6 Applications of Integrals

The previous chapter dealt exclusively with the computation of definite and indefinite integrals as well as some discussion of their properties and interpretations. It is now time to start looking at some applications of integrals. Note as well that we should probably say applications of **definite** integrals as that is really what we'll be looking at in this section.

In addition, we should note that there are a lot of different applications of (definite) integrals out there. We will look at the ones that can easily be done with the knowledge we have at our disposal at this point. Once we have covered the next chapter, Integration Techniques, we will be able to take a look at a few more applications of integrals. At this point we would not be able to compute many of the integrals that arise in those later applications.

In this chapter we'll take a look at using integrals to compute the average value of a function and the work required to move an object over a given distance. In addition we will take a look at a couple of geometric applications of integrals. In particular we will use integrals to compute the area that is between two curves and note that this application should not be too surprising given one of the major interpretations of the definite integral. We will also see how to compute the volume of some solids. We will compute the volume of solids of revolution, *i.e.* a solid obtained by rotating a curve about a given axis. In addition, we will compute the volume of some slightly more general solids in which the cross sections can be easily described with nice 2D geometric formulas (*i.e.* rectangles, triangles, circles, *etc.*).

6.1 Average Function Value

The first application of integrals that we'll take a look at is the average value of a function. The following fact tells us how to compute this.

Average Function Value

The average value of a continuous function $f(x)$ over the interval $[a, b]$ is given by,

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

To see a justification of this formula see the Proof of Various Integral Properties section of the Extras appendix.

Let's work a couple of quick examples.

Example 1

Determine the average value of each of the following functions on the given interval.

(a) $f(t) = t^2 - 5t + 6\cos(\pi t)$ on $\left[-1, \frac{5}{2}\right]$

(b) $R(z) = \sin(2z)\,\mathbf{e}^{1-\cos(2z)}$ on $[-\pi, \pi]$

Solution

(a) $f(t) = t^2 - 5t + 6\cos(\pi t)$ on $\left[-1, \frac{5}{2}\right]$

There's really not a whole lot to do in this problem other than just use the formula.

$$\begin{aligned} f_{avg} &= \frac{1}{\frac{5}{2}-(-1)}\int_{-1}^{\frac{5}{2}} t^2 - 5t + 6\cos(\pi t)\,dt \\ &= \frac{2}{7}\left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + \frac{6}{\pi}\sin(\pi t)\right)\Bigg|_{-1}^{\frac{5}{2}} \\ &= \frac{12}{7\pi} - \frac{13}{6} \\ &= -1.620993 \end{aligned}$$

You caught the substitution needed for the third term right?

So, the average value of this function of the given interval is -1.620993.

(b) $R(z) = \sin(2z)\,\mathbf{e}^{1-\cos(2z)}$ on $[-\pi, \pi]$

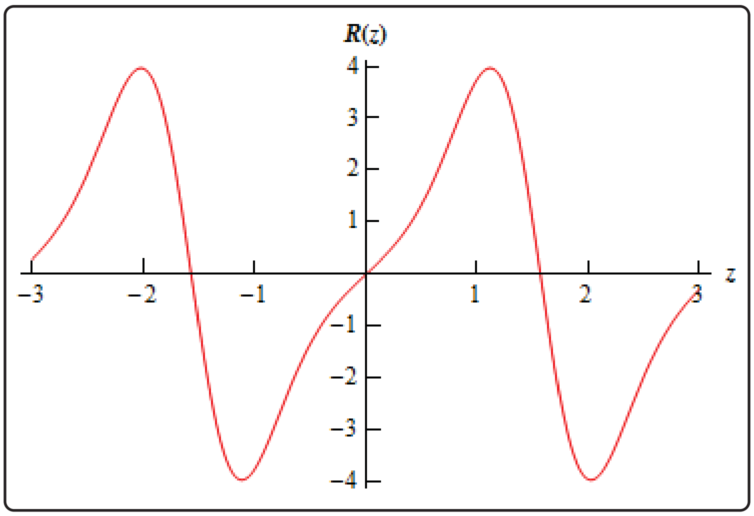
Again, not much to do here other than use the formula. Note that the integral will need the following substitution.

$$u = 1 - \cos(2z)$$

Here is the average value of this function,

$$\begin{aligned} R_{avg} &= \frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} \sin(2z)\,\mathbf{e}^{1-\cos(2z)}\,dz \\ &= \frac{1}{4\pi}\mathbf{e}^{1-\cos(2z)}\Big|_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

So, in this case the average function value is zero. Do not get excited about getting zero here. It will happen on occasion. In fact, if you look at the graph of the function on this interval it's not too hard to see that this is the correct answer.



There is also a theorem that is related to the average function value.

The Mean Value Theorem for Integrals

If $f(x)$ is a continuous function on $[a, b]$ then there is a number c in $[a, b]$ such that,

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

Note that this is very similar to the Mean Value Theorem that we saw in the Derivatives Applications chapter. See the Proof of Various Integral Properties section of the Extras appendix for the proof.

Note that one way to think of this theorem is the following. First rewrite the result as,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

and from this we can see that this theorem is telling us that there is a number $a < c < b$ such that $f_{avg} = f(c)$. Or, in other words, if $f(x)$ is a continuous function then somewhere in $[a, b]$ the function will take on its average value.

Let's take a quick look at an example using this theorem.

Example 2

Determine the number c that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^2 + 3x + 2$ on the interval $[1, 4]$.

Solution

First let's notice that the function is a polynomial and so is continuous on the given interval. This means that we can use the Mean Value Theorem. So, let's do that.

$$\begin{aligned}
\int_1^4 x^2 + 3x + 2\,dx &= \left(c^2 + 3c + 2\right)(4-1)\\
\left.\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right)\right|_1^4 &= 3\left(c^2 + 3c + 2\right)\\
\frac{99}{2} &= 3c^2 + 9c + 6\\
0 &= 3c^2 + 9c - \frac{87}{2}
\end{aligned}$$

This is a quadratic equation that we can solve. Using the quadratic formula we get the following two solutions,

$$\begin{aligned}
c &= \frac{-3+\sqrt{67}}{2} = 2.593\\
c &= \frac{-3-\sqrt{67}}{2} = -5.593
\end{aligned}$$

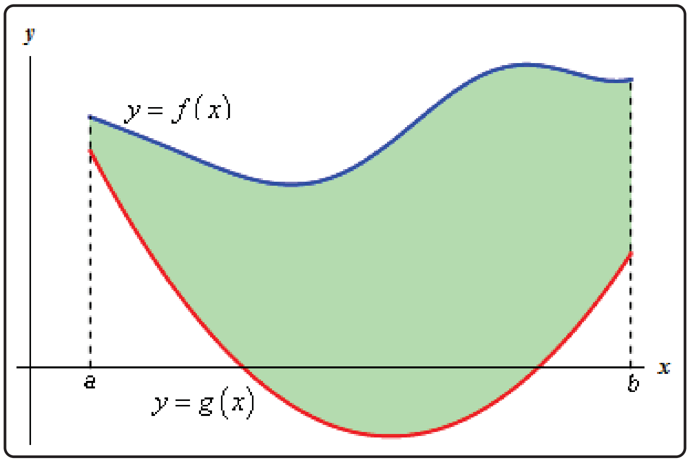
Clearly the second number is not in the interval and so that isn't the one that we're after. The first however is in the interval and so that's the number we want.

Note that it is possible for both numbers to be in the interval so don't expect only one to be in the interval.

6.2 Area Between Curves

In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

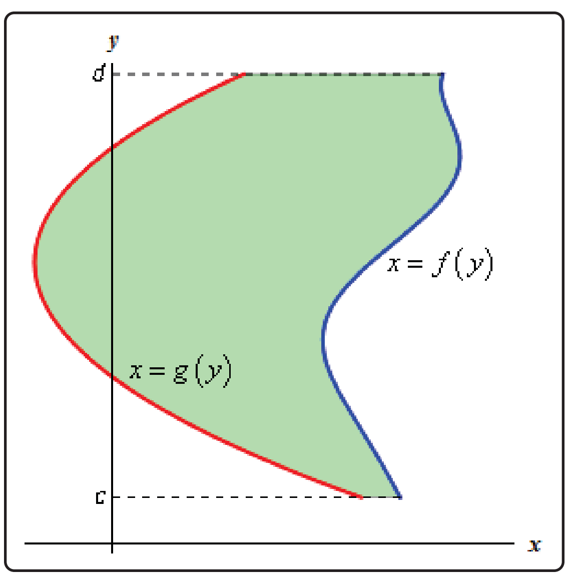
In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



In the Area and Volume Formulas section of the Extras appendix we derived the following formula for the area in this case.

$$A = \int_a^b f(x) - g(x)\, dx \tag{6.1}$$

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$.



In this case the formula is,

$$A = \int_c^d f(y) - g(y)\, dy \tag{6.2}$$

Now Equation 6.1 and Equation 6.2 are perfectly serviceable formulas, however, it is sometimes easy to forget that these always require the first function to be the larger of the two functions. So, instead of these formulas we will instead use the following "word" formulas to make sure that we remember that the area is always the "larger" function minus the "smaller" function.

In the first case we will use,

Area Between Curves, Case 1

$$A = \int_a^b \left(\begin{array}{c}\text{upper}\\\text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\\text{function}\end{array}\right) dx, \qquad a \le x \le b \tag{6.3}$$

In the second case we will use,

Area Between Curves, Case 2

$$A = \int_c^d \left(\begin{array}{c}\text{right}\\\text{function}\end{array}\right) - \left(\begin{array}{c}\text{left}\\\text{function}\end{array}\right) dy, \qquad c \le y \le d \tag{6.4}$$

Using these formulas will always force us to think about what is going on with each problem and to make sure that we've got the correct order of functions when we go to use the formula.

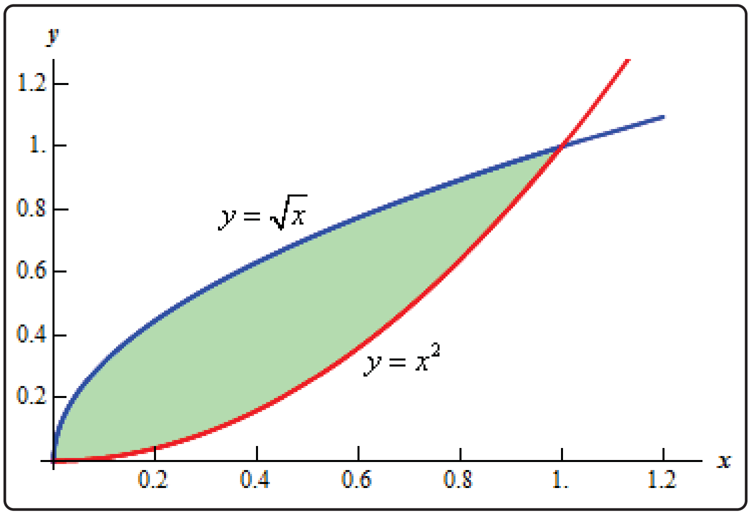
Let's work an example.

Example 1

Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

Solution

First of all, just what do we mean by "area enclosed by". This means that the region we're interested in must have one of the two curves on every boundary of the region. So, here is a graph of the two functions with the enclosed region shaded.



Note that we don't take any part of the region to the right of the rightmost intersection point of these two graphs. In this region there is no boundary on the right side and so this region is not part of the enclosed area. Remember that one of the given functions must be on the boundary of the enclosed region.

Also, from this graph it's clear that the upper function will be dependent on the range of x's that we use. Because of this you should always sketch of a graph of the region. Without a sketch it's often easy to mistake which of the two functions is the larger. In this case most would probably say that $y = x^2$ is the upper function and they would be right for the vast majority of the x's. However, in this case it is the lower of the two functions.

The limits of integration for this will be the intersection points of the two curves. In this case it's pretty easy to see that they will intersect at $x = 0$ and $x = 1$ so these are the limits of integration.

So, the integral that we'll need to compute to find the area is,

$$\begin{aligned} A &= \int_a^b \left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right) dx \\ &= \int_0^1 \sqrt{x} - x^2 \, dx \\ &= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right)\Big|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

Before moving on to the next example, there are a couple of important things to note.

First, in almost all of these problems a graph is pretty much required. Often the bounding region, which will give the limits of integration, is difficult to determine without a graph.

Also, it can often be difficult to determine which of the functions is the upper function and which is the lower function without a graph. This is especially true in cases like the last example where the answer to that question actually depended upon the range of x's that we were using.

Finally, unlike the area under a curve that we looked at in the previous chapter the area between two curves will always be positive. If we get a negative number or zero we can be sure that we've made a mistake somewhere and will need to go back and find it.

