion, see situations like the previous example where the limit at a point and the function value at a point are different. This won't always happen of course. There are times where the function value and the limit at a point are the same and we will eventually see some examples of those. It is important however, to not get excited about things when the function and the limit do not take the same value at a point. It happens sometimes so we will need to be able to deal with those cases when they arise.

Let's take a look another example to try and beat this idea into the ground.

Example 3

Estimate the value of the following limit.

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\theta}$$

Solution

First don't get excited about the θ in function. It's just a letter, just like x is a letter! It's a Greek letter, but it's a letter and you will be asked to deal with Greek letters on occasion so it's a good idea to start getting used to them at this point.

Now, also notice that if we plug in $\theta = 0$ that we will get division by zero and so the function doesn't exist at this point. Actually, we get $0/0$ at this point, but because of the division by zero this function does not exist at $\theta = 0$.

So, as we did in the first example let's get a table of values and see what if we can guess what value the function is heading in towards.

θ	$f(\theta)$	θ	$f(\theta)$
1	0.45969769	-1	-0.45969769
0.1	0.04995835	-0.1	-0.04995835
0.01	0.00499996	-0.01	-0.00499996
0.001	0.00049999	-0.001	-0.00049999

Okay, it looks like the function is moving in towards a value of zero as θ moves in towards 0, from both sides of course.

Therefore, the we will guess that the limit has the value,

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\theta} = 0$$

So, once again, the limit had a value even though the function didn't exist at the point we were interested in.

It's now time to work a couple of more examples that will lead us into the next idea about limits that we're going to want to discuss.

Example 4

Estimate the value of the following limit.

$$\lim_{t \rightarrow 0} \cos\left(\frac{\pi}{t}\right)$$

Solution

Let's build up a table of values and see what's going on with our function in this case.

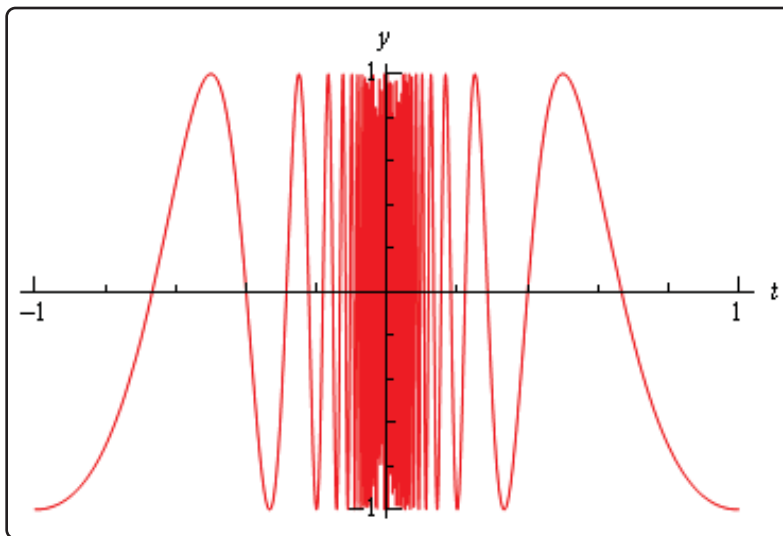
t	$f(t)$	t	$f(t)$
1	-1	-1	-1
0.1	1	-0.1	1
0.01	1	-0.01	1
0.001	1	-0.001	1

Now, if we were to guess the limit from this table we would guess that the limit is 1. However, if we did make this guess we would be wrong. Consider any of the following function evaluations.

$$f\left(\frac{1}{2001}\right) = -1 \quad f\left(\frac{2}{2001}\right) = 0 \quad f\left(\frac{4}{4001}\right) = \frac{\sqrt{2}}{2}$$

In all three of these function evaluations we evaluated the function at a number that is less than 0.001 and got three totally different numbers. Recall that the definition of the limit that we're working with requires that the function be approaching a single value (our guess) as t gets closer and closer to the point in question. It doesn't say that only some of the function values must be getting closer to the guess. It says that all the function values must be getting closer and closer to our guess.

To see what's happening here a graph of the function would be convenient.



From this graph we can see that as we move in towards $t = 0$ the function starts oscillating wildly and in fact the oscillations increases in speed the closer to $t = 0$ that we get. Recall from our definition of the limit that in order for a limit to exist the function must be settling down in towards a single value as we get closer to the point in question.

This function clearly does not settle in towards a single number and so this limit **does not exist!**

This last example points out the drawback of just picking values of the variable and using a table of function values to estimate the value of a limit. The values of the variable that we chose in the previous example were valid and in fact were probably values that many would have picked. In fact, they were exactly the same values we used in the problem before this one and they worked in that problem!

When using a table of values there will always be the possibility that we aren't choosing the correct values and that we will guess incorrectly for our limit. This is something that we should always keep in mind when doing this to guess the value of limits. In fact, this is such a problem that after this section we will never use a table of values to guess the value of a limit again.

This last example also has shown us that limits do not have to exist. To that point we've only seen limits that existed, but that just doesn't always have to be the case.

Let's take a look at one more example in this section.

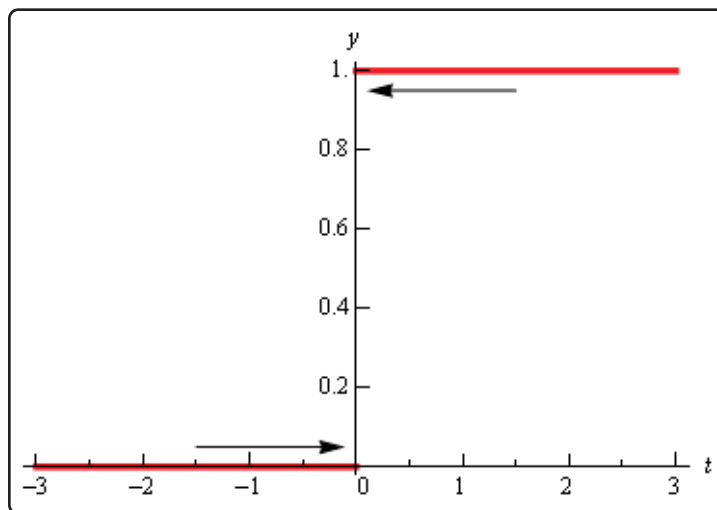
Example 5

Estimate the value of the following limit.

$$\lim_{t \rightarrow 0} H(t) \quad \text{where,} \quad H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Solution

This function is often called either the **Heaviside** or **step** function. We could use a table of values to estimate the limit, but it's probably just as quick in this case to use the graph so let's do that. Below is the graph of this function.



We can see from the graph that if we approach $t = 0$ from the right side the function is moving in towards a y value of 1. Well actually it's just staying at 1, but in the terminology that we've been using in this section it's moving in towards 1...

Also, if we move in towards $t = 0$ from the left the function is moving in towards a y value of 0.

According to our definition of the limit the function needs to move in towards a single value as we move in towards $t = a$ (from both sides). This isn't happening in this case and so in this example we will also say that the limit doesn't exist.

Note that the limit in this example is a little different from the previous example. In the previous example the function did not settle down to a single number as we moved in towards $t = 0$. In this example however, the function does settle down to a single number as $t = 0$ on either side. The problem is that the number is different on each side of $t = 0$. This is an idea that we'll look at in a little more detail in the next section.

Let's summarize what we (hopefully) learned in this section. In the first three examples we saw that limits do not care what the function is actually doing at the point in question. They only are concerned with what is happening around the point. In fact, we can have limits at $x = a$ even if the function itself does not exist at that point. Likewise, even if a function exists at a point there is no reason (at this point) to think that the limit will have the same value as the function at that point. Sometimes the limit and the function will have the same value at a point and other times they won't have the same value.

Next, in the third and fourth examples we saw the main reason for not using a table of values to guess the value of a limit. In those examples we used exactly the same set of values, however they only worked in one of the examples. Using tables of values to guess the value of limits is simply not a good way to get the value of a limit. This is the only section in which we will do this. Tables of values should always be your last choice in finding values of limits.

The last two examples showed us that not all limits will in fact exist. We should not get locked into the idea that limits will always exist. In most calculus courses we work with limits that almost always exist and so it's easy to start thinking that limits always exist. Limits don't always exist and so don't get into the habit of assuming that they will.

Finally, we saw in the fourth example that the only way to deal with the limit was to graph the function. Sometimes this is the only way, however this example also illustrated the drawback of using graphs. In order to use a graph to guess the value of the limit you need to be able to actually sketch the graph. For many functions this is not that easy to do.

There is another drawback in using graphs. Even if you have the graph it's only going to be useful if the y value is approaching an integer. If the y value is approaching say $\frac{-15}{123}$ there is no way that you're going to be able to guess that value from the graph and we are usually going to want exact values for our limits.

So, while graphs of functions can, on occasion, make your life easier in guessing values of limits they are again probably not the best way to get values of limits. They are only going to be useful if you can get your hands on it and the value of the limit is a "nice" number.

The natural question then is why did we even talk about using tables and/or graphs to estimate limits if they aren't the best way. There were a couple of reasons.

First, they can help us get a better understanding of what limits are and what they can tell us. If we don't do at least a couple of limits in this way we might not get all that good of an idea on just what limits are.

The second reason for doing limits in this way is to point out their drawback so that we aren't tempted to use them all the time!

We will eventually talk about how we really do limits. However, there is one more topic that we need to discuss before doing that. Since this section has already gone on for a while we will talk about this in the next section.

2.3 One-Sided Limits

In the final two examples in the previous section we saw two limits that did not exist. However, the reason for each of the limits not existing was different for each of the examples.

We saw that

$$\lim_{t \rightarrow 0} \cos\left(\frac{\pi}{t}\right)$$

did not exist because the function did not settle down to a single value as t approached $t = 0$. The closer to $t = 0$ we moved the more wildly the function oscillated and in order for a limit to exist the function must settle down to a single value.

However, we saw that

$$\lim_{t \rightarrow 0} H(t) \quad \text{where,} \quad H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

did not exist not because the function didn't settle down to a single number as we moved in towards $t = 0$, but instead because it settled into two different numbers depending on which side of $t = 0$ we were on.

In this case the function was a very well-behaved function, unlike the first function. The only problem was that, as we approached $t = 0$, the function was moving in towards different numbers on each side. We would like a way to differentiate between these two examples.

We do this with **one-sided limits**. As the name implies, with one-sided limits we will only be looking at one side of the point in question. Here are the definitions for the two one sided limits.

Right-handed limit

We say

$$\lim_{x \rightarrow a^+} f(x) = L$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently close to a with $x > a$ without actually letting x be a .

Left-handed limit

We say

$$\lim_{x \rightarrow a^-} f(x) = L$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently close to a with $x < a$ without actually letting x be a .

Note that the change in notation is very minor and in fact might be missed if you aren't paying attention. The only difference is the bit that is under the "lim" part of the limit. For the right-handed limit we now have $x \rightarrow a^+$ (note the "+") which means that we know will only look at $x > a$. Likewise, for the left-handed limit we have $x \rightarrow a^-$ (note the "-") which means that we will only be looking at $x < a$.

Also, note that as with the "normal" limit (*i.e.* the limits from the previous section) we still need the function to settle down to a single number in order for the limit to exist. The only difference this time is that the function only needs to settle down to a single number on either the right side of $x = a$ or the left side of $x = a$ depending on the one-sided limit we're dealing with.

So, when we are looking at limits it's now important to pay very close attention to see whether we are doing a normal limit or one of the one-sided limits. Let's now take a look at the some of the problems from the last section and look at one-sided limits instead of the normal limit.

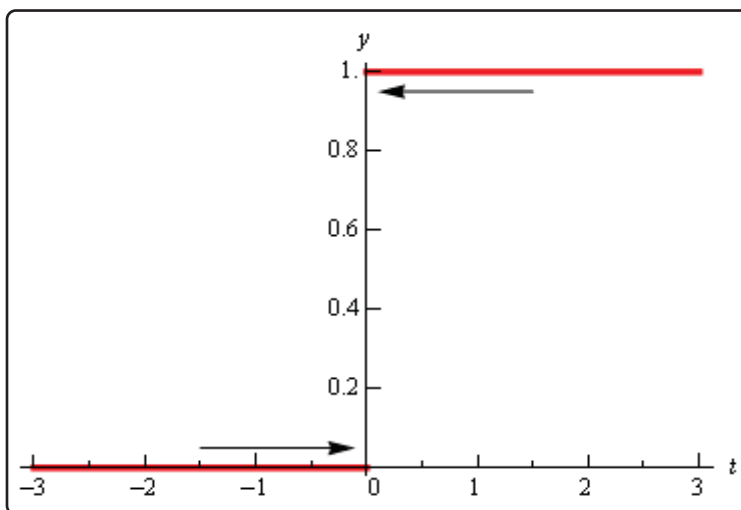
Example 1

Estimate the value of the following limits.

$$\lim_{t \rightarrow 0^+} H(t) \quad \text{and} \quad \lim_{t \rightarrow 0^-} H(t) \quad \text{where} \quad H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Solution

To remind us what this function looks like here's the graph.