Note as well that sometimes instead of saying region enclosed by we will say region bounded by. They mean the same thing.

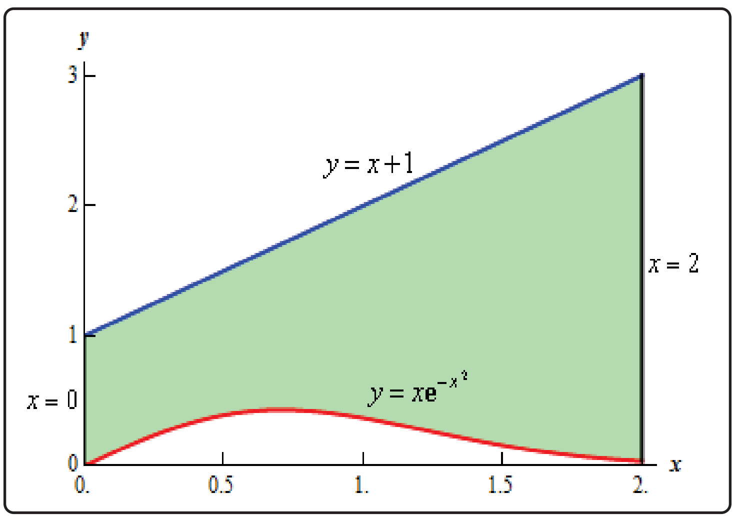
Let's work some more examples.

Example 2

Determine the area of the region bounded by $y = x\mathbf{e}^{-x^2}$, $y = x + 1$, $x = 2$, and the y-axis.

Solution

In this case the last two pieces of information, $x = 2$ and the y-axis, tell us the right and left boundaries of the region. Also, recall that the y-axis is given by the line $x = 0$. Here is the graph with the enclosed region shaded in.



Here, unlike the first example, the two curves don't meet. Instead we rely on two vertical lines to bound the left and right sides of the region as we noted above

Here is the integral that will give the area.

$$\begin{aligned} A &= \int_a^b \left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right) dx \\ &= \int_0^2 x + 1 - x\mathbf{e}^{-x^2}\, dx \\ &= \left(\frac{1}{2}x^2 + x + \frac{1}{2}\mathbf{e}^{-x^2}\right)\Bigg|_0^2 \\ &= \frac{7}{2} + \frac{\mathbf{e}^{-4}}{2} = 3.5092 \end{aligned}$$

Example 3

Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.

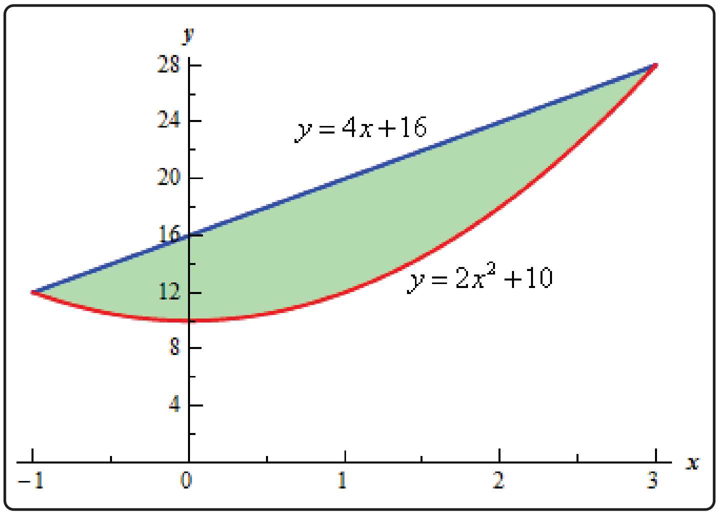
Solution

In this case the intersection points (which we'll need eventually) are not going to be easily identified from the graph so let's go ahead and get them now. Note that for most of these problems you'll not be able to accurately identify the intersection points from the graph and so you'll need to be able to determine them by hand. In this case we can get the intersection points by setting the two equations equal.

$$\begin{aligned} 2x^2 + 10 &= 4x + 16 \\ 2x^2 - 4x - 6 &= 0 \\ 2(x+1)(x-3) &= 0 \end{aligned}$$

So, it looks like the two curves will intersect at $x = -1$ and $x = 3$. If we need them we can get the y values corresponding to each of these by plugging the values back into either of the equations. We'll leave it to you to verify that the coordinates of the two intersection points on the graph are $(-1, 12)$ and $(3, 28)$.

Note as well that if you aren't good at graphing knowing the intersection points can help in at least getting the graph started. Here is a graph of the region.



With the graph we can now identify the upper and lower function and so we can now find the enclosed area.

$$\begin{aligned} A &= \int_a^b \left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right) dx \\ &= \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right)\, dx \\ &= \int_{-1}^{3} -2x^2 + 4x + 6\, dx \\ &= \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Big|_{-1}^{3} \\ &= \frac{64}{3} \end{aligned}$$

Be careful with parenthesis in these problems. One of the more common mistakes students make with these problems is to neglect parenthesis on the second term.

Example 4

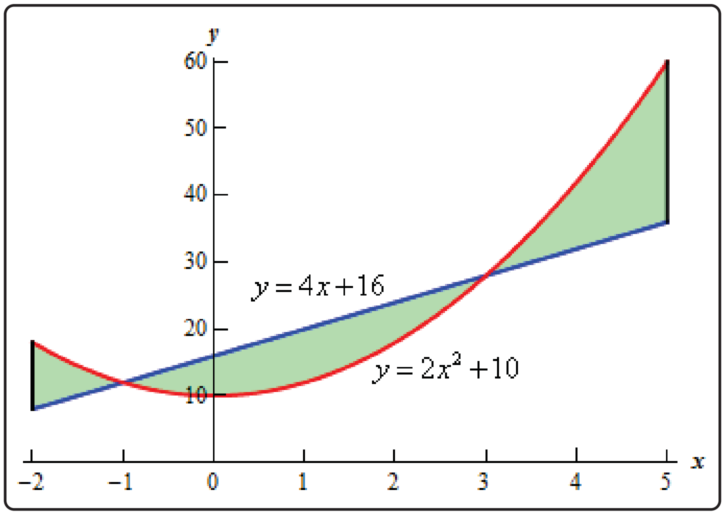
Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$.

Solution

So, the functions used in this problem are identical to the functions from the first problem. The difference is that we've extended the bounded region out from the intersection points.

Since these are the same functions we used in the previous example we won't bother finding the intersection points again.

Here is a graph of this region.



Okay, we have a small problem here. Our formula requires that one function always be the upper function and the other function always be the lower function and we clearly do not have that here. However, this actually isn't the problem that it might at first appear to be. There are three regions in which one function is always the upper function and the other is always the lower function. So, all that we need to do is find the area of each of the three regions, which we can do, and then add them all up.

Here is the area.

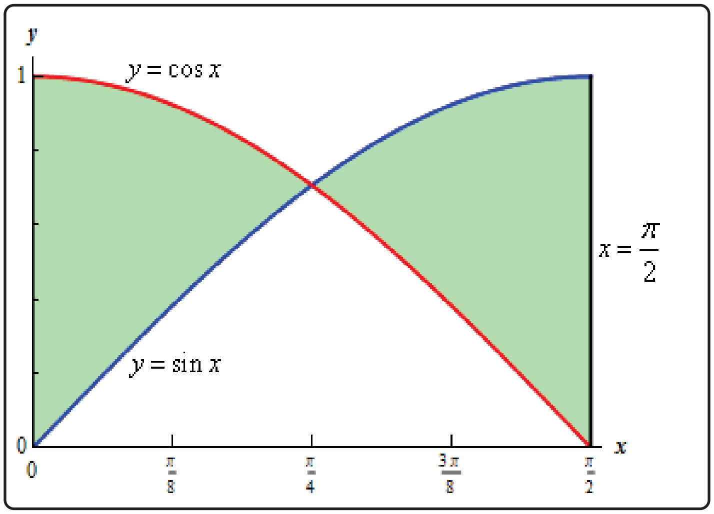
$$\begin{aligned}
A &= \int_{-2}^{-1} 2x^2 + 10 - (4x + 16)\,dx + \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right)\,dx \\
&\qquad\qquad\qquad\qquad + \int_{3}^{5} 2x^2 + 10 - (4x + 16)\,dx \\
&= \int_{-2}^{-1} 2x^2 - 4x - 6\,dx + \int_{-1}^{3} -2x^2 + 4x + 6\,dx + \int_{3}^{5} 2x^2 - 4x - 6\,dx \\
&= \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{-2}^{-1} + \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Bigg|_{-1}^{3} + \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{3}^{5} \\
&= \frac{14}{3} + \frac{64}{3} + \frac{64}{3} \\
&= \frac{142}{3}
\end{aligned}$$

Example 5

Determine the area of the region enclosed by $y = \sin(x)$, $y = \cos(x)$, $x = \frac{\pi}{2}$, and the y-axis.

Solution

First let's get a graph of the region.



So, we have another situation where we will need to do two integrals to get the area. The intersection point will be where

$$\sin(x) = \cos(x)$$

in the interval. We'll leave it to you to verify that this will be $x = \frac{\pi}{4}$. The area is then,

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x)\,dx + \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)\,dx \\ &= \left(\sin(x) + \cos(x)\right)\Big|_0^{\frac{\pi}{4}} + \left(-\cos(x) - \sin(x)\right)\Big|_{\pi/4}^{\pi/2} \\ &= \sqrt{2} - 1 + \left(\sqrt{2} - 1\right) \\ &= 2\sqrt{2} - 2 = 0.828427 \end{aligned}$$

We will need to be careful with this next example.

Example 6

Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$.

Solution

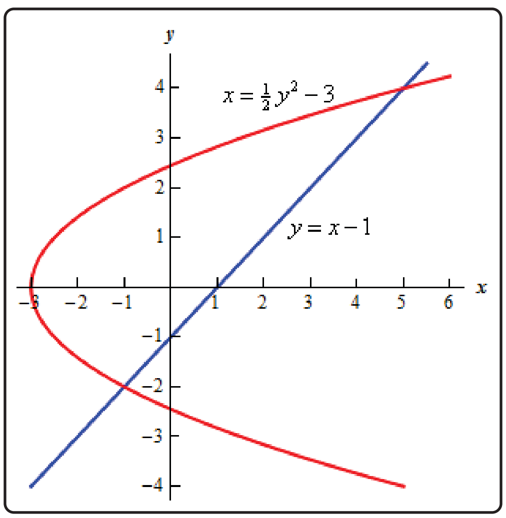
Don't let the first equation get you upset. We will have to deal with these kinds of equations occasionally so we'll need to get used to dealing with them.

As always, it will help if we have the intersection points for the two curves. In this case we'll get the intersection points by solving the second equation for x and then setting them equal. Here is that work,

$$\begin{aligned} y + 1 &= \frac{1}{2}y^2 - 3 \\ 2y + 2 &= y^2 - 6 \\ 0 &= y^2 - 2y - 8 \\ 0 &= (y - 4)(y + 2) \end{aligned}$$

So, it looks like the two curves will intersect at $y = -2$ and $y = 4$ or if we need the full coordinates they will be : $(-1, -2)$ and $(5, 4)$.

Here is a sketch of the two curves.



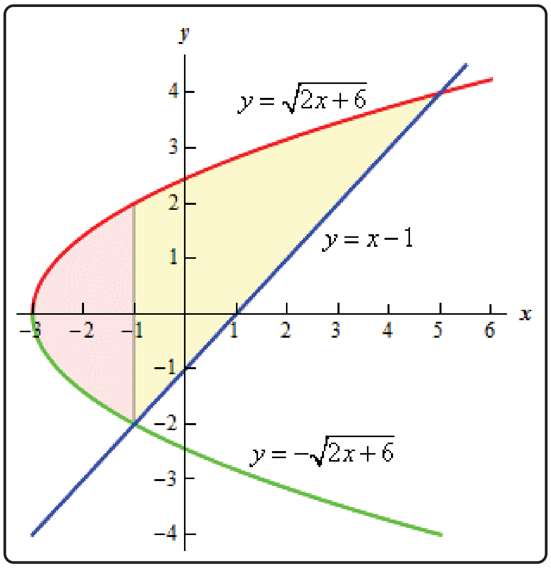
Now, we will have a serious problem at this point if we aren't careful. To this point we've been using an upper function and a lower function. To do that here notice that there are actually two portions of the region that will have different lower functions. In the range $[-3, -1]$ the parabola is actually both the upper and the lower function.

To use the formula that we've been using to this point we need to solve the parabola for y. This gives,

$$y = \pm\sqrt{2x+6}$$

where the "$+$" gives the upper portion of the parabola and the "-" gives the lower portion.

Here is a sketch of the complete area with each region shaded that we'd need if we were going to use the first formula.



The integrals for the area would then be,

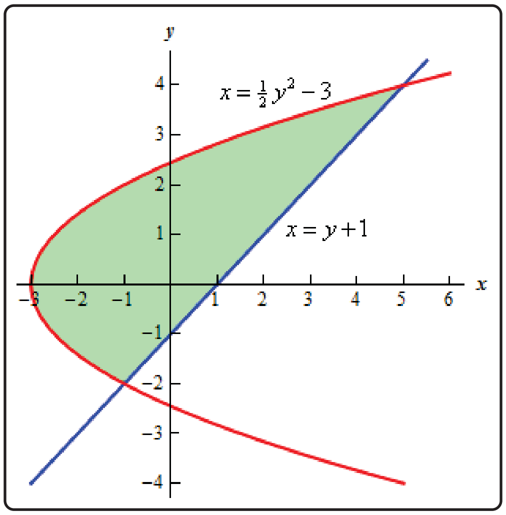
$$\begin{aligned} A &= \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right)\,dx + \int_{-1}^{5} \sqrt{2x+6} - (x-1)\,dx \\ &= \int_{-3}^{-1} 2\sqrt{2x+6}\,dx + \int_{-1}^{5} \sqrt{2x+6} - x + 1\,dx \\ &= \int_{-3}^{-1} 2\sqrt{2x+6}\,dx + \int_{-1}^{5} \sqrt{2x+6}\,dx + \int_{-1}^{5} -x + 1\,dx \\ &= \frac{2}{3}u^{\frac{3}{2}}\Big|_{0}^{4} + \frac{1}{3}u^{\frac{3}{2}}\Big|_{4}^{16} + \left(-\frac{1}{2}x^2 + x\right)\Big|_{-1}^{5} \\ &= 18 \end{aligned}$$

While these integrals aren't terribly difficult they are more difficult than they need to be.

Recall that there is another formula for determining the area. It is,

$$A = \int_{c}^{d} \left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right) dy, \qquad c \le y \le d$$

and in our case we do have one function that is always on the left and the other is always on the right. So, in this case this is definitely the way to go. Note that we will need to rewrite the equation of the line since it will need to be in the form $x = f(y)$ but that is easy enough to do. Here is the graph for using this formula.



The area is,

$$\begin{aligned} A &= \int_c^d \left(\begin{array}{c}\text{right}\\\text{function}\end{array}\right) - \left(\begin{array}{c}\text{left}\\\text{function}\end{array}\right) dy \\ &= \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy \\ &= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\, dy \\ &= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right)\Big|_{-2}^{4} \\ &= 18 \end{aligned}$$

This is the same that we got using the first formula and this was definitely easier than the first method.

So, in this last example we've seen a case where we could use either formula to find the area. However, the second was definitely easier.

Students often come into a calculus class with the idea that the only easy way to work with functions is to use them in the form $y = f(x)$. However, as we've seen in this previous example there are definitely times when it will be easier to work with functions in the form $x = f(y)$. In fact, there are going to be occasions when this will be the only way in which a problem can be worked so make sure that you can deal with functions in this form.

Let's take a look at one more example to make sure we can deal with functions in this form.

Example 7

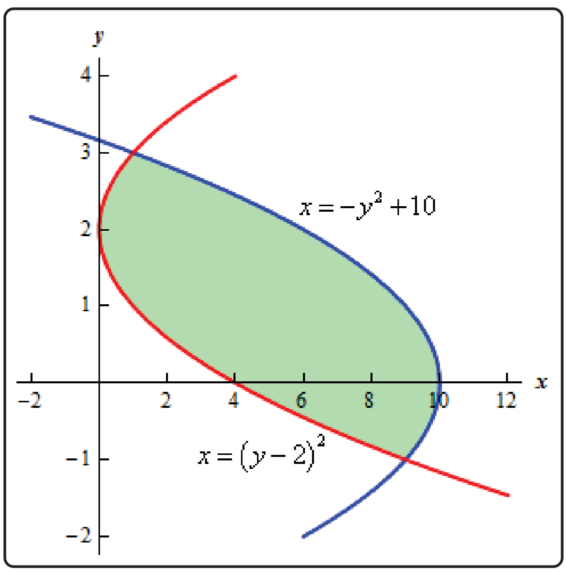
Determine the area of the region bounded by $x = -y^2 + 10$ and $x = (y-2)^2$.

Solution

First, we will need intersection points.

$$\begin{aligned} -y^2 + 10 &= (y-2)^2 \\ -y^2 + 10 &= y^2 - 4y + 4 \\ 0 &= 2y^2 - 4y - 6 \\ 0 &= 2(y+1)(y-3) \end{aligned}$$

The intersection points are $y = -1$ and $y = 3$. Here is a sketch of the region.



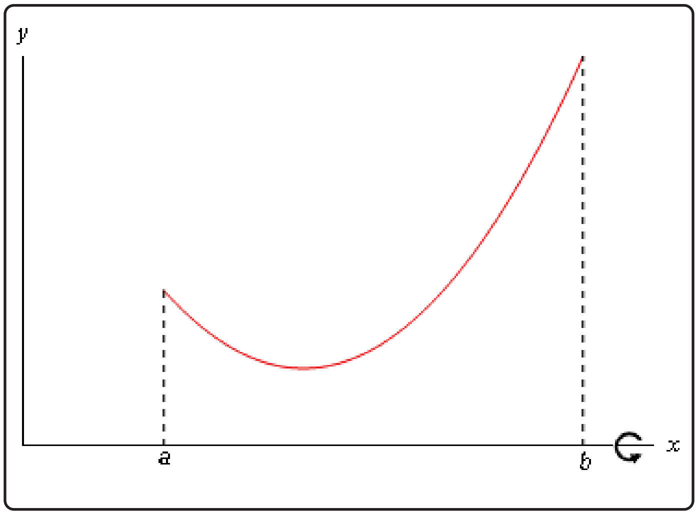
This is definitely a region where the second area formula will be easier. If we used the first formula there would be three different regions that we'd have to look at.

The area in this case is,

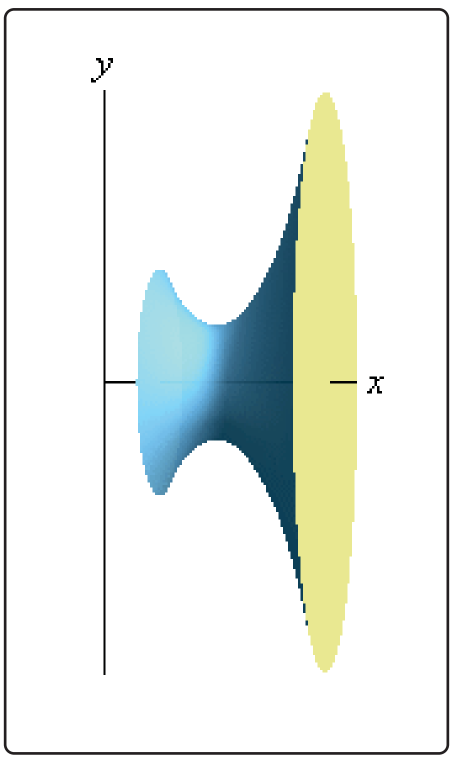
$$\begin{aligned} A &= \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy \\ &= \int_{-1}^{3} -y^2 + 10 - (y-2)^2 \, dy \\ &= \int_{-1}^{3} -2y^2 + 4y + 6 \, dy \\ &= \left(-\frac{2}{3}y^3 + 2y^2 + 6y \right) \Big|_{-1}^{3} = \frac{64}{3} \end{aligned}$$

6.3 Solids of Revolution / Method of Rings

In this section we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y = f(x)$, on an interval $[a, b]$.



We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this discussion let's rotate the curve about the x-axis, although it could be any vertical or horizontal axis. Doing this for the curve above gives the following three dimensional region.



What we want to do over the course of the next two sections is to determine the volume of this object.

In the Area and Volume Formulas section of the Extras appendix we derived the following formulas for the volume of this solid.

Volume Formulas

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

where, $A(x)$ and $A(y)$ are the cross-sectional area functions of the solid. There are many ways to get the cross-sectional area and we'll see two (or three depending on how you look at it) over the next two sections. Whether we will use $A(x)$ or $A(y)$ will depend upon the method and the axis of rotation used for each problem.

One of the easier methods for getting the cross-sectional area is to cut the object perpendicular to the axis of rotation. Doing this the cross section will be either a solid disk if the object is solid (as our above example is) or a ring if we've hollowed out a portion of the solid (we will see this eventually).

In the case that we get a solid disk the area is,

$$A = \pi(\text{radius})^2$$

where the radius will depend upon the function and the axis of rotation.

In the case that we get a ring the area is,

Area of Ring

$$A = \pi\left(\left(\begin{array}{c}\text{outer}\\ \text{radius}\end{array}\right)^2 - \left(\begin{array}{c}\text{inner}\\ \text{radius}\end{array}\right)^2\right)$$

where again both of the radii will depend on the functions given and the axis of rotation. Note as well that in the case of a solid disk we can think of the inner radius as zero and we'll arrive at the correct formula for a solid disk and so this is a much more general formula to use.

Also, in both cases, whether the area is a function of x or a function of y will depend upon the axis of rotation as we will see.

This method is often called the **method of disks** or the **method of rings**.

Let's do an example.

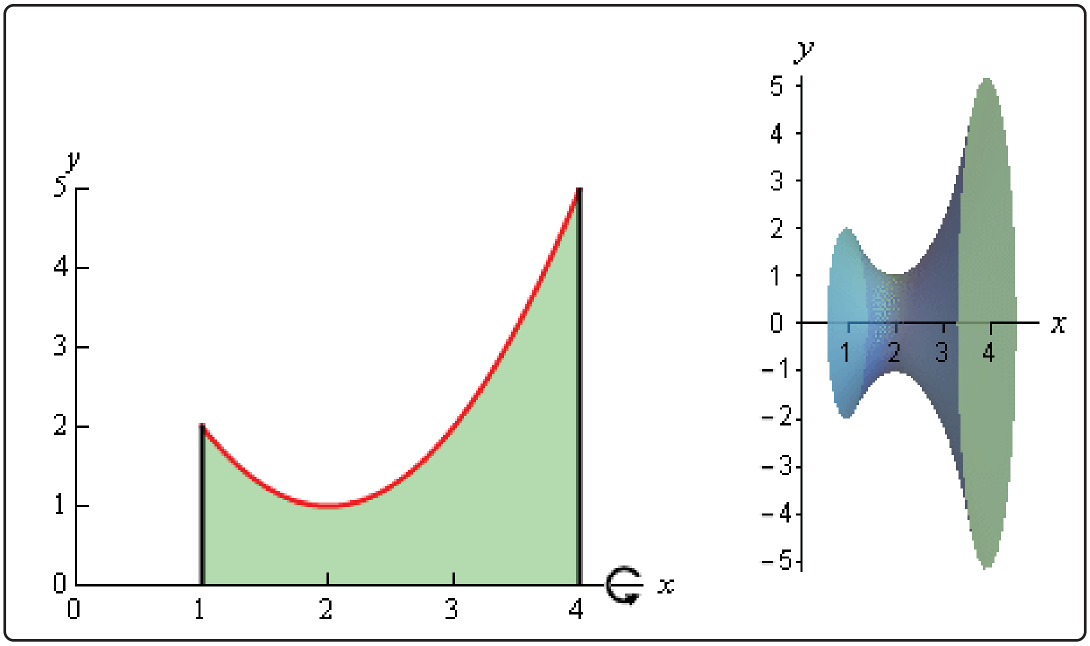
Example 1

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and the x-axis about the x-axis.