So, we can see that if we stay to the right of $t = 0$ (*i.e.* $t > 0$) then the function is moving in towards a value of 1 as we get closer and closer to $t = 0$, but staying to the right. We can

therefore say that the right-handed limit is,

$$\lim_{t \rightarrow 0^+} H(t) = 1$$

Likewise, if we stay to the left of $t = 0$ (i.e. $t < 0$) the function is moving in towards a value of 0 as we get closer and closer to $t = 0$, but staying to the left. Therefore, the left-handed limit is,

$$\lim_{t \rightarrow 0^-} H(t) = 0$$

In this example we do get one-sided limits even though the normal limit itself doesn't exist.

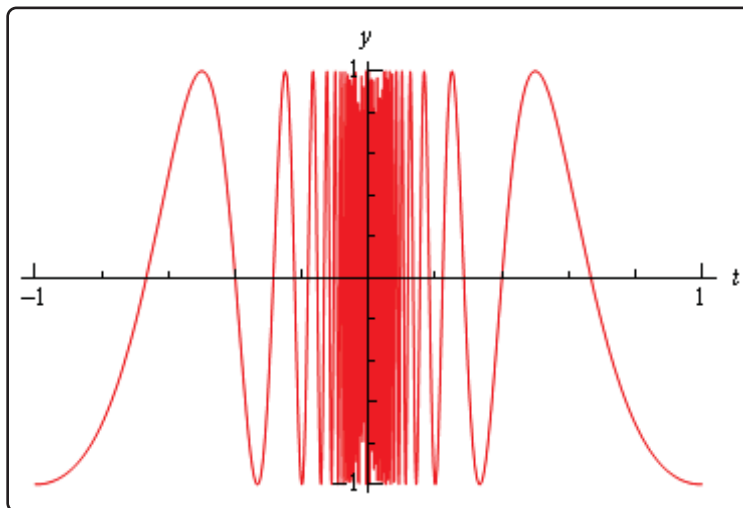
Example 2

Estimate the value of the following limits.

$$\lim_{t \rightarrow 0^+} \cos\left(\frac{\pi}{t}\right) \qquad \lim_{t \rightarrow 0^-} \cos\left(\frac{\pi}{t}\right)$$

Solution

From the graph of this function shown below,



we can see that both of the one-sided limits suffer the same problem that the normal limit did in the previous section. The function does not settle down to a single number on either side of $t = 0$. Therefore, neither the left-handed nor the right-handed limit will exist in this case.

So, one-sided limits don't have to exist just as normal limits aren't guaranteed to exist.

Let's take a look at another example from the previous section.

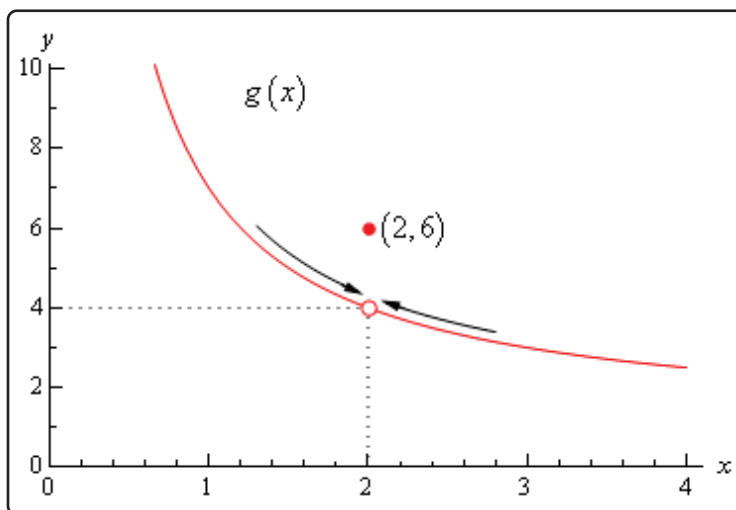
Example 3

Estimate the value of the following limits.

$$\lim_{x \rightarrow 2^+} g(x) \quad \text{and} \quad \lim_{x \rightarrow 2^-} g(x) \quad \text{where} \quad g(x) = \begin{cases} \frac{x^2 + 4x - 12}{x^2 - 2x} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$$

Solution

So, as we've done with the previous two examples, let's remind ourselves of the graph of this function.



In this case regardless of which side of $x = 2$ we are on the function is always approaching a value of 4 and so we get,

$$\lim_{x \rightarrow 2^+} g(x) = 4 \quad \lim_{x \rightarrow 2^-} g(x) = 4$$

Note that one-sided limits do not care about what's happening at the point any more than normal limits do. They are still only concerned with what is going on around the point. The only real difference between one-sided limits and normal limits is the range of x 's that we look at when determining the value of the limit.

Now let's take a look at the first and last example in this section to get a very nice fact about the relationship between one-sided limits and normal limits. In the last example the one-sided limits as

well as the normal limit existed and all three had a value of 4. In the first example the two one-sided limits both existed, but did not have the same value and the normal limit did not exist.

The relationship between one-sided limits and normal limits can be summarized by the following fact.

Fact

Given a function $f(x)$ if,

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

then the normal limit will exist and

$$\lim_{x \rightarrow a} f(x) = L$$

Likewise, if

$$\lim_{x \rightarrow a} f(x) = L$$

then,

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

This fact can be turned around to also say that if the two one-sided limits have different values, *i.e.*,

$$\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$$

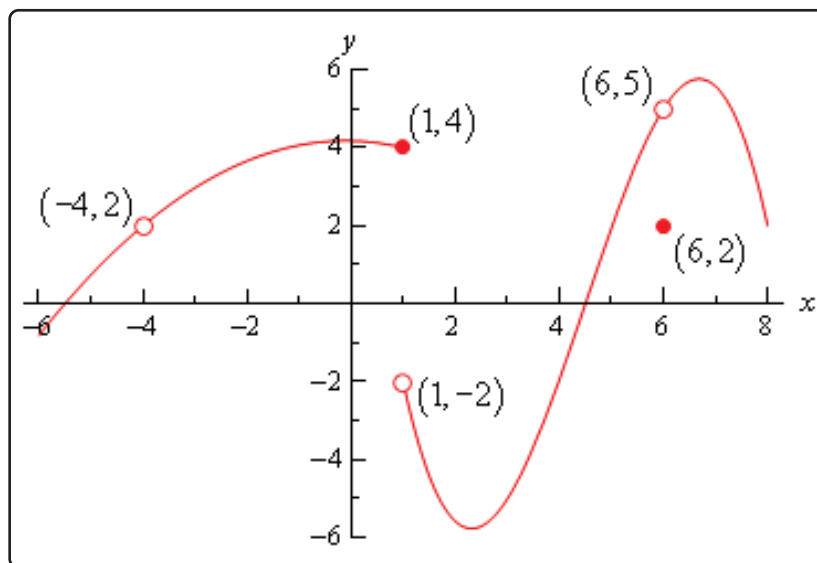
then the normal limit will not exist.

This should make some sense. If the normal limit did exist then by the fact the two one-sided limits would have to exist and have the same value by the above fact. So, if the two one-sided limits have different values (or don't even exist) then the normal limit simply can't exist.

Let's take a look at one more example to make sure that we've got all the ideas about limits down that we've looked at in the last couple of sections.

Example 4

Given the following graph,



compute each of the following.

- | | | | |
|-------------|--------------------------------------|--------------------------------------|------------------------------------|
| (a) $f(-4)$ | (b) $\lim_{x \rightarrow -4^-} f(x)$ | (c) $\lim_{x \rightarrow -4^+} f(x)$ | (d) $\lim_{x \rightarrow -4} f(x)$ |
| (e) $f(1)$ | (f) $\lim_{x \rightarrow 1^-} f(x)$ | (g) $\lim_{x \rightarrow 1^+} f(x)$ | (h) $\lim_{x \rightarrow 1} f(x)$ |
| (i) $f(6)$ | (j) $\lim_{x \rightarrow 6^-} f(x)$ | (k) $\lim_{x \rightarrow 6^+} f(x)$ | (l) $\lim_{x \rightarrow 6} f(x)$ |

Solution

- (a) $f(-4)$ doesn't exist. There is no closed dot for this value of x and so the function doesn't exist at this point.
- (b) $\lim_{x \rightarrow -4^-} f(x) = 2$ The function is approaching a value of 2 as x moves in towards -4 from the left.
- (c) $\lim_{x \rightarrow -4^+} f(x) = 2$ The function is approaching a value of 2 as x moves in towards -4 from the right.

- (d) $\lim_{x \rightarrow -4} f(x) = 2$ We can do this one of two ways. Either we can use the fact here and notice that the two one-sided limits are the same and so the normal limit must exist and have the same value as the one-sided limits or just get the answer from the graph.
- Also recall that a limit can exist at a point even if the function doesn't exist at that point.
- (e) $f(1) = 4$. The function will take on the y value where the closed dot is.
- (f) $\lim_{x \rightarrow 1^-} f(x) = 4$ The function is approaching a value of 4 as x moves in towards 1 from the left.
- (g) $\lim_{x \rightarrow 1^+} f(x) = -2$ The function is approaching a value of -2 as x moves in towards 1 from the right. Remember that the limit does NOT care about what the function is actually doing at the point, it only cares about what the function is doing around the point. In this case, always staying to the right of $x = 1$, the function is approaching a value of -2 and so the limit is -2 . The limit is not 4, as that is value of the function at the point and again the limit doesn't care about that!
- (h) $\lim_{x \rightarrow 1} f(x)$ doesn't exist. The two one-sided limits both exist, however they are different and so the normal limit doesn't exist.
- (i) $f(6) = 2$. The function will take on the y value where the closed dot is.
- (j) $\lim_{x \rightarrow 6^-} f(x) = 5$ The function is approaching a value of 5 as x moves in towards 6 from the left.
- (k) $\lim_{x \rightarrow 6^+} f(x) = 5$ The function is approaching a value of 5 as x moves in towards 6 from the right.
- (l) $\lim_{x \rightarrow 6} f(x) = 5$ Again, we can use either the graph or the fact to get this. Also, once more remember that the limit doesn't care what is happening at the point and so it's possible for the limit to have a different value than the function at a point. When dealing with limits we've always got to remember that limits simply do not care about what the function is doing at the point in question. Limits are only concerned with what the function is doing around the point.

Hopefully over the last couple of sections you've gotten an idea on how limits work and what they can tell us about functions. Some of these ideas will be important in later sections so it's important that you have a good grasp on them.

2.4 Limit Properties

The time has almost come for us to actually compute some limits. However, before we do that we will need some properties of limits that will make our life somewhat easier. So, let's take a look at those first. The proof of some of these properties can be found in the [Proof of Various Limit Properties](#) section of the Extras appendix.

Properties

First, we will assume that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist and that c is any constant. Then,

$$1. \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

In other words, we can “factor” a multiplicative constant out of a limit.

$$2. \lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

So, to take the limit of a sum or difference all we need to do is take the limit of the individual parts and then put them back together with the appropriate sign. This is also not limited to two functions. This fact will work no matter how many functions we've got separated by “+” or “-”.

$$3. \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$$

We take the limits of products in the same way that we can take the limit of sums or differences. Just take the limit of the pieces and then put them back together. Also, as with sums or differences, this fact is not limited to just two functions.

$$4. \lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ provided } \lim_{x \rightarrow a} g(x) \neq 0$$

As noted in the statement we only need to worry about the limit in the denominator being zero when we do the limit of a quotient. If it were zero we would end up with a division by zero error and we need to avoid that.

$$5. \lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n, \text{ where } n \text{ is any real number}$$

In this property n can be any real number (positive, negative, integer, fraction, irrational, zero, *etc.*). In the case that n is an integer this rule can be thought of as an extended case of **3**.

For example, consider the case of $n = 2$.

$$\begin{aligned} \lim_{x \rightarrow a} [f(x)]^2 &= \lim_{x \rightarrow a} [f(x) f(x)] \\ &= \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} f(x) && \text{using property 3} \\ &= \left[\lim_{x \rightarrow a} f(x) \right]^2 \end{aligned}$$

The same can be done for any integer n .

$$6. \lim_{x \rightarrow a} \left[\sqrt[n]{f(x)} \right] = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$$

This is just a special case of the previous example.

$$\begin{aligned} \lim_{x \rightarrow a} \left[\sqrt[n]{f(x)} \right] &= \lim_{x \rightarrow a} [f(x)]^{\frac{1}{n}} \\ &= \left[\lim_{x \rightarrow a} f(x) \right]^{\frac{1}{n}} \\ &= \sqrt[n]{\lim_{x \rightarrow a} f(x)} \end{aligned}$$

$$7. \lim_{x \rightarrow a} c = c, \quad c \text{ is any real number}$$

In other words, the limit of a constant is just the constant. You should be able to convince yourself of this by drawing the graph of $f(x) = c$.

$$8. \lim_{x \rightarrow a} x = a$$

As with the last one you should be able to convince yourself of this by drawing the graph of $f(x) = x$.

$$9. \lim_{x \rightarrow a} x^n = a^n$$

This is really just a special case of property **5** using $f(x) = x$.