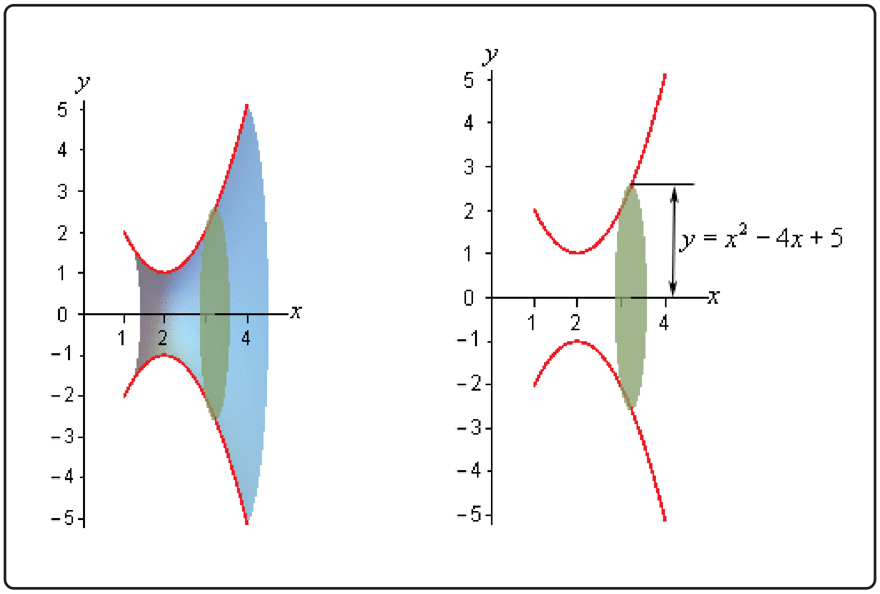
Solution

The first thing to do is get a sketch of the bounding region and the solid obtained by rotating the region about the x-axis. We don't need a picture perfect sketch of the curves we just need something that will allow us to get a feel for what the bounded region looks like so we can get a quick sketch of the solid. With that in mind we can note that the first equation is just a parabola with vertex $(2, 1)$ (you do remember how to get the vertex of a parabola right?) and opens upward and so we don't really need to put a lot of time into sketching it.

Here are both of these sketches.



Okay, to get a cross section we cut the solid at any x. Below are a couple of sketches showing a typical cross section. The sketch on the right shows a cut away of the object with a typical cross section without the caps. The sketch on the left shows just the curve we're rotating as well as its mirror image along the bottom of the solid.



In this case the radius is simply the distance from the x-axis to the curve and this is nothing more than the function value at that particular x as shown above. The cross-sectional area is then,

$$A(x) = \pi\left(x^2 - 4x + 5\right)^2 = \pi\left(x^4 - 8x^3 + 26x^2 - 40x + 25\right)$$

Next, we need to determine the limits of integration. Working from left to right the first cross section will occur at $x = 1$ and the last cross section will occur at $x = 4$. These are the limits of integration.

The volume of this solid is then,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \pi \int_1^4 x^4 - 8x^3 + 26x^2 - 40x + 25\,dx \\ &= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Bigg|_1^4 \\ &= \frac{78\pi}{5} \end{aligned}$$

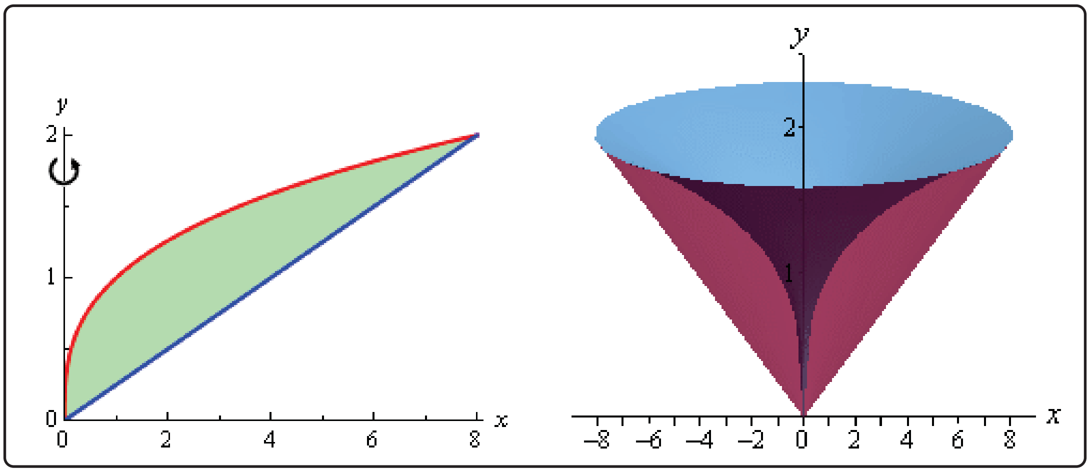
In the above example the object was a solid object, but the more interesting objects are those that are not solid so let's take a look at one of those.

Example 2

Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant about the y-axis.

Solution

First, let's get a graph of the bounding region and a graph of the object. Remember that we only want the portion of the bounding region that lies in the first quadrant. There is a portion of the bounding region that is in the third quadrant as well, but we don't want that for this problem.

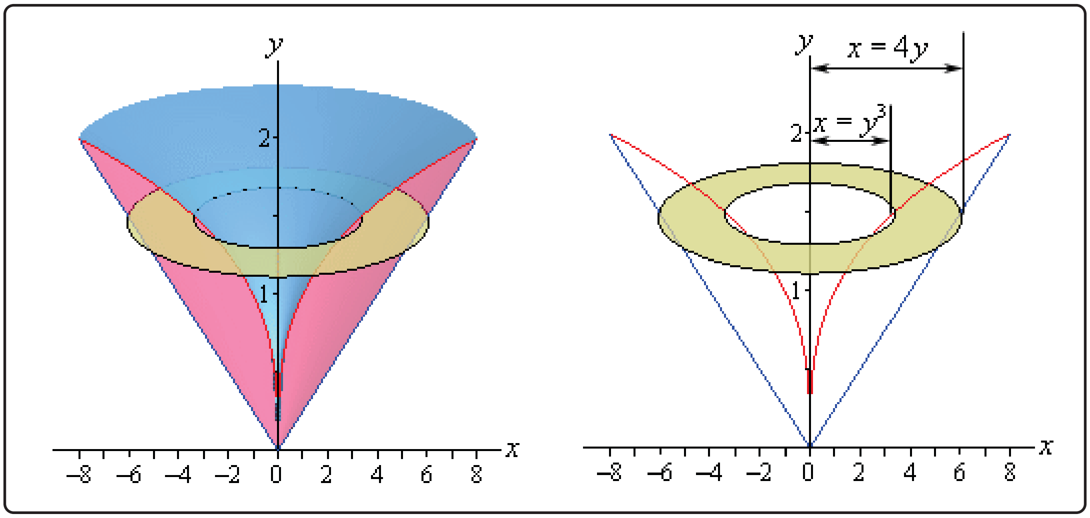


There are a couple of things to note with this problem. First, we are only looking for the volume of the "walls" of this solid, not the complete interior as we did in the last example.

Next, we will get our cross section by cutting the object perpendicular to the axis of rotation. The cross section will be a ring (remember we are only looking at the walls) for this example and it will be horizontal at some y. This means that the inner and outer radius for the ring will be x values and so we will need to rewrite our functions into the form $x = f(y)$. Here are the functions written in the correct form for this example.

$$\begin{aligned} y &= \sqrt[3]{x} & \Rightarrow \qquad & x = y^3 \\ y &= \frac{x}{4} & \Rightarrow \qquad & x = 4y \end{aligned}$$

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.



The inner radius in this case is the distance from the y-axis to the inner curve while the outer radius is the distance from the y-axis to the outer curve. Both of these are then x distances and so are given by the equations of the curves as shown above.

The cross-sectional area is then,

$$A(y) = \pi\left((4y)^2 - \left(y^3\right)^2\right) = \pi\left(16y^2 - y^6\right)$$

Working from the bottom of the solid to the top we can see that the first cross-section will occur at $y = 0$ and the last cross-section will occur at $y = 2$. These will be the limits of integration. The volume is then,

$$\begin{aligned} V &= \int_c^d A(y)\, dy \\ &= \pi\int_0^2 16y^2 - y^6\, dy \\ &= \pi\left(\frac{16}{3}y^3 - \frac{1}{7}y^7\right)\Big|_0^2 \\ &= \frac{512\pi}{21} \end{aligned}$$

With these two examples out of the way we can now make a generalization about this method. If we rotate about a horizontal axis (the x-axis for example) then the cross-sectional area will be a function of x. Likewise, if we rotate about a vertical axis (the y-axis for example) then the cross-sectional area will be a function of y.

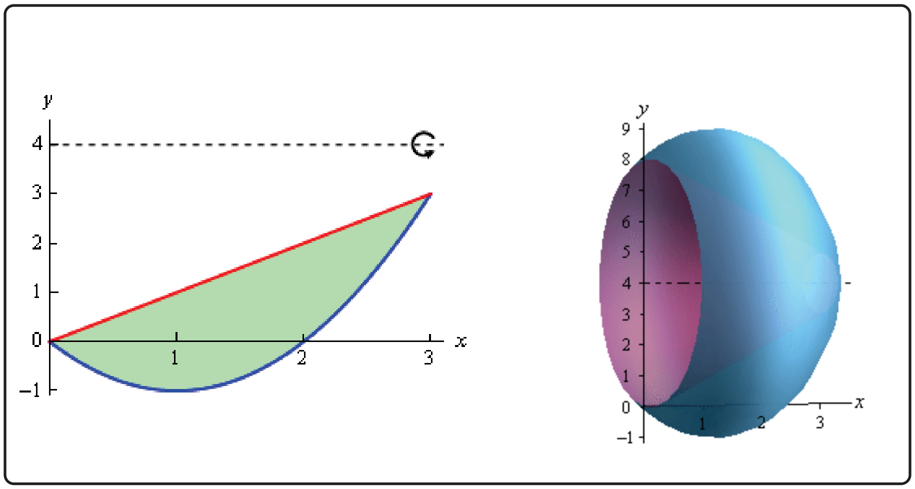
The remaining two examples in this section will make sure that we don't get too used to the idea of always rotating about the x or y-axis.

Example 3

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

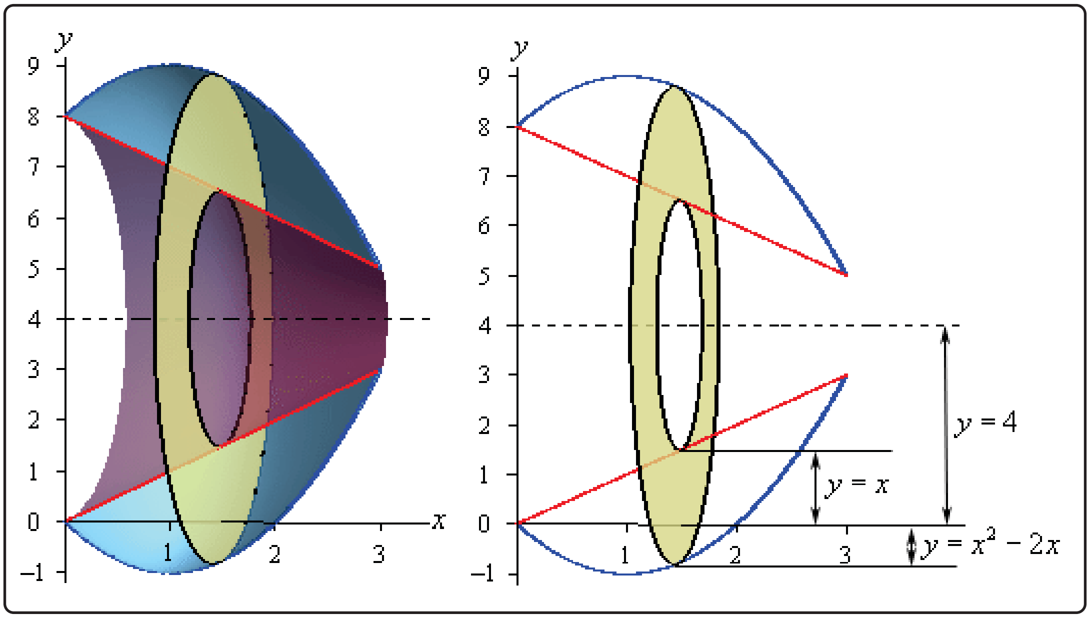
Solution

First let's get the bounding region and the solid graphed.



Again, we are going to be looking for the volume of the walls of this object. Also, since we are rotating about a horizontal axis we know that the cross-sectional area will be a function of x.

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.



Now, we're going to have to be careful here in determining the inner and outer radius as they aren't going to be quite as simple they were in the previous two examples.

Let's start with the inner radius as this one is a little clearer. First, the inner radius is NOT x. The distance from the x-axis to the inner edge of the ring is x, but we want the radius and that is the distance from the axis of rotation to the inner edge of the ring. So, we know that the distance from the axis of rotation to the x-axis is 4 and the distance from the x-axis to the inner ring is x. The inner radius must then be the difference between these two. Or,

$$\text{inner radius} = 4 - x$$

The outer radius works the same way. The outer radius is,

$$\text{outer radius} = 4 - \left(x^2 - 2x\right) = -x^2 + 2x + 4$$

Note that given the location of the typical ring in the sketch above the formula for the outer radius may not look quite right but it is in fact correct. As sketched the outer edge of the ring is below the x-axis and at this point the value of the function will be negative and so when we do the subtraction in the formula for the outer radius we'll actually be subtracting off a negative number which has the net effect of adding this distance onto 4 and that gives the correct outer radius. Likewise, if the outer edge is above the x-axis, the function value will be positive and so we'll be doing an honest subtraction here and again we'll get the correct radius in this case.