Note that all these properties also hold for the two one-sided limits as well we just didn't write them down with one sided limits to save on space.

Let's compute a limit or two using these properties. The next couple of examples will lead us to some truly useful facts about limits that we will use on a continual basis.

Example 1

Compute the value of the following limit.

$$\lim_{x \rightarrow -2} (3x^2 + 5x - 9)$$

Solution

This first time through we will use only the properties above to compute the limit.

First, we will use property **2** to break up the limit into three separate limits. We will then use property **1** to bring the constants out of the first two limits. Doing this gives us,

$$\begin{aligned} \lim_{x \rightarrow -2} (3x^2 + 5x - 9) &= \lim_{x \rightarrow -2} 3x^2 + \lim_{x \rightarrow -2} 5x - \lim_{x \rightarrow -2} 9 \\ &= 3 \lim_{x \rightarrow -2} x^2 + 5 \lim_{x \rightarrow -2} x - \lim_{x \rightarrow -2} 9 \end{aligned}$$

We can now use properties 7 through 9 to actually compute the limit.

$$\begin{aligned}\lim_{x \rightarrow -2} (3x^2 + 5x - 9) &= 3 \lim_{x \rightarrow -2} x^2 + 5 \lim_{x \rightarrow -2} x - \lim_{x \rightarrow -2} 9 \\ &= 3(-2)^2 + 5(-2) - 9 \\ &= -7\end{aligned}$$

Now, let's notice that if we had defined

$$p(x) = 3x^2 + 5x - 9$$

then the proceeding example would have been,

$$\begin{aligned}\lim_{x \rightarrow -2} p(x) &= \lim_{x \rightarrow -2} (3x^2 + 5x - 9) \\ &= 3(-2)^2 + 5(-2) - 9 \\ &= -7 \\ &= p(-2)\end{aligned}$$

In other words, in this case we see that the limit is the same value that we'd get by just evaluating the function at the point in question. This seems to violate one of the main concepts about limits that we've seen to this point.

In the previous two sections we made a big deal about the fact that limits do not care about what is happening at the point in question. They only care about what is happening around the point. So how does the previous example fit into this since it appears to violate this main idea about limits?

Despite appearances the limit still doesn't care about what the function is doing at $x = -2$. In this case the function that we've got is simply "nice enough" so that what is happening around the point is exactly the same as what is happening at the point. **Eventually** we will formalize up just what is meant by "nice enough". At this point let's not worry too much about what "nice enough" is. Let's just take advantage of the fact that some functions will be "nice enough", whatever that means.

The function in the last example was a polynomial. It turns out that all polynomials are "nice enough" so that what is happening around the point is exactly the same as what is happening at the point. This leads to the following fact.

Fact

If $p(x)$ is a polynomial then,

$$\lim_{x \rightarrow a} p(x) = p(a)$$

By the end of this section we will generalize this out considerably to most of the functions that we'll be seeing throughout this course.

Let's take a look at another example.

Example 2

Evaluate the following limit.

$$\lim_{z \rightarrow 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1}$$

Solution

First notice that we can use property 4 to write the limit as,

$$\lim_{z \rightarrow 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{\lim_{z \rightarrow 1} 6 - 3z + 10z^2}{\lim_{z \rightarrow 1} -2z^4 + 7z^3 + 1}$$

Well, actually we should be a little careful. We can do that provided the limit of the denominator isn't zero. As we will see however, it isn't in this case so we're okay.

Now, both the numerator and denominator are polynomials so we can use the fact above to compute the limits of the numerator and the denominator and hence the limit itself.

$$\begin{aligned} \lim_{z \rightarrow 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} &= \frac{6 - 3(1) + 10(1)^2}{-2(1)^4 + 7(1)^3 + 1} \\ &= \frac{13}{6} \end{aligned}$$

Notice that the limit of the denominator wasn't zero and so our use of property 4 was legitimate.

In the previous example, as with polynomials, all we really did was evaluate the function at the point in question. So, it appears that there is a fairly large class of functions for which this can be done. Let's generalize the fact from above a little.

Fact

Provided $f(x)$ is "nice enough" we have,

$$\lim_{x \rightarrow a} f(x) = f(a)$$

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

Again, we will formalize up just what we mean by “nice enough” **eventually**. At this point all we want to do is worry about which functions are “nice enough”. Some functions are “nice enough” for all x while others will only be “nice enough” for certain values of x . It will all depend on the function.

As noted in the statement, this fact also holds for the two one-sided limits as well as the normal limit.

Here is a list of some of the more common functions that are “nice enough”.

- Polynomials are nice enough for all x 's.
- If $f(x) = \frac{p(x)}{q(x)}$ then $f(x)$ will be nice enough provided both $p(x)$ and $q(x)$ are nice enough and if we don't get division by zero at the point we're evaluating at.
- $\cos(x)$, $\sin(x)$ are nice enough for all x 's
- $\sec(x)$, $\tan(x)$ are nice enough provided $x \neq \dots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$. In other words secant and tangent are nice enough everywhere cosine isn't zero. To see why recall that these are both really rational functions and that cosine is in the denominator of both then go back up and look at the second bullet above.
- $\csc(x)$, $\cot(x)$ are nice enough provided $x \neq \dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots$. In other words cosecant and cotangent are nice enough everywhere sine isn't zero.
- $\sqrt[n]{x}$ is nice enough for all x if n is odd.
- $\sqrt[n]{x}$ is nice enough for $x \geq 0$ if n is even. Here we require $x \geq 0$ to avoid having to deal with complex values.
- a^x , e^x are nice enough for all x 's.
- $\log_b x$, $\ln x$ are nice enough for $x > 0$. Remember we can only plug positive numbers into logarithms and not zero or negative numbers.
- Any sum, difference or product of the above functions will also be nice enough. Quotients will be nice enough provided we don't get division by zero upon evaluating the limit.

The last bullet is important. This means that for any combination of these functions all we need to do is evaluate the function at the point in question, making sure that none of the restrictions are violated. This means that we can now do a large number of limits.

Example 3

Evaluate the following limit.

$$\lim_{x \rightarrow 3} \left(-\sqrt[5]{x} + \frac{e^x}{1 + \ln(x)} + \sin(x) \cos(x) \right)$$

Solution

This is a combination of several of the functions listed above and none of the restrictions are violated so all we need to do is plug in $x = 3$ into the function to get the limit.

$$\begin{aligned} \lim_{x \rightarrow 3} \left(-\sqrt[5]{x} + \frac{e^x}{1 + \ln(x)} + \sin(x) \cos(x) \right) &= -\sqrt[5]{3} + \frac{e^3}{1 + \ln(3)} + \sin(3) \cos(3) \\ &= 8.1854272743 \end{aligned}$$

Not a very pretty answer, but we can now do the limit.

2.5 Computing Limits

In the previous [section](#) we saw that there is a large class of functions that allows us to use

$$\lim_{x \rightarrow a} f(x) = f(a)$$

to compute limits. However, there are also many limits for which this won't work easily. The purpose of this section is to develop techniques for dealing with some of these limits that will not allow us to just use this fact.

Let's first go back and take a look at one of the first limits that we looked at and compute its exact value and verify our guess for the limit.

Example 1

Evaluate the following limit.

$$\lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

Solution

First let's notice that if we try to plug in $x = 2$ we get,

$$\lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \frac{0}{0}$$

So, we can't just plug in $x = 2$ to evaluate the limit. So, we're going to have to do something else.

The first thing that we should always do when evaluating limits is to simplify the function as much as possible. In this case that means factoring both the numerator and denominator. Doing this gives,

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{x^2 - 2x} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 6)}{x(x - 2)} \\ &= \lim_{x \rightarrow 2} \frac{x + 6}{x} \end{aligned}$$

So, upon factoring we saw that we could cancel an $x - 2$ from both the numerator and the denominator. Upon doing this we now have a new rational expression that we can plug $x = 2$ into because we lost the division by zero problem. Therefore, the limit is,

$$\lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \lim_{x \rightarrow 2} \frac{x + 6}{x} = \frac{8}{2} = 4$$

Note that this is in fact what we guessed the limit to be.

Before leaving this example let's discuss the fact that we couldn't plug $x = 2$ into our original limit but once we did the simplification we just plugged in $x = 2$ to get the answer. At first glance this may appear to be a contradiction.

In the original limit we couldn't plug in $x = 2$ because that gave us the $0/0$ situation that we couldn't do anything with. Upon doing the simplification we can note that,

$$\frac{x^2 + 4x - 12}{x^2 - 2x} = \frac{x + 6}{x} \quad \text{provided } x \neq 2$$

In other words, the two equations give identical values except at $x = 2$ and because limits are only concerned with that is going on around the point $x = 2$ the limit of the two equations will be equal. More importantly, in the simplified version we get a "nice enough" equation and so what is happening around $x = 2$ is identical to what is happening at $x = 2$.

We can therefore take the limit of the simplified version simply by plugging in $x = 2$ even though we couldn't plug $x = 2$ into the original equation and the value of the limit of the simplified equation will be the same as the limit of the original equation.

On a side note, the $0/0$ we initially got in the previous example is called an **indeterminate form**. This means that we don't really know what it will be until we do some more work. Typically, zero in the denominator means it's undefined. However, that will only be true if the numerator isn't also zero. Also, zero in the numerator usually means that the fraction is zero, unless the denominator is also zero. Likewise, anything divided by itself is 1, unless we're talking about zero.

So, there are really three competing "rules" here and it's not clear which one will win out. It's also possible that none of them will win out and we will get something totally different from undefined, zero, or one. We might, for instance, get a value of 4 out of this, to pick a number completely at random.

When simply evaluating an equation $0/0$ is undefined. However, in taking the limit, if we get $0/0$ we can get a variety of answers and the only way to know which one is correct is to actually compute the limit.

There are many more kinds of indeterminate forms and we will be discussing indeterminate forms at length in the next chapter.

Let's take a look at a couple of more examples.

Example 2

Evaluate the following limit.

$$\lim_{h \rightarrow 0} \frac{2(-3 + h)^2 - 18}{h}$$

Solution

In this case we also get $0/0$ and factoring is not really an option. However, there is still some simplification that we can do.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{2(-3 + h)^2 - 18}{h} &= \lim_{h \rightarrow 0} \frac{2(9 - 6h + h^2) - 18}{h} \\ &= \lim_{h \rightarrow 0} \frac{18 - 12h + 2h^2 - 18}{h} \\ &= \lim_{h \rightarrow 0} \frac{-12h + 2h^2}{h} \end{aligned}$$

So, upon multiplying out the first term we get a little cancellation and now notice that we can factor an h out of both terms in the numerator which will cancel against the h in the denominator and the division by zero problem goes away and we can then evaluate the limit.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{2(-3 + h)^2 - 18}{h} &= \lim_{h \rightarrow 0} \frac{-12h + 2h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(-12 + 2h)}{h} \\ &= \lim_{h \rightarrow 0} -12 + 2h = -12 \end{aligned}$$

Example 3

Evaluate the following limit.

$$\lim_{t \rightarrow 4} \frac{t - \sqrt{3t + 4}}{4 - t}$$

Solution

This limit is going to be a little more work than the previous two. Once again however note that we get the indeterminate form $0/0$ if we try to just evaluate the limit. Also note that neither of the two examples will be of any help here, at least initially. We can't factor the equation and we can't just multiply something out to get the equation to simplify.

When there is a square root in the numerator or denominator we can try to rationalize and see if that helps. Recall that rationalizing makes use of the fact that

$$(a + b)(a - b) = a^2 - b^2$$

So, if either the first and/or the second term have a square root in them the rationalizing will eliminate the root(s). This *might* help in evaluating the limit.

Let's try rationalizing the numerator in this case.

$$\lim_{t \rightarrow 4} \frac{t - \sqrt{3t + 4}}{4 - t} = \lim_{t \rightarrow 4} \frac{(t - \sqrt{3t + 4})(t + \sqrt{3t + 4})}{(4 - t)(t + \sqrt{3t + 4})}$$

Remember that to rationalize we just take the numerator (since that's what we're rationalizing), change the sign on the second term and multiply the numerator and denominator by this new term.

Next, we multiply the numerator out being careful to watch minus signs.

$$\begin{aligned} \lim_{t \rightarrow 4} \frac{t - \sqrt{3t + 4}}{4 - t} &= \lim_{t \rightarrow 4} \frac{t^2 - (3t + 4)}{(4 - t)(t + \sqrt{3t + 4})} \\ &= \lim_{t \rightarrow 4} \frac{t^2 - 3t - 4}{(4 - t)(t + \sqrt{3t + 4})} \end{aligned}$$

Notice that we didn't multiply the denominator out as well. Most students come out of an Algebra class having it beaten into their heads to always multiply this stuff out. However, in this case multiplying out will make the problem very difficult and in the end you'll just end up factoring it back out anyway.