The cross-sectional area for this case is,

$$A(x) = \pi\left(\left(-x^2 + 2x + 4\right)^2 - (4 - x)^2\right) = \pi\left(x^4 - 4x^3 - 5x^2 + 24x\right)$$

The first ring will occur at $x = 0$ and the last ring will occur at $x = 3$ and so these are our limits of integration. The volume is then,

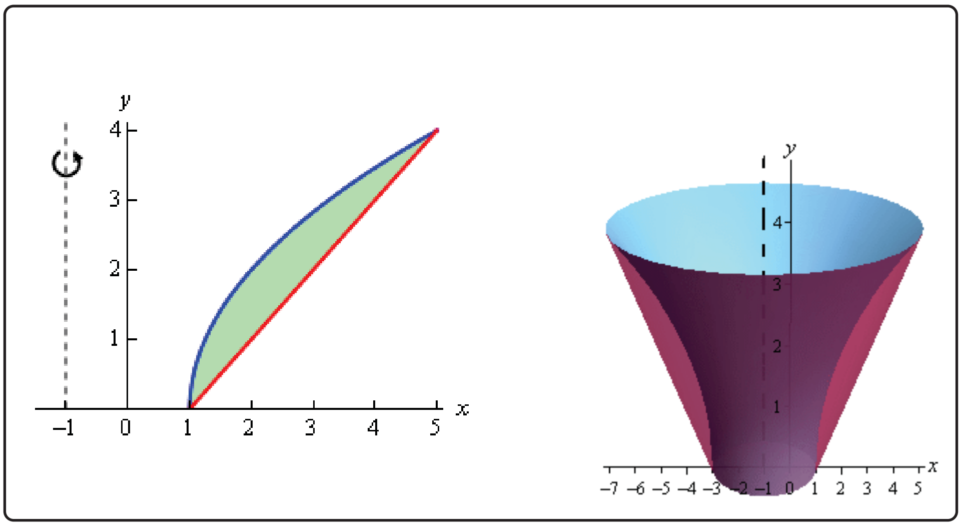
$$\begin{aligned} V &= \int_a^b A(x)\, dx \\ &= \pi \int_0^3 x^4 - 4x^3 - 5x^2 + 24x\, dx \\ &= \pi \left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2 \right) \Big|_0^3 \\ &= \frac{153\pi}{5} \end{aligned}$$

Example 4

Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = -1$.

Solution

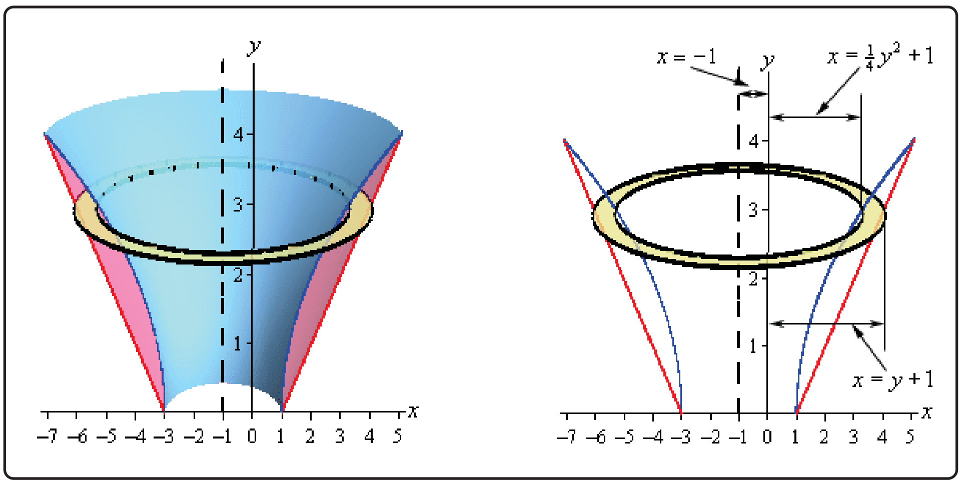
As with the previous examples, let's first graph the bounded region and the solid.



Now, let's notice that since we are rotating about a vertical axis and so the cross-sectional area will be a function of y. This also means that we are going to have to rewrite the functions to also get them in terms of y.

$$y = 2\sqrt{x-1} \qquad \Rightarrow \qquad x = \frac{y^2}{4} + 1$$
$$y = x - 1 \qquad \Rightarrow \qquad x = y + 1$$

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.



The inner and outer radius for this case is both similar and different from the previous example. This example is similar in the sense that the radii are not just the functions. In this example the functions are the distances from the y-axis to the edges of the rings. The center of the ring however is a distance of 1 from the y-axis. This means that the distance from the center to the edges is a distance from the axis of rotation to the y-axis (a distance of 1) and then from the y-axis to the edge of the rings.

So, the radii are then the functions plus 1 and that is what makes this example different from the previous example. Here we had to add the distance to the function value whereas in the previous example we needed to subtract the function from this distance. Note that without sketches the radii on these problems can be difficult to get.

So, in summary, we've got the following for the inner and outer radius for this example.

$$\text{outer radius} = y + 1 + 1 = y + 2$$
$$\text{inner radius} = \frac{y^2}{4} + 1 + 1 = \frac{y^2}{4} + 2$$

The cross-sectional area is then,

$$A\left(y\right) = \pi\left(\left(y+2\right)^2 - \left(\frac{y^2}{4}+2\right)^2\right) = \pi\left(4y - \frac{y^4}{16}\right)$$

The first ring will occur at $y = 0$ and the final ring will occur at $y = 4$ and so these will be our limits of integration.

The volume is,

$$\begin{aligned} V &= \int_c^d A\left(y\right)\,dy \\ &= \pi\int_0^4 4y - \frac{y^4}{16}\,dy \\ &= \pi\left(2y^2 - \frac{1}{80}y^5\right)\Big|_0^4 \\ &= \frac{96\pi}{5} \end{aligned}$$

6.4 Solids of Revolution / Method of Cylinders

In the previous section we started looking at finding volumes of solids of revolution. In that section we took cross sections that were rings or disks, found the cross-sectional area and then used the following formulas to find the volume of the solid.

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

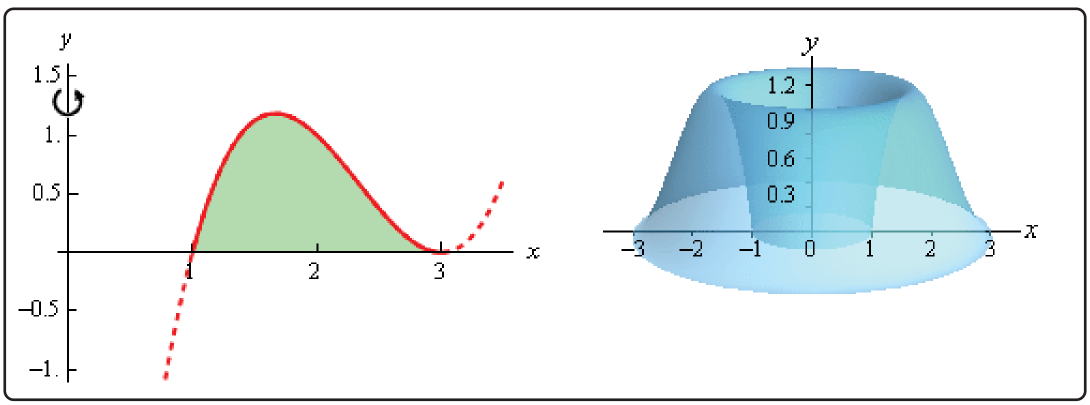
In the previous section we only used cross sections that were in the shape of a disk or a ring. This however does not always need to be the case. We can use any shape for the cross sections as long as it can be expanded or contracted to completely cover the solid we're looking at. This is a good thing because as our first example will show us we can't always use rings/disks.

Example 1

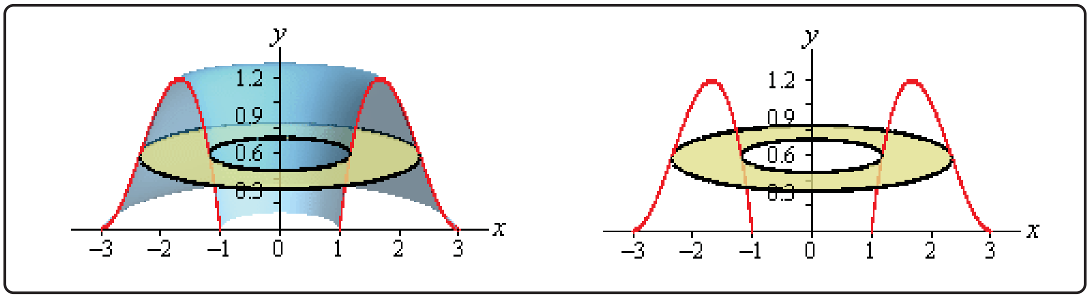
Determine the volume of the solid obtained by rotating the region bounded by $y = (x-1)(x-3)^2$ and the x-axis about the y-axis.

Solution

As we did in the previous section, let's first graph the bounded region and solid. Note that the bounded region here is the shaded portion shown. The curve is extended out a little past this for the purposes of illustrating what the curve looks like.



So, we've basically got something that's roughly doughnut shaped. If we were to use rings on this solid here is what a typical ring would look like.



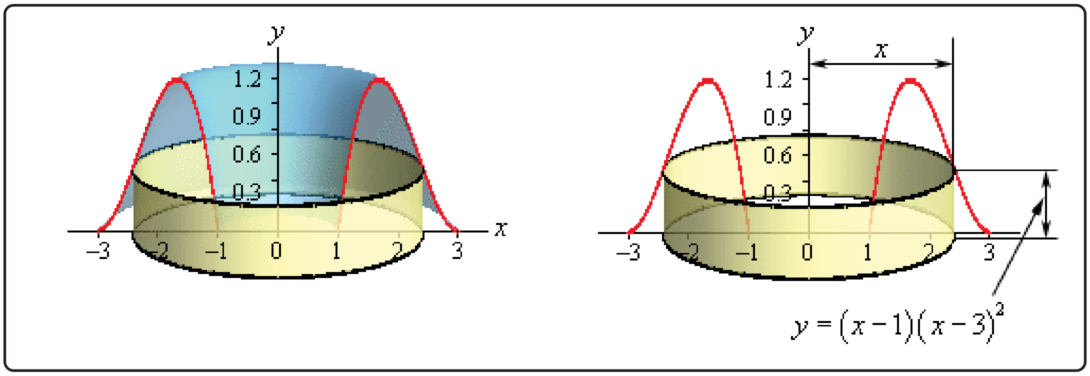
This leads to several problems. First, both the inner and outer radius are defined by the same function. This, in itself, can be dealt with on occasion as we saw in a example in the Area Between Curves section. However, this usually means more work than other methods so it's often not the best approach.

This leads to the second problem we got here. In order to use rings we would need to put this function into the form $x = f(y)$. That is NOT easy in general for a cubic polynomial and in other cases may not even be possible to do. Even when it is possible to do this the resulting equation is often significantly messier than the original which can also cause problems.

The last problem with rings in this case is not so much a problem as it's just added work. If we were to use rings the limit would be y limits and this means that we will need to know how high the graph goes. To this point the limits of integration have always been intersection points that were fairly easy to find. However, in this case the highest point is not an intersection point, but instead a relative maximum. We spent several sections in the Applications of Derivatives chapter talking about how to find maximum values of functions. However, finding them can, on occasion, take some work.

So, we've seen three problems with rings in this case that will either increase our work load or outright prevent us from using rings.

What we need to do is to find a different way to cut the solid that will give us a cross-sectional area that we can work with. One way to do this is to think of our solid as a lump of cookie dough and instead of cutting it perpendicular to the axis of rotation we could instead center a cylindrical cookie cutter on the axis of rotation and push this down into the solid. Doing this would give the following picture,



Doing this gives us a cylinder or shell in the object and we can easily find its surface area. The surface area of this cylinder is,

$$\begin{aligned} A(x) &= 2\pi\,(\text{radius})\,(\text{height}) \\ &= 2\pi\,(x)\left((x-1)(x-3)^2\right) \\ &= 2\pi\left(x^4 - 7x^3 + 15x^2 - 9x\right) \end{aligned}$$

Notice as well that as we increase the radius of the cylinder we will completely cover the solid and so we can use this in our formula to find the volume of this solid. All we need are limits of integration. The first cylinder will cut into the solid at $x = 1$ and as we increase x to $x = 3$ we will completely cover both sides of the solid since expanding the cylinder in one direction will automatically expand it in the other direction as well.

The volume of this solid is then,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= 2\pi \int_1^3 x^4 - 7x^3 + 15x^2 - 9x\,dx \\ &= 2\pi\left(\frac{1}{5}x^5 - \frac{7}{4}x^4 + 5x^3 - \frac{9}{2}x^2\right)\Bigg|_1^3 \\ &= \frac{24\pi}{5} \end{aligned}$$

The method used in the last example is called the **method of cylinders** or **method of shells**. The formula for the area in all cases will be,

Area of Cylinder

$$A = 2\pi\,(\text{radius})\,(\text{height})$$

There are a couple of important differences between this method and the method of rings/disks that we should note before moving on. First, rotation about a vertical axis will give an area that is a function of x and rotation about a horizontal axis will give an area that is a function of y. This is exactly opposite of the method of rings/disks.

Second, we don't take the complete range of x or y for the limits of integration as we did in the previous section. Instead we take a range of x or y that will cover one side of the solid. As we noted in the first example if we expand out the radius to cover one side we will automatically expand in the other direction as well to cover the other side.

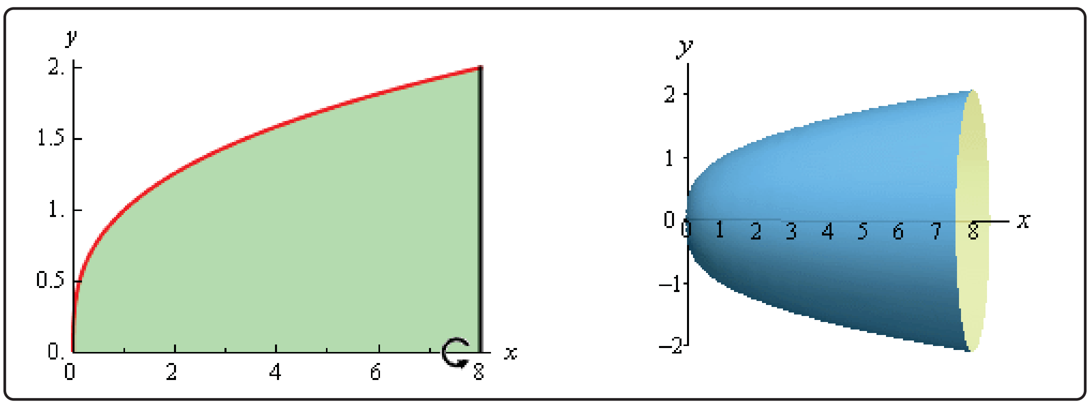
Let's take a look at another example.

Example 2

Determine the volume of the solid obtained by rotating the region bounded by $y = \sqrt[3]{x}$, $x = 8$ and the x-axis about the x-axis.

Solution

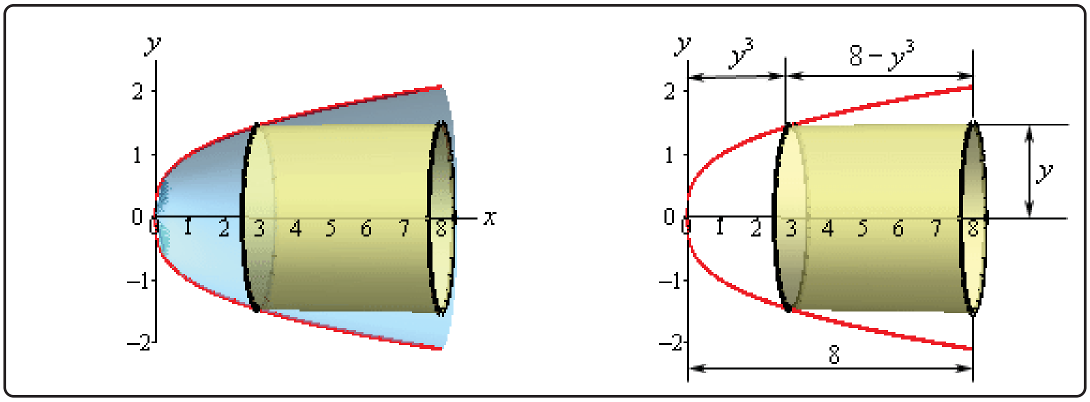
First let's get a graph of the bounded region and the solid.



Okay, we are rotating about a horizontal axis. This means that the area will be a function of y and so our equation will also need to be written in $x = f(y)$ form.

$$y = \sqrt[3]{x} \qquad \Rightarrow \qquad x = y^3$$

As we did in the ring/disk section let's take a couple of looks at a typical cylinder. The sketch on the left shows a typical cylinder with the back half of the object also in the sketch to give the right sketch some context. The sketch on the right contains a typical cylinder and only the curves that define the edge of the solid.



In this case the width of the cylinder is not the function value as it was in the previous example. In this case the function value is the distance between the edge of the cylinder and the y-axis. The distance from the edge out to the line is $x = 8$ and so the width is then $8 - y^3$. The cross-sectional area in this case is,

$$\begin{aligned} A(y) &= 2\pi\,(\text{radius})\,(\text{width}) \\ &= 2\pi\,(y)\left(8 - y^3\right) \\ &= 2\pi\left(8y - y^4\right) \end{aligned}$$

The first cylinder will cut into the solid at $y = 0$ and the final cylinder will cut in at $y = 2$ and so these are our limits of integration.

The volume of this solid is,

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= 2\pi\int_0^2 8y - y^4\,dy \\ &= 2\pi\left(4y^2 - \frac{1}{5}y^5\right)\Bigg|_0^2 \\ &= \frac{96\pi}{5} \end{aligned}$$

The remaining examples in this section will have axis of rotation about axis other than the x and

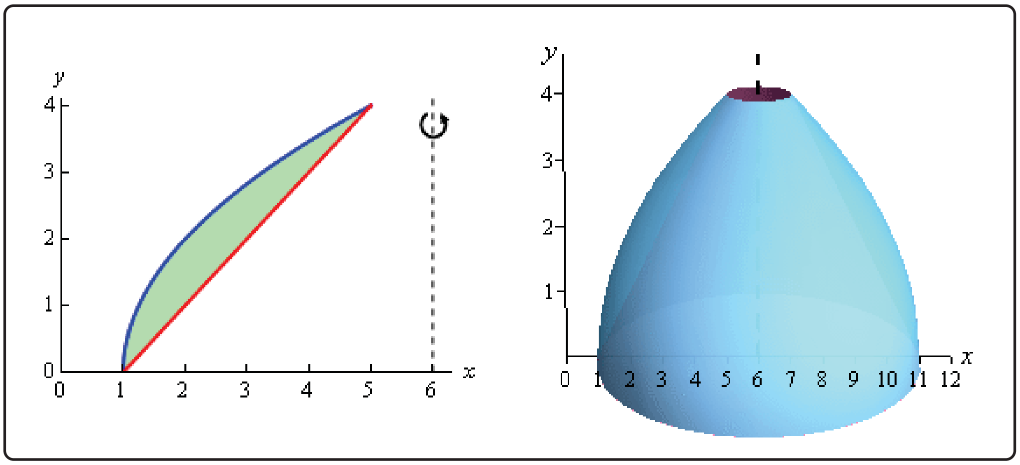
y-axis. As with the method of rings/disks we will need to be a little careful with these.

Example 3

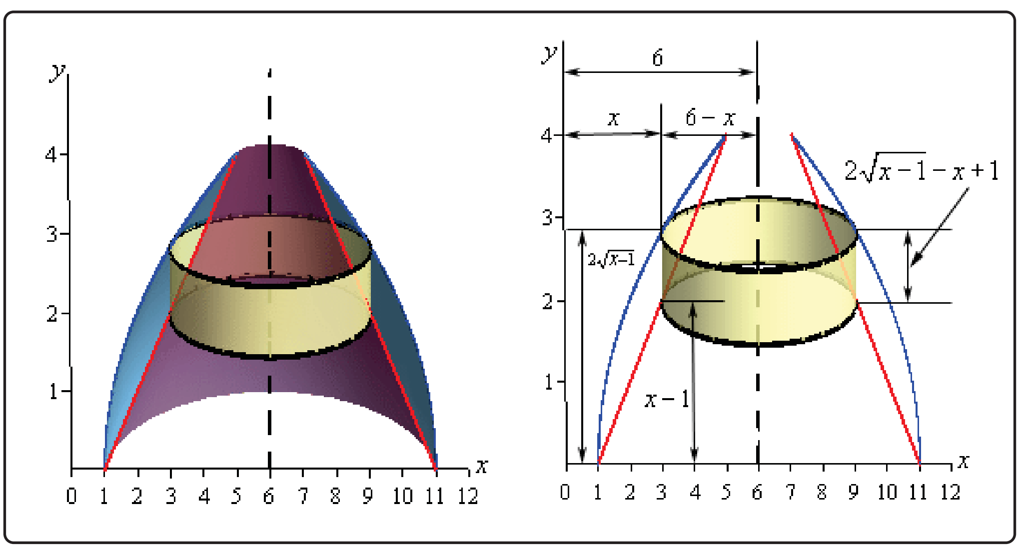
Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = 6$.

Solution

Here's a graph of the bounded region and solid.



Here are our sketches of a typical cylinder. Again, the sketch on the left is here to provide some context for the sketch on the right.



Okay, there is a lot going on in the sketch to the left. First notice that the radius is not just an x or y as it was in the previous two cases. In this case x is the distance from the *y*-axis to the edge of the cylinder and we need the distance from the axis of rotation to the edge of the cylinder. That means that the radius of this cylinder is $6 - x$.

Secondly, the height of the cylinder is the difference of the two functions in this case.

The cross-sectional area is then,

$$\begin{aligned} A(x) &= 2\pi\,(\text{radius})\,(\text{height}) \\ &= 2\pi\,(6-x)\left(2\sqrt{x-1}-x+1\right) \\ &= 2\pi\left(x^2-7x+6+12\sqrt{x-1}-2x\sqrt{x-1}\right) \end{aligned}$$

Now the first cylinder will cut into the solid at $x = 1$ and the final cylinder will cut into the solid at $x = 5$ so there are our limits.

Here is the volume.

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= 2\pi\int_1^5 x^2-7x+6+12\sqrt{x-1}-2x\sqrt{x-1}\,dx \\ &= 2\pi\left(\frac{1}{3}x^3-\frac{7}{2}x^2+6x+8(x-1)^{\frac{3}{2}}-\frac{4}{3}(x-1)^{\frac{3}{2}}-\frac{4}{5}(x-1)^{\frac{5}{2}}\right)\Bigg|_1^5 \\ &= 2\pi\left(\frac{136}{15}\right) \\ &= \frac{272\pi}{15} \end{aligned}$$

The integration of the last term is a little tricky so let's do that here. It will use the substitution,

$$u = x-1 \qquad du = dx \qquad x = u+1$$

$$\begin{aligned} \int 2x\sqrt{x-1}\,dx &= 2\int (u+1)\,u^{\frac{1}{2}}\,du \\ &= 2\int u^{\frac{3}{2}}+u^{\frac{1}{2}}\,du \\ &= 2\left(\frac{2}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}\right)+c \\ &= \frac{4}{5}(x-1)^{\frac{5}{2}}+\frac{4}{3}(x-1)^{\frac{3}{2}}+c \end{aligned}$$

We saw one of these kinds of substitutions back in the substitution section.

Example 4

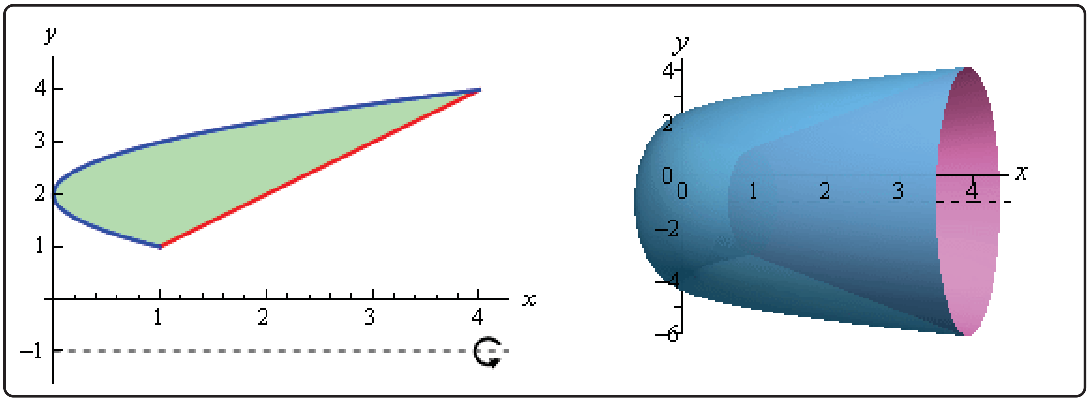
Determine the volume of the solid obtained by rotating the region bounded by $x = (y-2)^2$ and $y = x$ about the line $y = -1$.