At this stage we are almost done. Notice that we can factor the numerator so let's do that.

$$\lim_{t \rightarrow 4} \frac{t - \sqrt{3t + 4}}{4 - t} = \lim_{t \rightarrow 4} \frac{(t - 4)(t + 1)}{(4 - t)(t + \sqrt{3t + 4})}$$

Now all we need to do is notice that if we factor a “-1” out of the first term in the denominator we can do some canceling. At that point the division by zero problem will go away and we can evaluate the limit

$$\begin{aligned} \lim_{t \rightarrow 4} \frac{t - \sqrt{3t + 4}}{4 - t} &= \lim_{t \rightarrow 4} \frac{(t - 4)(t + 1)}{-(t - 4)(t + \sqrt{3t + 4})} \\ &= \lim_{t \rightarrow 4} \frac{t + 1}{-(t + \sqrt{3t + 4})} \\ &= -\frac{5}{8} \end{aligned}$$

Note that if we had multiplied the denominator out we would not have been able to do this canceling and in all likelihood would not have even seen that some canceling could have been done.

So, we've taken a look at a couple of limits in which evaluation gave the indeterminate form $0/0$ and we now have a couple of things to try in these cases.

Let's take a look at another kind of problem that can arise in computing some limits involving piecewise functions.

Example 4

Given the function,

$$g(y) = \begin{cases} y^2 + 5 & \text{if } y < -2 \\ 1 - 3y & \text{if } y \geq -2 \end{cases}$$

Compute the following limits.

(a) $\lim_{y \rightarrow 6} g(y)$

(b) $\lim_{y \rightarrow -2} g(y)$

Solution

(a) $\lim_{y \rightarrow 6} g(y)$

In this case there really isn't a whole lot to do. In doing limits recall that we must always look at what's happening on both sides of the point in question as we move in towards it. In this case $y = 6$ is completely inside the second interval for the function and so there are values of y on both sides of $y = 6$ that are also inside this interval. This means that we can just use the fact to evaluate this limit.

$$\begin{aligned} \lim_{y \rightarrow 6} g(y) &= \lim_{y \rightarrow 6} (1 - 3y) \\ &= -17 \end{aligned}$$

(b) $\lim_{y \rightarrow -2} g(y)$

This part is the real point to this problem. In this case the point that we want to take the limit for is the cutoff point for the two intervals. In other words, we can't just plug $y = -2$ into the second portion because this interval does not contain values of y to the left of $y = -2$ and we need to know what is happening on both sides of the point.

To do this part we are going to have to remember the fact from the section on [one-sided limits](#) that says that if the two one-sided limits exist and are the same then the normal limit will also exist and have the same value.

Notice that both of the one-sided limits can be done here since we are only going to be looking at one side of the point in question. So, let's do the two one-sided limits

and see what we get.

$$\begin{aligned}\lim_{y \rightarrow -2^-} g(y) &= \lim_{y \rightarrow -2^-} (y^2 + 5) && \text{since } y \rightarrow -2^- \text{ implies } y < -2 \\ &= 9\end{aligned}$$

$$\begin{aligned}\lim_{y \rightarrow -2^+} g(y) &= \lim_{y \rightarrow -2^+} (1 - 3y) && \text{since } y \rightarrow -2^+ \text{ implies } y > -2 \\ &= 7\end{aligned}$$

So, in this case we can see that,

$$\lim_{y \rightarrow -2^-} g(y) = 9 \neq 7 = \lim_{y \rightarrow -2^+} g(y)$$

and so since the two one sided limits aren't the same

$$\lim_{y \rightarrow -2} g(y)$$

doesn't exist.

Note that a very simple change to the function will make the limit at $y = -2$ exist so don't get in into your head that limits at these cutoff points in piecewise function don't ever exist as the following example will show.

Example 5

Evaluate the following limit.

$$\lim_{y \rightarrow -2} g(y) \quad \text{where, } g(y) = \begin{cases} y^2 + 5 & \text{if } y < -2 \\ 3 - 3y & \text{if } y \geq -2 \end{cases}$$

Solution

The two one-sided limits this time are,

$$\begin{aligned}\lim_{y \rightarrow -2^-} g(y) &= \lim_{y \rightarrow -2^-} (y^2 + 5) && \text{since } y \rightarrow -2^- \text{ implies } y < -2 \\ &= 9\end{aligned}$$

$$\begin{aligned}\lim_{y \rightarrow -2^+} g(y) &= \lim_{y \rightarrow -2^+} (3 - 3y) && \text{since } y \rightarrow -2^+ \text{ implies } y > -2 \\ &= 9\end{aligned}$$

The one-sided limits are the same so we get,

$$\lim_{y \rightarrow -2} g(y) = 9$$

There is one more limit that we need to do. However, we will need a new fact about limits that will help us to do this.

Fact

If $f(x) \leq g(x)$ for all x on $[a, b]$ (except possibly at $x = c$) and $a \leq c \leq b$ then,

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x)$$

Note that this fact should make some sense to you if we assume that both functions are nice enough. If both of the functions are “nice enough” to use the limit evaluation [fact](#) then we have,

$$\lim_{x \rightarrow c} f(x) = f(c) \leq g(c) = \lim_{x \rightarrow c} g(x)$$

The inequality is true because we know that c is somewhere between a and b and in that range we also know $f(x) \leq g(x)$.

Note that we don't really need the two functions to be nice enough for the fact to be true, but it does provide a nice way to give a quick “justification” for the fact.

Also, note that we said that we assumed that $f(x) \leq g(x)$ for all x on $[a, b]$ (except possibly at $x = c$). Because limits do not care what is actually happening at $x = c$ we don't really need the inequality to hold at that specific point. We only need it to hold around $x = c$ since that is what the limit is concerned about.

We can take this fact one step farther to get the following theorem.

Squeeze Theorem

Suppose that for all x on $[a, b]$ (except possibly at $x = c$) we have,

$$f(x) \leq h(x) \leq g(x)$$

Also suppose that,

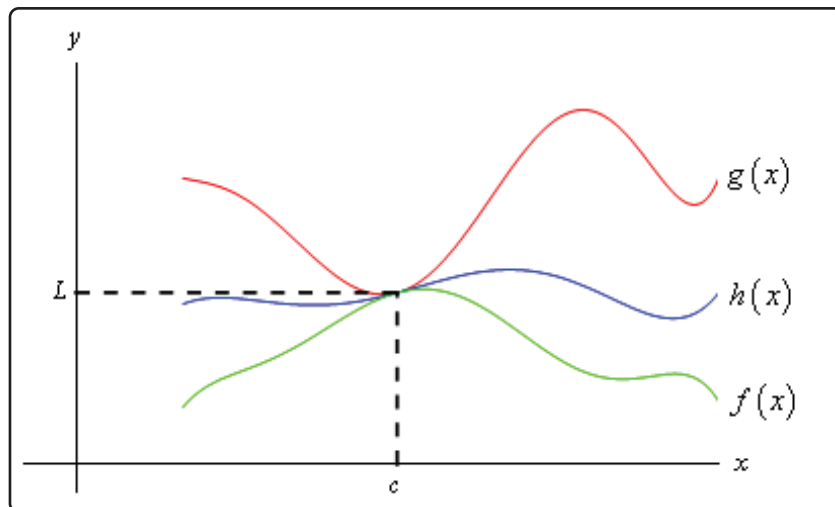
$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = L$$

for some $a \leq c \leq b$. Then,

$$\lim_{x \rightarrow c} h(x) = L$$

As with the previous fact we only need to know that $f(x) \leq h(x) \leq g(x)$ is true around $x = c$ because we are working with limits and they are only concerned with what is going on around $x = c$ and not what is actually happening at $x = c$.

Now, if we again assume that all three functions are nice enough (again this isn't required to make the Squeeze Theorem true, it only helps with the visualization) then we can get a quick sketch of what the Squeeze Theorem is telling us. The following figure illustrates what is happening in this theorem.



From the figure we can see that if the limits of $f(x)$ and $g(x)$ are equal at $x = c$ then the function values must also be equal at $x = c$ (this is where we're using the fact that we assumed the functions were "nice enough", which isn't really required for the Theorem). However, because $h(x)$ is "squeezed" between $f(x)$ and $g(x)$ at this point then $h(x)$ must have the same value. Therefore, the limit of $h(x)$ at this point must also be the same.

The Squeeze theorem is also known as the **Sandwich Theorem** and the **Pinching Theorem**.

So, how do we use this theorem to help us with limits? Let's take a look at the following example to see the theorem in action.

Example 6

Evaluate the following limit.

$$\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right)$$

Solution

In this example none of the previous examples can help us. There's no factoring or simplifying to do. We can't rationalize and one-sided limits won't work. There's even a question as to whether this limit will exist since we have division by zero inside the cosine at $x = 0$.

The first thing to notice is that we know the following fact about cosine.

$$-1 \leq \cos(x) \leq 1$$

Our function doesn't have just an x in the cosine, but as long as we avoid $x = 0$ we can say the same thing for our cosine.

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

It's okay for us to ignore $x = 0$ here because we are taking a limit and we know that limits don't care about what's actually going on at the point in question, $x = 0$ in this case.

Now if we have the above inequality for our cosine we can just multiply everything by an x^2 and get the following.

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2$$

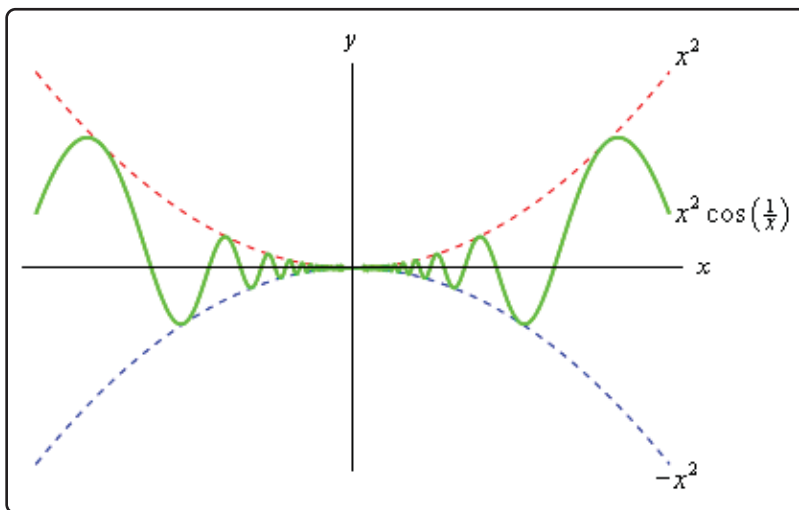
In other words we've managed to squeeze the function that we were interested in between two other functions that are very easy to deal with. So, the limits of the two outer functions are.

$$\lim_{x \rightarrow 0} x^2 = 0 \qquad \lim_{x \rightarrow 0} (-x^2) = 0$$

These are the same and so by the Squeeze theorem we must also have,

$$\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) = 0$$

We can verify this with the graph of the three functions. This is shown below.



In this section we've seen several tools that we can use to help us to compute limits in which we can't just evaluate the function at the point in question. As we will see many of the limits that we'll be doing in later sections will require one or more of these tools.

2.6 Infinite Limits

In this section we will take a look at limits whose value is infinity or minus infinity. These kinds of limit will show up fairly regularly in later sections and in other courses and so you'll need to be able to deal with them when you run across them.

The first thing we should probably do here is to define just what we mean when we say that a limit has a value of infinity or minus infinity.

Definition

We say

$$\lim_{x \rightarrow a} f(x) = \infty$$

if we can make $f(x)$ arbitrarily large for all x sufficiently close to $x = a$, from both sides, without actually letting $x = a$.

We say

$$\lim_{x \rightarrow a} f(x) = -\infty$$

if we can make $f(x)$ arbitrarily large and negative for all x sufficiently close to $x = a$, from both sides, without actually letting $x = a$.

These definitions can be appropriately modified for the one-sided limits as well. To see a more precise and mathematical definition of this kind of limit see the [The Definition of the Limit](#) section at the end of this chapter.

Let's start off with a fairly typical example illustrating infinite limits.

Example 1

Evaluate each of the following limits.

$$\lim_{x \rightarrow 0^+} \frac{1}{x}$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x}$$

$$\lim_{x \rightarrow 0} \frac{1}{x}$$

Solution

So, we're going to be taking a look at a couple of one-sided limits as well as the normal limit here. In all three cases notice that we can't just plug in $x = 0$. If we did we would get division by zero. Also recall that the definitions above can be easily modified to give similar definitions for the two one-sided limits which we'll be needing here.

Now, there are several ways we could proceed here to get values for these limits. One way is to plug in some points and see what value the function is approaching. In the preceding

section we said that we were no longer going to do this, but in this case it is a good way to illustrate just what's going on with this function.

So, here is a table of values of x 's from both the left and the right. Using these values we'll be able to estimate the value of the two one-sided limits and once we have that done we can use the [fact](#) that the normal limit will exist only if the two one-sided limits exist and have the same value.