Solution

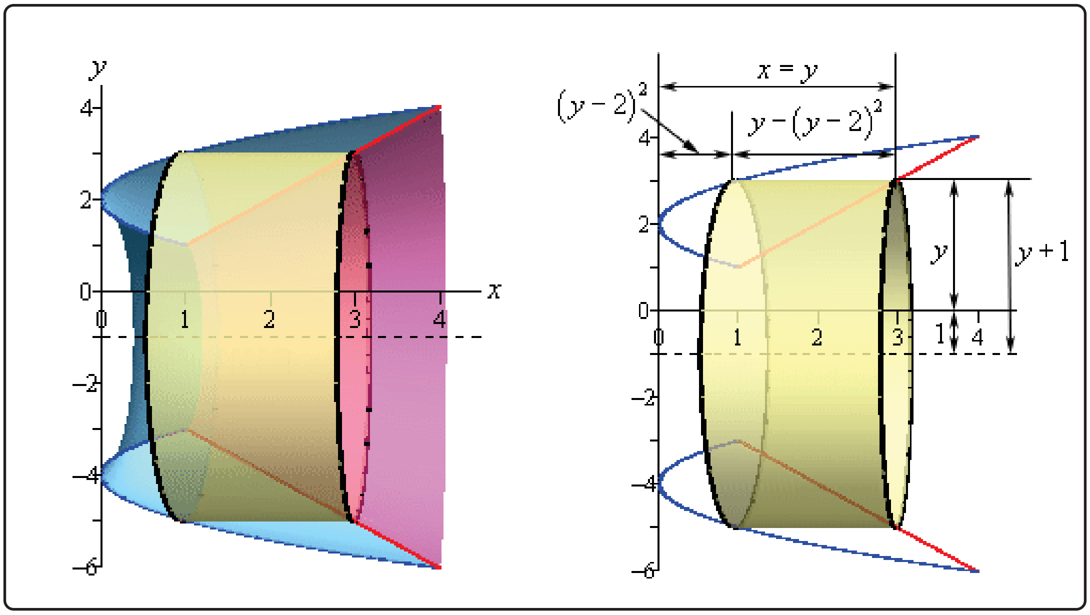
We should first get the intersection points there.

$$\begin{aligned} y &= (y-2)^2 \\ y &= y^2 - 4y + 4 \\ 0 &= y^2 - 5y + 4 \\ 0 &= (y-4)(y-1) \end{aligned}$$

So, the two curves will intersect at $y = 1$ and $y = 4$. Here is a sketch of the bounded region and the solid.



Here are our sketches of a typical cylinder. The sketch on the left is here to provide some context for the sketch on the right.



Here's the cross-sectional area for this cylinder.

$$\begin{aligned} A(y) &= 2\pi\,(\text{radius})\,(\text{width}) \\ &= 2\pi\,(y+1)\left(y-(y-2)^2\right) \\ &= 2\pi\left(-y^3+4y^2+y-4\right) \end{aligned}$$

The first cylinder will cut into the solid at $y = 1$ and the final cylinder will cut in at $y = 4$. The volume is then,

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= 2\pi\int_1^4 -y^3+4y^2+y-4\,dy \\ &= 2\pi\left(-\frac{1}{4}y^4+\frac{4}{3}y^3+\frac{1}{2}y^2-4y\right)\Bigg|_1^4 \\ &= \frac{63\pi}{2} \end{aligned}$$

6.5 More Volume Problems

In this section we're going to take a look at some more volume problems. However, the problems we'll be looking at here will not be solids of revolution as we looked at in the previous two sections. There are many solids out there that cannot be generated as solids of revolution, or at least not easily and so we need to take a look at how to do some of these problems.

Now, having said that these will not be solids of revolutions they will still be worked in pretty much the same manner. For each solid we'll need to determine the cross-sectional area, either $A\left(x\right)$ or $A\left(y\right)$, and they use the formulas we used in the previous two sections,

Volume Formulas

$$V = \int_{a}^{b}{{A\left(x \right)\,dx}} \hspace{1in} V = \int_{c}^{d}{{A\left(y \right)\,dy}}$$

The "hard" part of these problems will be determining what the cross-sectional area for each solid is. Each problem will be different and so each cross-sectional area will be determined by different means.

Also, before we proceed with any examples we need to acknowledge that the integrals in this section might look a little tricky at first. There are going to be very few numbers in these problems. All of the examples in this section are going to be more general derivation of volume formulas for certain solids. As such we'll be working with things like circles of radius r and we'll not be giving a specific value of r and we'll have heights of h instead of specific heights, *etc*.

All the letters in the integrals are going to make the integrals look a little tricky, but all you have to remember is that the r's and the h's are just letters being used to represent a fixed quantity for the problem, *i.e.* it is a constant. So, when we integrate we only need to worry about the letter in the differential as that is the variable we're actually integrating with respect to. All other letters in the integral should be thought of as constants. If you have trouble doing that, just think about what you'd do if the r was a 2 or the h was a 3 for example.

Let's start with a simple example that we don't actually need to do an integral that will illustrate how these problems work in general and will get us used to seeing multiple letters in integrals.

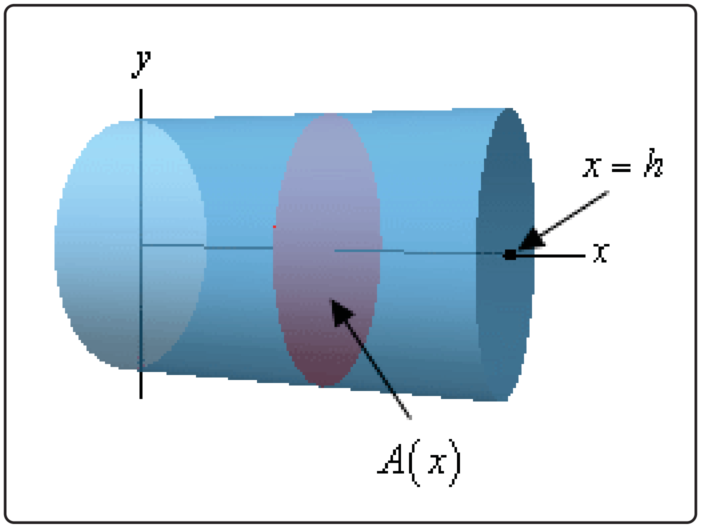
Example 1

Find the volume of a cylinder of radius r and height h.

Solution

Now, as we mentioned before starting this example we really don't need to use an integral to find this volume, but it is a good example to illustrate the method we'll need to use for these types of problems.

We'll start off with the sketch of the cylinder below.



We'll center the cylinder on the x-axis and the cylinder will start at $x = 0$ and end at $x = h$ as shown. Note that we're only choosing this particular set up to get an integral in terms of x and to make the limits nice to deal with. There are many other orientations that we could use.

What we need here is to get a formula for the cross-sectional area at any x. In this case the cross-sectional area is constant and will be a disk of radius r. Therefore, for any x we'll have the following cross-sectional area,

$$A(x) = \pi r^2$$

Next the limits for the integral will be $0 \le x \le h$ since that is the range of x in which the cylinder lives. Here is the integral for the volume,

$$V = \int_0^h \pi r^2 \, dx = \pi r^2 \int_0^h dx = \pi r^2 x\Big|_0^h = \pi r^2 h$$

So, we get the expected formula.

Also, recall we are using r to represent the radius of the cylinder. While r can clearly take different values it will never change once we start the problem. Cylinders do not change their radius in the middle of a problem and so as we move along the center of the cylinder (*i.e.* the x-axis) r is a fixed number and won't change. In other words, it is a constant that will not change as we change the x. Therefore, because we integrated with respect to x the r will be a constant as far as the integral is concerned. The r can then be pulled out of the integral as shown (although that's not required, we just did it to make the point). At this point we're just integrating dx and we know how to do that.

When we evaluate the integral remember that the limits are x values and so we plug into the x and NOT the r. Again, remember that r is just a letter that is being used to represent the radius of the cylinder and, once we start the integration, is assumed to be a fixed constant.

As noted before we started this example if you're having trouble with the r just think of what you'd do if there was a 2 there instead of an r. In this problem, because we're integrating with respect to x, both the 2 and the r will behave in the same manner. Note however that you should NEVER actually replace the r with a 2 as that WILL lead to a wrong answer. You should just think of what you would do IF the r was a 2.

So, to work these problems we'll first need to get a sketch of the solid with a set of x and y axes to help us see what's going on. At the very least we'll need the sketch to get the limits of the integral, but we will often need it to see just what the cross-sectional area is. Once we have the sketch we'll need to determine a formula for the cross-sectional area and then do the integral.

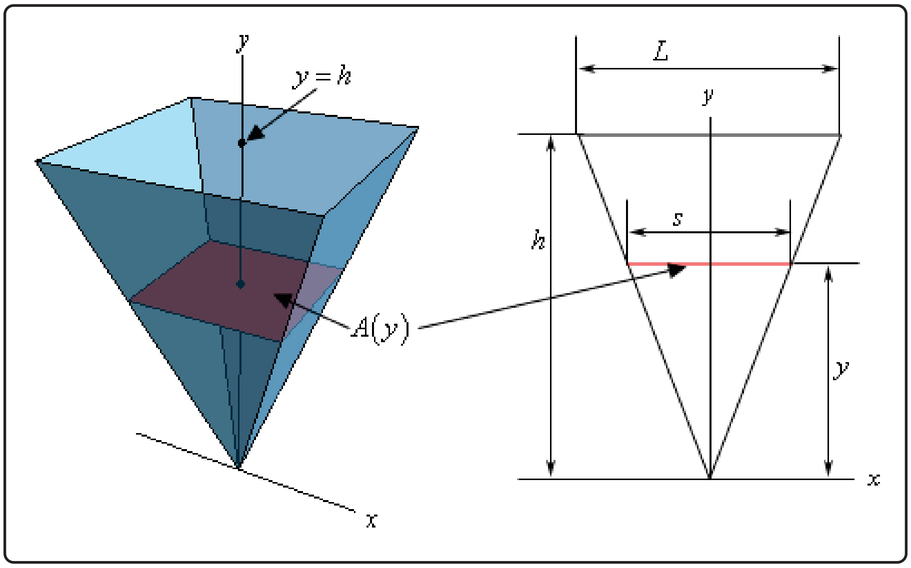
Let's work a couple of more complicated examples. In these examples the main issue is going to be determining what the cross-sectional areas are.

Example 2

Find the volume of a pyramid whose base is a square with sides of length L and whose height is h.

Solution

Let's start off with a sketch of the pyramid. In this case we'll center the pyramid on the y-axis and to make the equations easier we are going to position the point of the pyramid at the origin.



Now, as shown here the cross-sectional area will be a function of y and it will also be a square with sides of length s. The area of the square is easy, but we'll need to get the length of the side in terms of y. To determine this, consider the figure on the right above. If we look at the pyramid directly from the front we'll see that we have two similar triangles and we know that the ratio of any two sides of similar triangles must be equal. In other words, we know that,

$$\frac{s}{L} = \frac{y}{h} \qquad \Rightarrow \qquad s = \frac{y}{h}L = \frac{L}{h}y$$

So, the cross-sectional area is then,

$$A\left(y\right) = s^2 = \frac{L^2}{h^2}y^2$$

The limit for the integral will be $0 \le y \le h$ and the volume will be,

$$V = \int_0^h \frac{L^2}{h^2}y^2\,dy = \frac{L^2}{h^2}\int_0^h y^2\,dy = \frac{L^2}{h^2}\left(\frac{1}{3}y^3\right)\bigg|_0^h = \frac{1}{3}L^2h$$

Again, do not get excited about the L and the h in the integral. Once we start the problem if we change y they will not change and so they are constants as far as the integral is concerned and can get pulled out of the integral. Also, remember that when we evaluate will only plug the limits into the variable we integrated with respect to, y in this case.