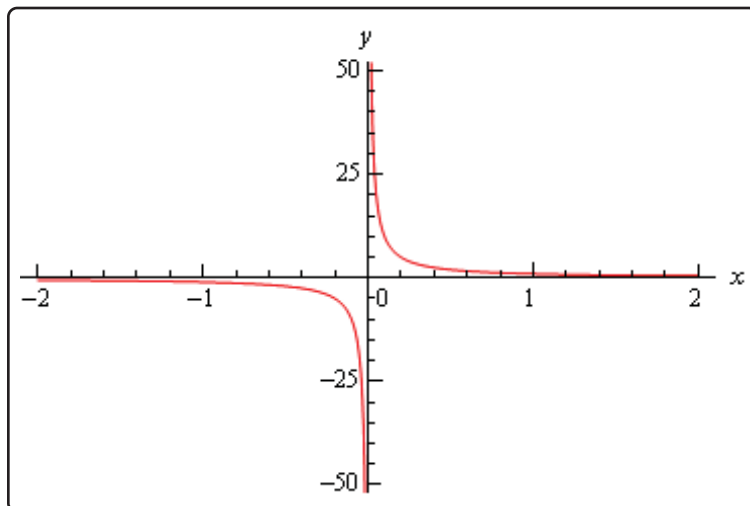
x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10000	0.0001	10000

From this table we can see that as we make x smaller and smaller the function $\frac{1}{x}$ gets larger and larger and will retain the same sign that x originally had. It should make sense that this trend will continue for any smaller value of x that we chose to use. The function is a constant (one in this case) divided by an increasingly small number. The resulting fraction should be an increasingly large number and as noted above the fraction will retain the same sign as x .

We can make the function as large and positive as we want for all x 's sufficiently close to zero while staying positive (*i.e.* on the right). Likewise, we can make the function as large and negative as we want for all x 's sufficiently close to zero while staying negative (*i.e.* on the left). So, from our definition above it looks like we should have the following values for the two one sided limits.

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \qquad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

Another way to see the values of the two one sided limits here is to graph the function. Again, in the previous section we mentioned that we won't do this too often as most functions are not something we can just quickly sketch out as well as the problems with accuracy in reading values off the graph. In this case however, it's not too hard to sketch a graph of the function and, in this case as we'll see accuracy is not really going to be an issue. So, here is a quick sketch of the graph.



So, we can see from this graph that the function does behave much as we predicted that it would from our table values. The closer x gets to zero from the right the larger (in the positive sense) the function gets, while the closer x gets to zero from the left the larger (in the negative sense) the function gets.

Finally, the normal limit, in this case, will not exist since the two one-sided limits have different values.

So, in summary here are the values of the three limits for this example.

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty \quad \lim_{x \rightarrow 0} \frac{1}{x} \text{ doesn't exist}$$

For most of the remaining examples in this section we'll attempt to "talk our way through" each limit. This means that we'll see if we can analyze what should happen to the function as we get very close to the point in question without actually plugging in any values into the function. For most of the following examples this kind of analysis shouldn't be all that difficult to do. We'll also verify our analysis with a quick graph.

So, let's do a couple more examples.

Example 2

Evaluate each of the following limits.

$$\lim_{x \rightarrow 0^+} \frac{6}{x^2} \quad \lim_{x \rightarrow 0^-} \frac{6}{x^2} \quad \lim_{x \rightarrow 0} \frac{6}{x^2}$$

Solution

As with the previous example let's start off by looking at the two one-sided limits. Once we have those we'll be able to determine a value for the normal limit.

So, let's take a look at the right-hand limit first and as noted above let's see if we can figure out what each limit will be doing without actually plugging in any values of x into the function. As we take smaller and smaller values of x , while staying positive, squaring them will only make them smaller (recall squaring a number between zero and one will make it smaller) and of course it will stay positive. So, we have a positive constant divided by an increasingly small positive number. The result should then be an increasingly large positive number. It looks like we should have the following value for the right-hand limit in this case,

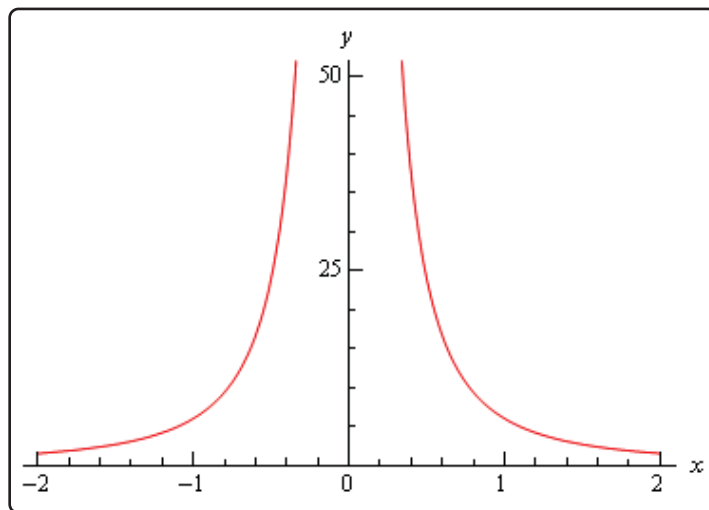
$$\lim_{x \rightarrow 0^+} \frac{6}{x^2} = \infty$$

Now, let's take a look at the left-hand limit. In this case we're going to take smaller and smaller values of x , while staying negative this time. When we square them they'll get smaller, but upon squaring the result is now positive. So, we have a positive constant divided by an increasingly small positive number. The result, as with the right-hand limit, will be an increasingly large positive number and so the left-hand limit will be,

$$\lim_{x \rightarrow 0^-} \frac{6}{x^2} = \infty$$

Now, in this example, unlike the first one, the normal limit will exist and be infinity since the two one-sided limits both exist and have the same value. So, in summary here are all the limits for this example as well as a quick graph verifying the limits.

$$\lim_{x \rightarrow 0^+} \frac{6}{x^2} = \infty \quad \lim_{x \rightarrow 0^-} \frac{6}{x^2} = \infty \quad \lim_{x \rightarrow 0} \frac{6}{x^2} = \infty$$



With this next example we'll move away from just an x in the denominator, but as we'll see in the next couple of examples they work pretty much the same way.

Example 3

Evaluate each of the following limits.

$$\lim_{x \rightarrow -2^+} \frac{-4}{x+2} \qquad \lim_{x \rightarrow -2^-} \frac{-4}{x+2} \qquad \lim_{x \rightarrow -2} \frac{-4}{x+2}$$

Solution

Let's again start with the right-hand limit. With the right-hand limit we know that we have,

$$x > -2 \qquad \Rightarrow \qquad x + 2 > 0$$

Also, as x gets closer and closer to -2 then $x + 2$ will be getting closer and closer to zero, while staying positive as noted above. So, for the right-hand limit, we'll have a negative constant divided by an increasingly small positive number. The result will be an increasingly large and negative number. So, it looks like the right-hand limit will be negative infinity.

For the left-hand limit we have,

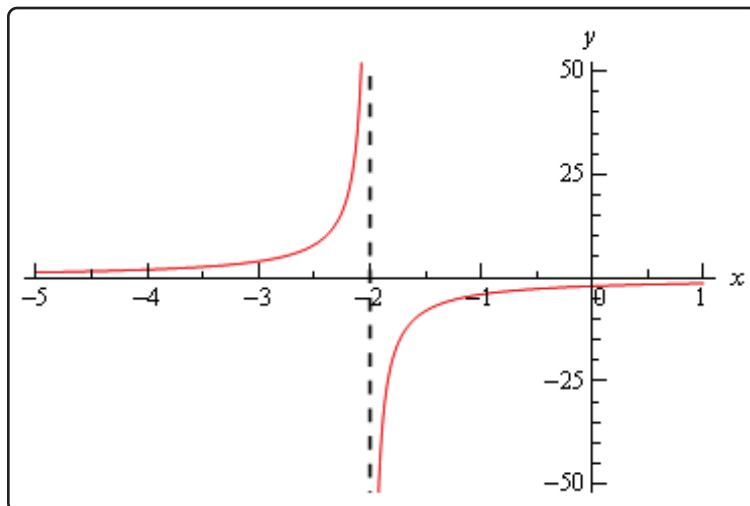
$$x < -2 \qquad \Rightarrow \qquad x + 2 < 0$$

and $x + 2$ will get closer and closer to zero (and be negative) as x gets closer and closer to -2 . In this case then we'll have a negative constant divided by an increasingly small negative number. The result will then be an increasingly large positive number and so it looks like the left-hand limit will be positive infinity.

Finally, since two one sided limits are not the same the normal limit won't exist.

Here are the official answers for this example as well as a quick graph of the function for verification purposes.

$$\lim_{x \rightarrow -2^+} \frac{-4}{x+2} = -\infty \qquad \lim_{x \rightarrow -2^-} \frac{-4}{x+2} = \infty \qquad \lim_{x \rightarrow -2} \frac{-4}{x+2} \text{ doesn't exist}$$



At this point we should briefly acknowledge the idea of vertical asymptotes. Each of the three previous graphs have had one. Recall from an Algebra class that a vertical asymptote is a vertical line (the dashed line at $x = -2$ in the previous example) in which the graph will go towards infinity and/or minus infinity on one or both sides of the line.

In an Algebra class they are a little difficult to define other than to say pretty much what we just said. Now that we have infinite limits under our belt we can easily define a vertical asymptote as follows,

Definition

The function $f(x)$ will have a vertical asymptote at $x = a$ if we have any of the following limits at $x = a$.

$$\lim_{x \rightarrow a^-} f(x) = \pm \infty \quad \lim_{x \rightarrow a^+} f(x) = \pm \infty \quad \lim_{x \rightarrow a} f(x) = \pm \infty$$

Note that it only requires one of the above limits for a function to have a vertical asymptote at $x = a$.

Using this definition we can see that the first two examples had vertical asymptotes at $x = 0$ while the third example had a vertical asymptote at $x = -2$.

We aren't really going to do a lot with vertical asymptotes here but wanted to mention them at this point since we'd reached a good point to do that.

Let's now take a look at a couple more examples of infinite limits that can cause some problems on occasion.

Example 4

Evaluate each of the following limits.

$$\lim_{x \rightarrow 4^+} \frac{3}{(4-x)^3}$$

$$\lim_{x \rightarrow 4^-} \frac{3}{(4-x)^3}$$

$$\lim_{x \rightarrow 4} \frac{3}{(4-x)^3}$$

Solution

Let's start with the right-hand limit. For this limit we have,

$$x > 4 \quad \Rightarrow \quad 4 - x < 0 \quad \Rightarrow \quad (4 - x)^3 < 0$$

also, $4 - x \rightarrow 0$ as $x \rightarrow 4$. So, we have a positive constant divided by an increasingly small negative number. The results will be an increasingly large negative number and so it looks like the right-hand limit will be negative infinity.

For the left-handed limit we have,

$$x < 4 \quad \Rightarrow \quad 4 - x > 0 \quad \Rightarrow \quad (4 - x)^3 > 0$$

and we still have, $4 - x \rightarrow 0$ as $x \rightarrow 4$. In this case we have a positive constant divided by an increasingly small positive number. The results will be an increasingly large positive number and so it looks like the left-hand limit will be positive infinity.

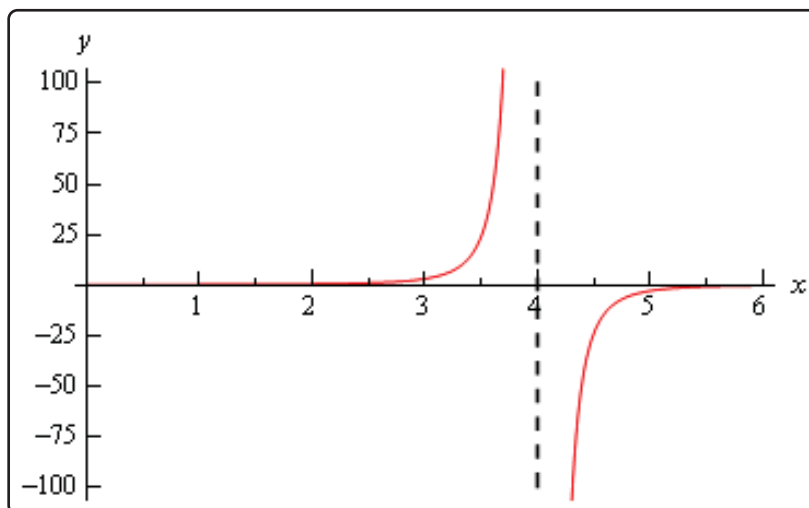
The normal limit will not exist since the two one-sided limits are not the same. The official answers to this example are then,

$$\lim_{x \rightarrow 4^+} \frac{3}{(4-x)^3} = -\infty$$

$$\lim_{x \rightarrow 4^-} \frac{3}{(4-x)^3} = \infty$$

$$\lim_{x \rightarrow 4} \frac{3}{(4-x)^3} \text{ doesn't exist}$$

Here is a quick sketch to verify our limits.



All the examples to this point have had a constant in the numerator and we should probably take a quick look at an example that doesn't have a constant in the numerator.

Example 5

Evaluate each of the following limits.

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3} \qquad \lim_{x \rightarrow 3^-} \frac{2x}{x-3} \qquad \lim_{x \rightarrow 3} \frac{2x}{x-3}$$

Solution

Let's take a look at the right-handed limit first. For this limit we'll have,

$$x > 3 \qquad \Rightarrow \qquad x - 3 > 0$$

The main difference here with this example is the behavior of the numerator as we let x get closer and closer to 3. In this case we have the following behavior for both the numerator and denominator.

$$x - 3 \rightarrow 0 \text{ and } 2x \rightarrow 6 \text{ as } x \rightarrow 3$$

So, as we let x get closer and closer to 3 (always staying on the right of course) the numerator, while not a constant, is getting closer and closer to a positive constant while the denominator is getting closer and closer to zero and will be positive since we are on the right side.

This means that we'll have a numerator that is getting closer and closer to a non-zero and positive constant divided by an increasingly smaller positive number and so the result should be an increasingly larger positive number. The right-hand limit should then be positive infinity.

For the left-hand limit we'll have,

$$x < 3 \qquad \Rightarrow \qquad x - 3 < 0$$

As with the right-hand limit we'll have the following behaviors for the numerator and the denominator,

$$x - 3 \rightarrow 0 \text{ and } 2x \rightarrow 6 \text{ as } x \rightarrow 3$$

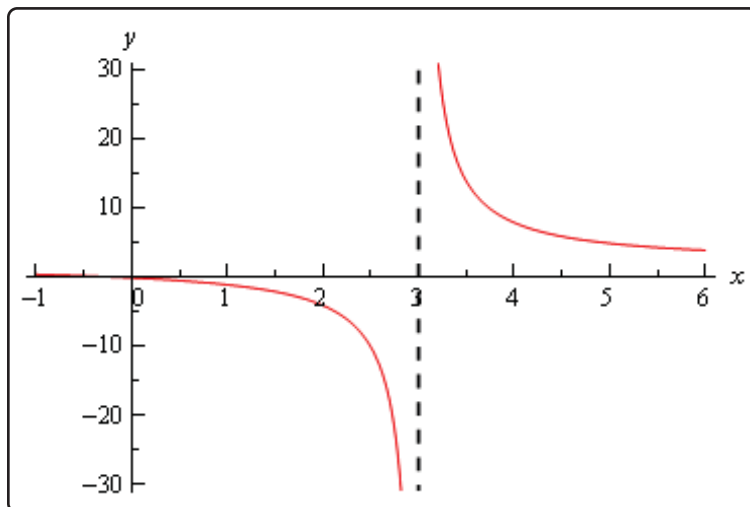
The main difference in this case is that the denominator will now be negative. So, we'll have a numerator that is approaching a positive, non-zero constant divided by an increasingly small negative number. The result will be an increasingly large and negative number.

The formal answers for this example are then,

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3} = \infty \qquad \lim_{x \rightarrow 3^-} \frac{2x}{x-3} = -\infty \qquad \lim_{x \rightarrow 3} \frac{2x}{x-3} \text{ doesn't exist}$$

As with most of the examples in this section the normal limit does not exist since the two one-sided limits are not the same.

Here's a quick graph to verify our limits.



So far all we've done is look at limits of rational expressions, let's do a couple of quick examples with some different functions.

Example 6

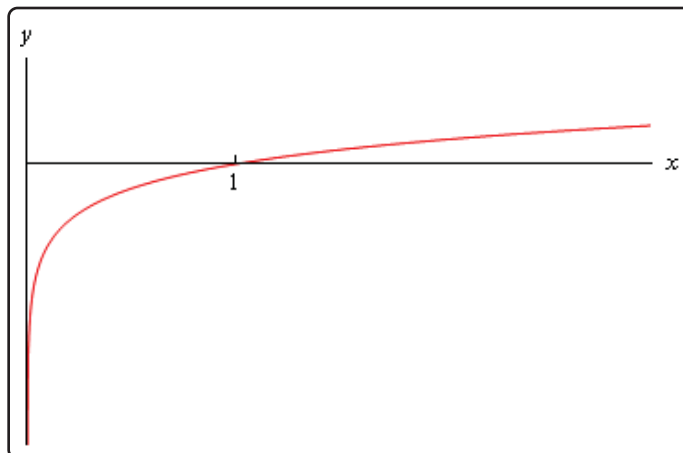
Evaluate

$$\lim_{x \rightarrow 0^+} \ln(x)$$

Solution

First, notice that we can only evaluate the right-handed limit here. We know that the domain of any logarithm is only the positive numbers and so we can't even talk about the left-handed limit because that would necessitate the use of negative numbers. Likewise, since we can't deal with the left-handed limit then we can't talk about the normal limit.

This limit is pretty simple to get from a quick sketch of the graph.



From this we can see that,

$$\lim_{x \rightarrow 0^+} \ln(x) = -\infty$$

Example 7

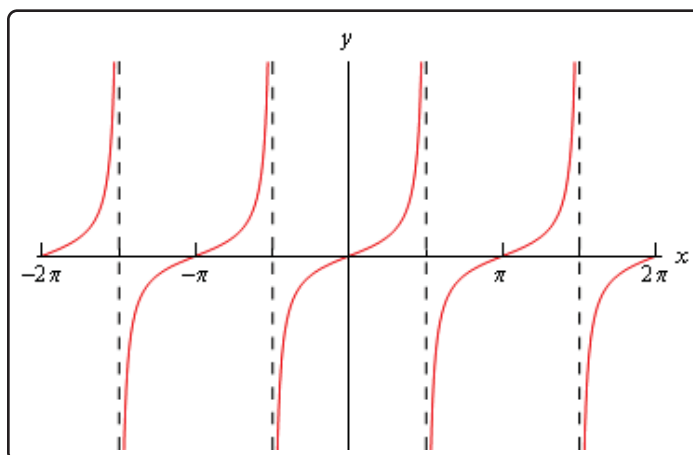
Evaluate both of the following limits.

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \tan(x)$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan(x)$$

Solution

Here's a quick sketch of the graph of the tangent function.



From this it's easy to see that we have the following values for each of these limits,

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \tan(x) = -\infty \qquad \lim_{x \rightarrow \frac{\pi}{2}^-} \tan(x) = \infty$$

Note that the normal limit will not exist because the two one-sided limits are not the same.

We'll leave this section with a few facts about infinite limits.

Facts

Given the functions $f(x)$ and $g(x)$ suppose we have,

$$\lim_{x \rightarrow c} f(x) = \infty \qquad \lim_{x \rightarrow c} g(x) = L$$

for some real numbers c and L . Then,

1. $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \infty$
2. If $L > 0$ then $\lim_{x \rightarrow c} [f(x)g(x)] = \infty$
3. If $L < 0$ then $\lim_{x \rightarrow c} [f(x)g(x)] = -\infty$
4. $\lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0$

To see the proof of this set of facts see the [Proof of Various Limit Properties](#) section in the Extras appendix.

Note as well that the above set of facts also holds for one-sided limits. They will also hold if $\lim_{x \rightarrow c} f(x) = -\infty$, with a change of sign on the infinities in the first three parts. The proofs of these changes to the facts are nearly identical to the proof of the original facts and so are left to the you.

2.7 Limits at Infinity, Part I

In the previous section we saw limits that were infinity and it's now time to take a look at limits at infinity. By limits at infinity we mean one of the following two limits.

$$\lim_{x \rightarrow \infty} f(x) \qquad \lim_{x \rightarrow -\infty} f(x)$$

In other words, we are going to be looking at what happens to a function if we let x get very large in either the positive or negative sense. Also, as we'll soon see, these limits may also have infinity as a value.

First, let's note that the set of [Facts](#) from the Infinite Limit section also hold if we replace the $\lim_{x \rightarrow c}$ with $\lim_{x \rightarrow \infty}$ or $\lim_{x \rightarrow -\infty}$. The proof of this is nearly identical to the [proof](#) of the original set of facts with only minor modifications to handle the change in the limit and so is left to you. We won't need these facts much over the next couple of sections but they will be required on occasion.

In fact, many of the limits that we're going to be looking at we will need the following two facts.

Fact 1

1. If r is a positive rational number and c is any real number then,

$$\lim_{x \rightarrow \infty} \frac{c}{x^r} = 0$$

2. If r is a positive rational number, c is any real number and x^r is defined for $x < 0$ then,

$$\lim_{x \rightarrow -\infty} \frac{c}{x^r} = 0$$

The first part of this fact should make sense if you think about it. Because we are requiring $r > 0$ we know that x^r will stay in the denominator. Next as we increase x then x^r will also increase. So, we have a constant divided by an increasingly large number and so the result will be increasingly small. Or, in the limit we will get zero.

The second part is nearly identical except we need to worry about x^r being defined for negative x . This condition is here to avoid cases such as $r = \frac{1}{2}$. If this r were allowed we'd be taking the square root of negative numbers which would be complex and we want to avoid that at this level.

Note as well that the sign of c will not affect the answer. Regardless of the sign of c we'll still have a constant divided by a very large number which will result in a very small number and the larger x get the smaller the fraction gets. The sign of c will affect which direction the fraction approaches zero (*i.e.* from the positive or negative side) but it still approaches zero.

If you think about it this is really a special case of the last Fact from the [Facts](#) in the previous section. However, to see a direct proof of this fact see the [Proof of Various Limit Properties](#) section in the Extras appendix.

Let's start off the examples with one that will lead us to a nice idea that we'll use on a regular basis about limits at infinity for polynomials.

Example 1

Evaluate each of the following limits.

(a) $\lim_{x \rightarrow \infty} (2x^4 - x^2 - 8x)$

(b) $\lim_{t \rightarrow -\infty} \left(\frac{1}{3}t^5 + 2t^3 - t^2 + 8\right)$

Solution

(a) $\lim_{x \rightarrow \infty} (2x^4 - x^2 - 8x)$

Our first thought here is probably to just “plug” infinity into the polynomial and “evaluate” each term to determine the value of the limit. It is pretty simple to see what each term will do in the limit and so this seems like an obvious step, especially since we've been doing that for other limits in previous sections.

So, let's see what we get if we do that. As x approaches infinity, then x to a power can only get larger and the coefficient on each term (the first and third) will only make the term even larger. So, if we look at what each term is doing in the limit we get the following,

$$\lim_{x \rightarrow \infty} (2x^4 - x^2 - 8x) = \infty - \infty - \infty$$

Now, we've got a small, but easily fixed, problem to deal with. We are probably tempted to say that the answer is zero (because we have an infinity minus an infinity) or maybe $-\infty$ (because we're subtracting two infinities off of one infinity). However, in both cases we'd be wrong. This is one of those **indeterminate forms** that we first started seeing in a previous [section](#).

Infinities just don't always behave as real numbers do when it comes to arithmetic. Without more work there is simply no way to know what $\infty - \infty$ will be and so we really need to be careful with this kind of problem. To read a little more about this see the [Types of Infinity](#) section in the Extras appendix.

So, we need a way to get around this problem. What we'll do here is factor the largest power of x out of the whole polynomial as follows,

$$\lim_{x \rightarrow \infty} (2x^4 - x^2 - 8x) = \lim_{x \rightarrow \infty} \left[x^4 \left(2 - \frac{1}{x^2} - \frac{8}{x^3} \right) \right]$$

If you're not sure you agree with the factoring above (there's a chance you haven't really been asked to do this kind of factoring prior to this) then recall that to check all

you need to do is multiply the x^4 back through the parenthesis to verify it was done correctly. Also, an easy way to remember how to do this kind of factoring is to note that the second term is just the original polynomial divided by x^4 . This will always work when factoring a power of x out of a polynomial.

Now for each of the terms we have,

$$\lim_{x \rightarrow \infty} x^4 = \infty \qquad \lim_{x \rightarrow \infty} \left(2 - \frac{1}{x^2} - \frac{8}{x^3} \right) = 2$$

The first limit is clearly infinity and for the second limit we'll use the fact above on the last two terms. Therefore using [Fact 2](#) from the previous section we see value of the limit will be,

$$\lim_{x \rightarrow \infty} (2x^4 - x^2 - 8x) = \infty$$

(b) $\lim_{t \rightarrow -\infty} \left(\frac{1}{3}t^5 + 2t^3 - t^2 + 8 \right)$

We'll work this part much quicker than the previous part. All we need to do is factor out the largest power of t to get the following,

$$\lim_{t \rightarrow -\infty} \left(\frac{1}{3}t^5 + 2t^3 - t^2 + 8 \right) = \lim_{t \rightarrow -\infty} \left[t^5 \left(\frac{1}{3} + \frac{2}{t^2} - \frac{1}{t^3} + \frac{8}{t^5} \right) \right]$$

Remember that all you need to do to get the factoring correct is divide the original polynomial by the power of t we're factoring out, t^5 in this case.

Now all we need to do is take the limit of the two terms. In the first don't forget that since we're going out towards $-\infty$ and we're raising t to the 5^{th} power that the limit will be negative (negative number raised to an odd power is still negative). In the second term we'll again make heavy use of the fact above to see that is a finite number.

Therefore, using a modification of the [Facts](#) from the previous section the value of the limit is,

$$\lim_{t \rightarrow -\infty} \left(\frac{1}{3}t^5 + 2t^3 - t^2 + 8 \right) = -\infty$$

Okay, now that we've seen how a couple of polynomials work we can give a simple fact about polynomials in general.

Fact 2

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is a polynomial of degree n (i.e. $a_n \neq 0$) then,

$$\lim_{x \rightarrow \infty} p(x) = \lim_{x \rightarrow \infty} a_n x^n \qquad \lim_{x \rightarrow -\infty} p(x) = \lim_{x \rightarrow -\infty} a_n x^n$$

What this fact is really saying is that when we take a limit at infinity for a polynomial all we need to really do is look at the term with the largest power and ask what that term is doing in the limit since the polynomial will have the same behavior.

You can see the proof in the [Proof of Various Limit Properties](#) section in the Extras appendix.

Let's now move into some more complicated limits.

Example 2

Evaluate both of the following limits.

$$\lim_{x \rightarrow \infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7}$$

$$\lim_{x \rightarrow -\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7}$$

Solution

First, the only difference between these two is that one is going to positive infinity and the other is going to negative infinity. Sometimes this small difference will affect the value of the limit and at other times it won't.

Let's start with the first limit and as with our first set of examples it might be tempting to just "plug" in the infinity. Since both the numerator and denominator are polynomials we can use the above fact to determine the behavior of each. Doing this gives,

$$\lim_{x \rightarrow \infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} = \frac{\infty}{-\infty}$$

This is yet another indeterminate form. In this case we might be tempted to say that the limit is infinity (because of the infinity in the numerator), zero (because of the infinity in the denominator) or -1 (because something divided by itself is one). There are three separate arithmetic "rules" at work here and without work there is no way to know which "rule" will be correct and to make matters worse it's possible that none of them may work and we might get a completely different answer, say $-\frac{2}{5}$ to pick a number completely at random.

So, when we have a polynomial divided by a polynomial we're going to proceed much as we did with only polynomials. We first identify the largest power of x in the denominator (and yes, we only look at the denominator for this) and we then factor this out of both the numerator and denominator. Doing this for the first limit gives,

$$\lim_{x \rightarrow \infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} = \lim_{x \rightarrow \infty} \frac{x^4 \left(2 - \frac{1}{x^2} + \frac{8}{x^3}\right)}{x^4 \left(-5 + \frac{7}{x^4}\right)}$$

Once we've done this we can cancel the x^4 from both the numerator and the denominator

and then use the Fact 1 above to take the limit of all the remaining terms. This gives,

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} &= \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x^2} + \frac{8}{x^3}}{-5 + \frac{7}{x^4}} \\ &= \frac{2 + 0 + 0}{-5 + 0} \\ &= -\frac{2}{5}\end{aligned}$$

In this case the indeterminate form was neither of the “obvious” choices of infinity, zero, or -1 so be careful with make these kinds of assumptions with this kind of indeterminate forms.

The second limit is done in a similar fashion. Notice however, that nowhere in the work for the first limit did we actually use the fact that the limit was going to plus infinity. In this case it doesn't matter which infinity we are going towards we will get the same value for the limit.

$$\lim_{x \rightarrow -\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} = -\frac{2}{5}$$

In the previous example the infinity that we were using in the limit didn't change the answer. This will not always be the case so don't make the assumption that this will always be the case.

Let's take a look at an example where we get different answers for each limit.

Example 3

Evaluate each of the following limits.

$$\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} \qquad \lim_{x \rightarrow -\infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x}$$

Solution

The square root in this problem won't change our work, but it will make the work a little messier.

Let's start with the first limit. In this case the largest power of x in the denominator is just an x . So, we need to factor an x out of the numerator and the denominator. When we are done factoring the x out we will need an x in both of the numerator and the denominator. To get this in the numerator we will have to factor an x^2 out of the square root so that after we take the square root we will get an x .

This is probably not something you're used to doing, but just remember that when it comes out of the square root it needs to be an x and the only way have an x come out of a square

root is to take the square root of x^2 and so that is what we'll need to factor out of the term under the radical. Here's the factoring work for this part,

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 \left(3 + \frac{6}{x^2}\right)}}{x \left(\frac{5}{x} - 2\right)} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2} \sqrt{3 + \frac{6}{x^2}}}{x \left(\frac{5}{x} - 2\right)}\end{aligned}$$

This is where we need to be really careful with the square root in the problem. Don't forget that

$$\sqrt{x^2} = |x|$$

Square roots are ALWAYS positive and so we need the absolute value bars on the x to make sure that it will give a positive answer. This is not something that most people ever remember seeing in an Algebra class and in fact it's not always given in an Algebra class. However, at this point it becomes absolutely vital that we know and use this fact. Using this fact the limit becomes,

$$\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} = \lim_{x \rightarrow \infty} \frac{|x| \sqrt{3 + \frac{6}{x^2}}}{x \left(\frac{5}{x} - 2\right)}$$

Now, we can't just cancel the x 's. We first will need to get rid of the absolute value bars. To do this let's recall the definition of absolute value.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In this case we are going out to plus infinity so we can safely assume that the x will be positive and so we can just drop the absolute value bars. The limit is then,

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} &= \lim_{x \rightarrow \infty} \frac{x \sqrt{3 + \frac{6}{x^2}}}{x \left(\frac{5}{x} - 2\right)} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{3 + \frac{6}{x^2}}}{\frac{5}{x} - 2} = \frac{\sqrt{3 + 0}}{0 - 2} = -\frac{\sqrt{3}}{2}\end{aligned}$$

Let's now take a look at the second limit (the one with negative infinity). In this case we will need to pay attention to the limit that we are using. The initial work will be the same up until we reach the following step.

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} = \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{3 + \frac{6}{x^2}}}{x \left(\frac{5}{x} - 2\right)}$$

In this limit we are going to minus infinity so in this case we can assume that x is negative. So, in order to drop the absolute value bars in this case we will need to tack on a minus sign as well. The limit is then,

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x} &= \lim_{x \rightarrow -\infty} \frac{-x\sqrt{3 + \frac{6}{x^2}}}{x\left(\frac{5}{x} - 2\right)} \\ &= \lim_{x \rightarrow -\infty} \frac{-\sqrt{3 + \frac{6}{x^2}}}{\frac{5}{x} - 2} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

So, as we saw in the last two examples sometimes the infinity in the limit will affect the answer and other times it won't. Note as well that it doesn't always just change the sign of the number. It can on occasion completely change the value. We'll see an example or two of this in the next section.

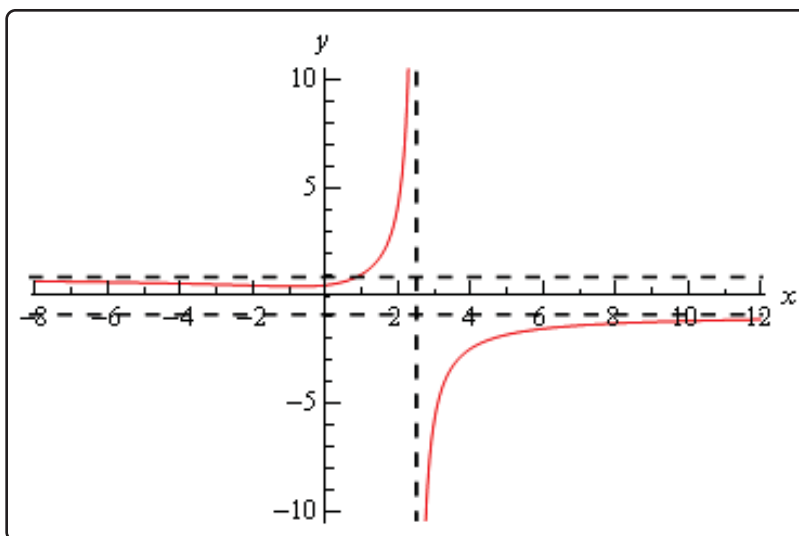
Before moving on to a couple of more examples let's revisit the idea of asymptotes that we first saw in the previous section. Just as we can have vertical asymptotes defined in terms of limits we can also have horizontal asymptotes defined in terms of limits.

Definition

The function $f(x)$ will have a horizontal asymptote at $y = L$ if either of the following are true.

$$\lim_{x \rightarrow \infty} f(x) = L \qquad \lim_{x \rightarrow -\infty} f(x) = L$$

We're not going to be doing much with asymptotes here, but it's an easy fact to give and we can use the previous example to illustrate all the asymptote ideas we've seen in the both this section and the previous section. The function in the last example will have two horizontal asymptotes. It will also have a vertical asymptote. Here is a graph of the function showing these.



Let's work another couple of examples involving rational expressions.

Example 4

Evaluate each of the following limits.

$$\lim_{z \rightarrow \infty} \frac{4z^2 + z^6}{1 - 5z^3}$$

$$\lim_{z \rightarrow -\infty} \frac{4z^2 + z^6}{1 - 5z^3}$$

Solution

Let's do the first limit and in this case it looks like we will factor a z^3 out of both the numerator and denominator. Remember that we only look at the denominator when determining the largest power of z here. There is a larger power of z in the numerator but we ignore it. We ONLY look at the denominator when doing this! So, doing the factoring gives,

$$\begin{aligned} \lim_{z \rightarrow \infty} \frac{4z^2 + z^6}{1 - 5z^3} &= \lim_{z \rightarrow \infty} \frac{z^3 \left(\frac{4}{z} + z^3 \right)}{z^3 \left(\frac{1}{z^3} - 5 \right)} \\ &= \lim_{z \rightarrow \infty} \frac{\frac{4}{z} + z^3}{\frac{1}{z^3} - 5} \end{aligned}$$

When we take the limit we'll need to be a little careful. The first term in the numerator and denominator will both be zero. However, the z^3 in the numerator will be going to plus infinity in the limit and so the limit is,

$$\lim_{z \rightarrow \infty} \frac{4z^2 + z^6}{1 - 5z^3} = \frac{\infty}{-5} = -\infty$$

The final limit is negative because we have a quotient of positive quantity and a negative quantity.

Now, let's take a look at the second limit. Note that the only difference in the work is at the final "evaluation" step and so we'll pick up the work there.

$$\lim_{z \rightarrow -\infty} \frac{4z^2 + z^6}{1 - 5z^3} = \lim_{z \rightarrow -\infty} \frac{\frac{4}{z} + z^3}{\frac{1}{z^3} - 5} = \frac{-\infty}{-5} = \infty$$

In this case the z^3 in the numerator gives negative infinity in the limit since we are going out to minus infinity and the power is odd. The answer is positive since we have a quotient of two negative numbers.

Example 5

Evaluate the following limit.

$$\lim_{t \rightarrow -\infty} \frac{t^2 - 5t - 9}{2t^4 + 3t^3}$$

Solution

In this case it looks like we will factor a t^4 out of both the numerator and denominator. Doing this gives,

$$\begin{aligned} \lim_{t \rightarrow -\infty} \frac{t^2 - 5t - 9}{2t^4 + 3t^3} &= \lim_{t \rightarrow -\infty} \frac{t^4 \left(\frac{1}{t^2} - \frac{5}{t^3} - \frac{9}{t^4} \right)}{t^4 \left(2 + \frac{3}{t} \right)} \\ &= \lim_{t \rightarrow -\infty} \frac{\frac{1}{t^2} - \frac{5}{t^3} - \frac{9}{t^4}}{2 + \frac{3}{t}} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

In this case using Fact 1 we can see that the numerator is zero and so since the denominator is also not zero the fraction, and hence the limit, will be zero.

In this section we concentrated on limits at infinity with functions that only involved polynomials and/or rational expression involving polynomials. There are many more types of functions that we could use here. That is the subject of the next section.

To see a precise and mathematical definition of this kind of limit see the [The Definition of the Limit](#) section at the end of this chapter.

2.9 Continuity

Over the last few sections we've been using the term "nice enough" to define those functions that we could evaluate limits by just evaluating the function at the point in question. It's now time to formally define what we mean by "nice enough".

Definition

A function $f(x)$ is said to be **continuous** at $x = a$ if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

A function is said to be continuous on the interval $[a, b]$ if it is continuous at each point in the interval.

Note that this definition is also implicitly assuming that both $f(a)$ and $\lim_{x \rightarrow a} f(x)$ exist. If either of these do not exist the function will not be continuous at $x = a$.

This definition can be turned around into the following fact.

Fact 1

If $f(x)$ is continuous at $x = a$ then,

$$\lim_{x \rightarrow a} f(x) = f(a)$$

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

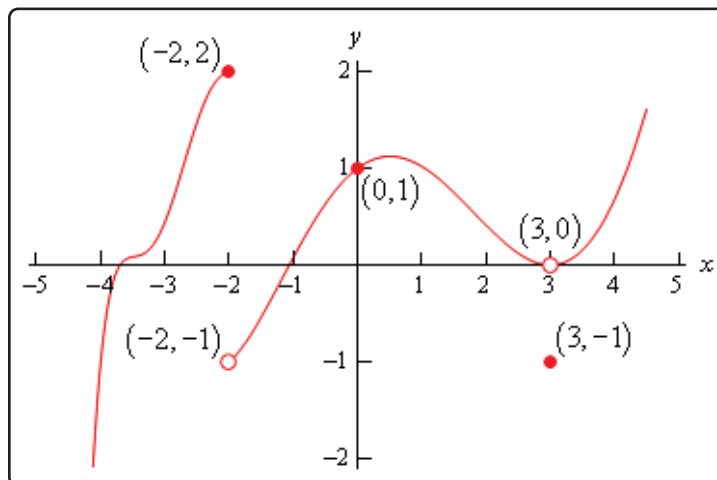
$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

This is exactly the same fact that we first put down [back](#) when we started looking at limits with the exception that we have replaced the phrase "nice enough" with continuous.

It's nice to finally know what we mean by "nice enough", however, the definition doesn't really tell us just what it means for a function to be continuous. Let's take a look at an example to help us understand just what it means for a function to be continuous.

Example 1

Given the graph of $f(x)$, shown below, determine if $f(x)$ is continuous at $x = -2$, $x = 0$, and $x = 3$.



Solution

To answer the question for each point we'll need to get both the limit at that point and the function value at that point. If they are equal the function is continuous at that point and if they aren't equal the function isn't continuous at that point.

First $x = -2$.

$$f(-2) = 2 \qquad \lim_{x \rightarrow -2} f(x) \text{ doesn't exist}$$

The function value and the limit aren't the same and so the function is not continuous at this point. This kind of discontinuity in a graph is called a **jump discontinuity**. Jump discontinuities occur where the graph has a break in it as this graph does and the values of the function to either side of the break are finite (*i.e.* the function doesn't go to infinity).

Now $x = 0$.

$$f(0) = 1 \qquad \lim_{x \rightarrow 0} f(x) = 1$$

The function is continuous at this point since the function and limit have the same value.

Finally $x = 3$.

$$f(3) = -1 \qquad \lim_{x \rightarrow 3} f(x) = 0$$

The function is not continuous at this point. This kind of discontinuity is called a **removable discontinuity**. Removable discontinuities are those where there is a hole in the graph as there is in this case.

From this example we can get a quick “working” definition of continuity. A function is continuous on an interval if we can draw the graph from start to finish without ever once picking up our pencil. The graph in the last example has only two discontinuities since there are only two places where we would have to pick up our pencil in sketching it.

In other words, a function is continuous if its graph has no holes or breaks in it.

For many functions it’s easy to determine where it won’t be continuous. Functions won’t be continuous where we have things like division by zero or logarithms of zero. Let’s take a quick look at an example of determining where a function is not continuous.

Example 2

Determine where the function below is not continuous.

$$h(t) = \frac{4t + 10}{t^2 - 2t - 15}$$

Solution

Rational functions are continuous everywhere except where we have division by zero. So all that we need to do is determine where the denominator is zero. That’s easy enough to determine by setting the denominator equal to zero and solving.

$$t^2 - 2t - 15 = (t - 5)(t + 3) = 0$$

So, the function will not be continuous at $t = -3$ and $t = 5$.

A nice consequence of continuity is the following fact.

Fact 2

If $f(x)$ is continuous at $x = b$ and $\lim_{x \rightarrow a} g(x) = b$ then,

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$$

To see a proof of this fact see the [Proof of Various Limit Properties](#) section in the Extras appendix. With this fact we can now do limits like the following example.

Example 3

Evaluate the following limit.

$$\lim_{x \rightarrow 0} e^{\sin x}$$

Solution

Since we know that exponentials are continuous everywhere we can use the fact above.

$$\lim_{x \rightarrow 0} e^{\sin x} = e^{\lim_{x \rightarrow 0} \sin x} = e^0 = 1$$

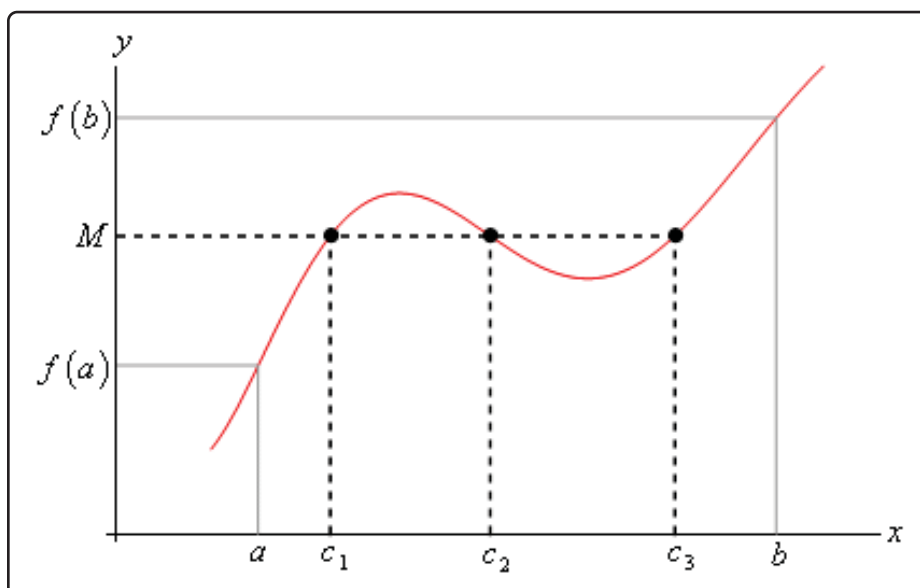
Another very nice consequence of continuity is the Intermediate Value Theorem.

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that,

1. $a < c < b$
2. $f(c) = M$

All the Intermediate Value Theorem is really saying is that a continuous function will take on all values between $f(a)$ and $f(b)$. Below is a graph of a continuous function that illustrates the Intermediate Value Theorem.



As we can see from this image if we pick any value, M , that is between the value of $f(a)$ and the value of $f(b)$ and draw a line straight out from this point the line will hit the graph in at least one point. In other words, somewhere between a and b the function will take on the value of M . Also, as the figure shows the function may take on the value at more than one place.

It's also important to note that the Intermediate Value Theorem only says that the function will take on the value of M somewhere between a and b . It doesn't say just what that value will be. It only says that it exists.

So, the Intermediate Value Theorem tells us that a function will take the value of M somewhere between a and b but it doesn't tell us where it will take the value nor does it tell us how many times it will take the value. These are important ideas to remember about the Intermediate Value Theorem.

A nice use of the Intermediate Value Theorem is to prove the existence of roots of equations as the following example shows.

Example 4

Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval $[-1, 2]$.

Solution

What we're really asking here is whether or not the function will take on the value

$$p(x) = 0$$

somewhere between -1 and 2 . In other words, we want to show that there is a number c such that $-1 < c < 2$ and $p(c) = 0$. However if we define $M = 0$ and acknowledge that $a = -1$ and $b = 2$ we can see that these two condition on c are exactly the conclusions of the Intermediate Value Theorem.

So, this problem is set up to use the Intermediate Value Theorem and in fact, all we need to do is to show that the function is continuous and that $M = 0$ is between $p(-1)$ and $p(2)$ (i.e. $p(-1) < 0 < p(2)$ or $p(2) < 0 < p(-1)$ and we'll be done.

To do this all we need to do is compute,

$$p(-1) = 8 \qquad p(2) = -19$$

So, we have,

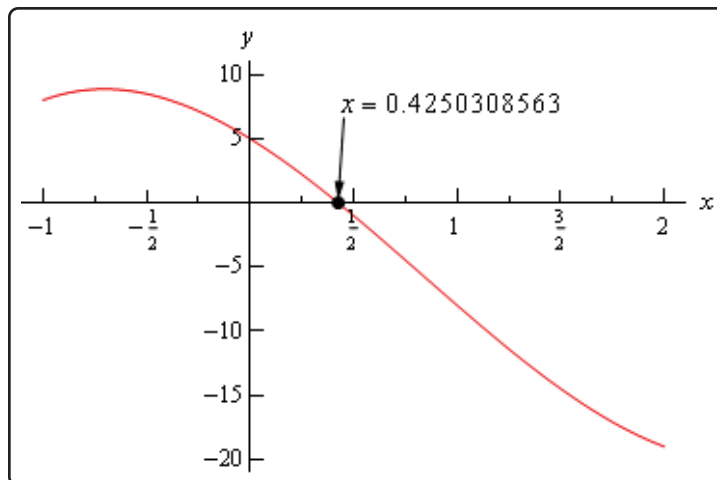
$$-19 = p(2) < 0 < p(-1) = 8$$

Therefore $M = 0$ is between $p(-1)$ and $p(2)$ and since $p(x)$ is a polynomial it's continuous everywhere and so in particular it's continuous on the interval $[-1, 2]$. So by the Intermediate Value Theorem there must be a number $-1 < c < 2$ so that,

$$p(c) = 0$$

Therefore, the polynomial does have a root between -1 and 2 .

For the sake of completeness here is a graph showing the root that we just proved existed. Note that we used a computer program to actually find the root and that the Intermediate Value Theorem did not tell us what this value was.



Let's take a look at another example of the Intermediate Value Theorem.

Example 5

If possible, determine if $f(x) = 20 \sin(x + 3) \cos\left(\frac{x^2}{2}\right)$ takes the following values in the interval $[0, 5]$.

- (a) Does $f(x) = 10$?
- (b) Does $f(x) = -10$?

Solution

Okay, so as with the previous example, we're being asked to determine, if possible, if the function takes on either of the two values above in the interval $[0, 5]$. First, let's notice that this is a continuous function and so we know that we can use the Intermediate Value Theorem to do this problem.

Now, for each part we will let M be the given value for that part and then we'll need to show that M lives between $f(0)$ and $f(5)$. If it does, then we can use the Intermediate Value Theorem to prove that the function will take the given value.

So, since we'll need the two function evaluations for each part let's give them here,

$$f(0) = 2.8224 \qquad f(5) = 19.7436$$

Now, let's take a look at each part.

(a) Does $f(x) = 10$?

Okay, in this case we'll define $M = 10$ and we can see that,

$$f(0) = 2.8224 < 10 < 19.7436 = f(5)$$

So, by the Intermediate Value Theorem there must be a number $0 \leq c \leq 5$ such that

$$f(c) = 10$$

(b) Does $f(x) = -10$?

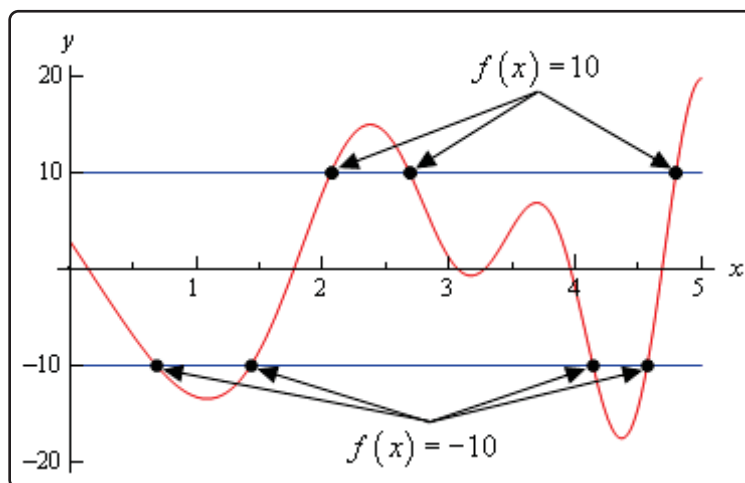
In this part we'll define $M = -10$. We now have a problem. In this part M does not live between $f(0)$ and $f(5)$. So, what does this mean for us? Does this mean that $f(x) \neq -10$ in $[0, 5]$?

Unfortunately for us, this doesn't mean anything. It is possible that $f(x) \neq -10$ in $[0, 5]$, but is it also possible that $f(x) = -10$ in $[0, 5]$. The Intermediate Value Theorem will only tell us that c 's will exist. The theorem will NOT tell us that c 's don't exist.

In this case it is not possible to determine if $f(x) = -10$ in $[0, 5]$ using the Intermediate Value Theorem.

Okay, as the previous example has shown, the Intermediate Value Theorem will not always be able to tell us what we want to know. Sometimes we can use it to verify that a function will take some value in a given interval and in other cases we won't be able to use it.

For completeness sake here is the graph of $f(x) = 20 \sin(x + 3) \cos\left(\frac{x^2}{2}\right)$ in the interval $[0, 5]$.



From this graph we can see that not only does $f(x) = -10$ in $[0, 5]$ it does so a total of 4 times!

Also note that as we verified in the first part of the previous example $f(x) = 10$ in $[0, 5]$ and in fact it does so a total of 3 times.

So, remember that the Intermediate Value Theorem will only verify that a function will take on a given value. It will never exclude a value from being taken by the function. Also, if we can use the Intermediate Value Theorem to verify that a function will take on a value it never tells us how many times the function will take on the value, it only tells us that it does take the value.