Before we proceed with some more complicated examples we should once again remind you to not get excited about the other letters in the integrals. If we're integrating with respect to x or y then all other letters in the formula that represent fixed quantities (*i.e.* radius, height, length of a side, *etc.*) are just constants and can be treated as such when doing the integral.

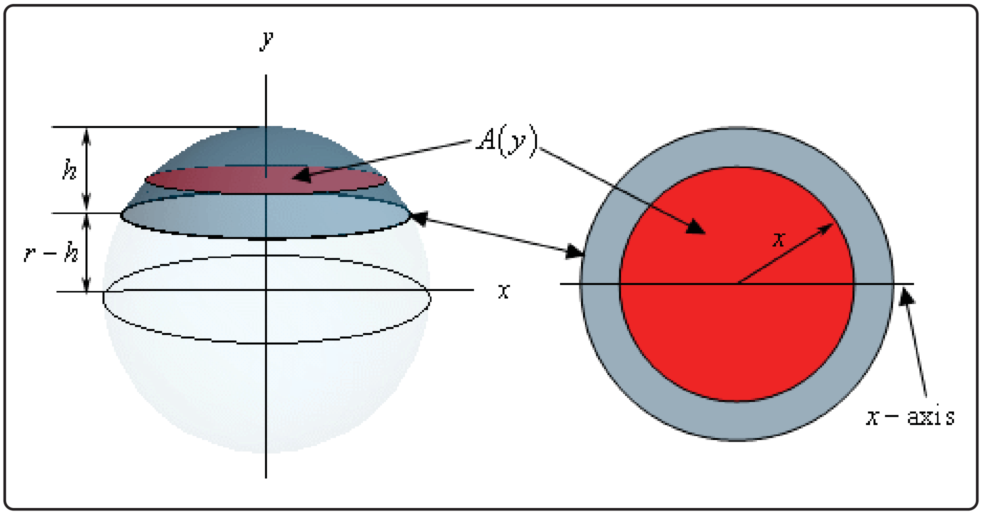
Now let's do some more examples.

Example 3

For a sphere of radius r find the volume of the cap of height h.

Solution

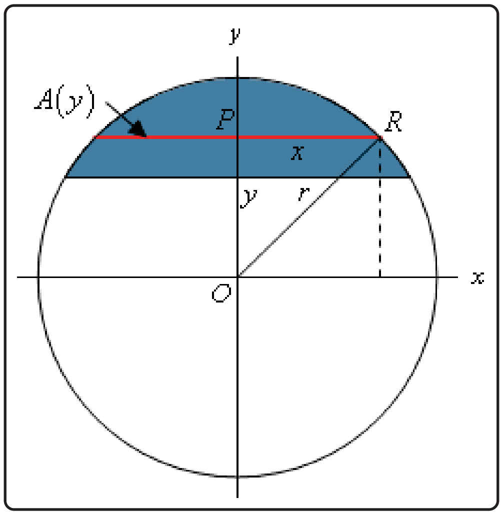
A sketch is probably best to illustrate what we're after here.



We are after the top portion of the sphere and the height of this is portion is h. In this case we'll use a cross-sectional area that starts at the bottom of the cap, which is at $y = r - h$, and moves up towards the top, which is at $y = r$. So, each cross-section will be a disk of radius x. It is a little easier to see that the radius will be x if we look at it from the top as shown in the sketch to the right above. The area of this disk is then,

$$\pi x^2$$

This is a problem however as we need the cross-sectional area in terms of y. So, what we really need to determine what x^2 will be for any given y at the cross-section. To get this let's look at the sphere from the front.



In particular look at the triangle POR. Because the point R lies on the sphere itself we can see that the length of the hypotenuse of this triangle (the line OR) is r, the radius of the sphere. The line PR has a length of x and the line OP has length y so by the Pythagorean Theorem we know,

$$x^2 + y^2 = r^2 \qquad \Rightarrow \qquad x^2 = r^2 - y^2$$

So, we now know what x^2 will be for any given y and so the cross-sectional area is,

$$A(y) = \pi\left(r^2 - y^2\right)$$

As we noted earlier the limits on y will be $r - h \le y \le r$ and so the volume is,

$$\begin{aligned} V &= \int_{r-h}^{r} \pi\left(r^2 - y^2\right)\,dy \\ &= \pi\left(r^2 y - \frac{1}{3}y^3\right)\bigg|_{r-h}^{r} \\ &= \pi\left(h^2 r - \frac{1}{3}h^3\right) = \pi h^2\left(r - \frac{1}{3}h\right) \end{aligned}$$

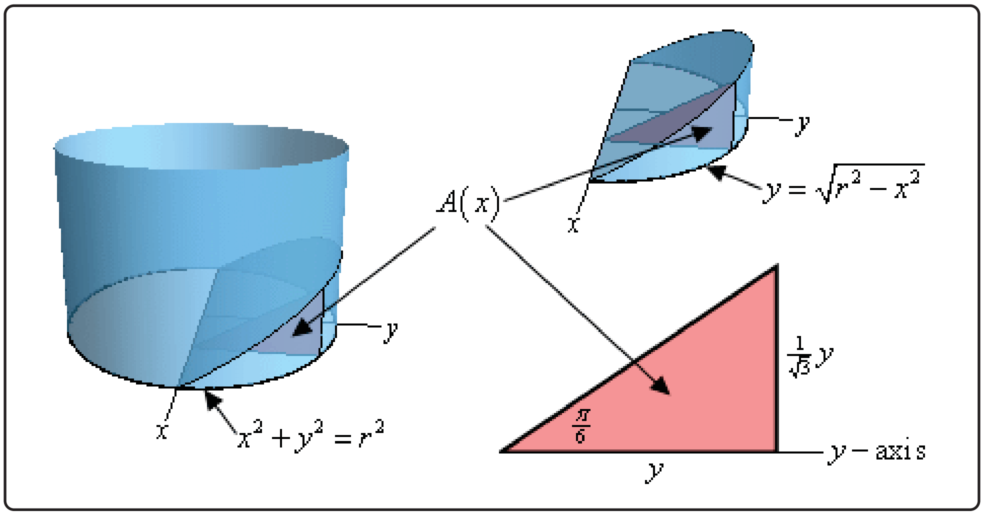
In the previous example we again saw an r in the integral. However, unlike the previous two examples it was not multiplied times the x or the y and so could not be pulled out of the integral. In this case it was like we were integrating $4 - y^2$ and we know how to integrate that. So, in this case we need to treat the r^2 like the 4 and so when we integrate that we'll pick up a y.

Example 4

Find the volume of a wedge cut out of a cylinder of radius r if the angle between the top and bottom of the wedge is $\frac{\pi}{6}$.

Solution

We should really start off with a sketch of just what we're looking for here.



On the left we see how the wedge is being cut out of the cylinder. The base of the cylinder is the circle give by $x^2 + y^2 = r^2$ and the angle between this circle and the top of the wedge is $\frac{\pi}{6}$. The sketch in the upper right position is the actual wedge itself. Given the orientation of the axes here we get the portion of the circle with positive y and so we can write the equation of the circle as $y = \sqrt{r^2 - x^2}$ since we only need the positive y values. Note as well that this is the reason for the way we oriented the axes here.

We get positive y's and we can write the equation of the circle as a function only of x's.

Now, as we can see in the two sketches of the wedge the cross-sectional area will be a right triangle and the area will be a function of x as we move from the back of the cylinder, at $x = -r$, to the front of the cylinder, at $x = r$.

The right angle of the triangle will be on the circle itself while the point on the x-axis will have an interior angle of $\frac{\pi}{6}$. The base of the triangle will have a length of y and using a little right triangle trig we see that the height of the rectangle is,

$$\text{height} = y \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} y$$

So, we now know the base and height of our triangle, in terms of y, and we have an equation for y in terms of x and so we can see that the area of the triangle, *i.e.* the cross-sectional area is,

$$A(x) = \frac{1}{2}(y)\left(\frac{1}{\sqrt{3}}y\right) = \frac{1}{2}\sqrt{r^2 - x^2}\left(\frac{1}{\sqrt{3}}\sqrt{r^2 - x^2}\right) = \frac{1}{2\sqrt{3}}\left(r^2 - x^2\right)$$

The limits on x are $-r \le x \le r$ and so the volume is then,

$$V = \int_{-r}^{r} \frac{1}{2\sqrt{3}}\left(r^2 - x^2\right) dx = \frac{1}{2\sqrt{3}}\left(r^2 x - \frac{1}{3}x^3\right)\Bigg|_{-r}^{r} = \frac{2r^3}{3\sqrt{3}}$$

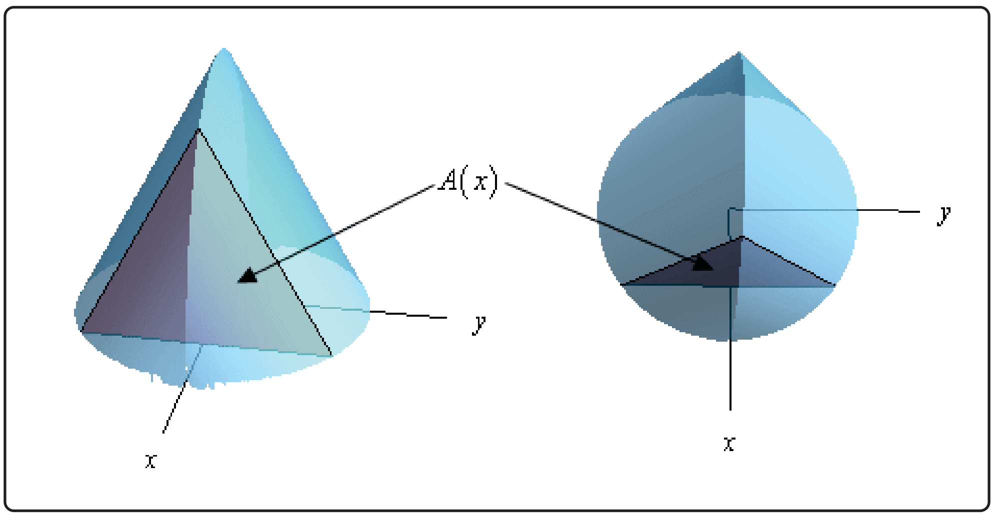
The next example is very similar to the previous one except it can be a little difficult to visualize the solid itself.

Example 5

Find the volume of the solid whose base is a disk of radius r and whose cross-sections are equilateral triangles.

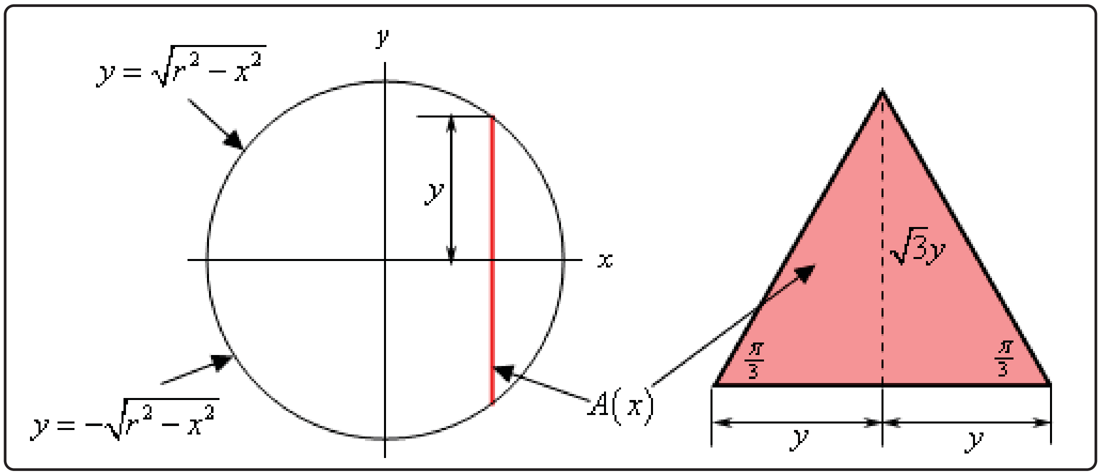
Solution

Let's start off with a couple of sketches of this solid. The sketch on the left is from the "front" of the solid and the sketch on the right is more from the top of the solid.



The base of this solid is the disk of radius r and we move from the back of the disk at $x = -r$ to the front of the disk at $x = r$ we form equilateral triangles to form the solid. A sample equilateral triangle, which is also the cross-sectional area, is shown above to hopefully make it a little clearer how the solid is formed.

Now, let's get a formula for the cross-sectional area. Let's start with the two sketches below.



In the left hand sketch we are looking at the solid from directly above and notice that we "reoriented" the sketch a little to put the x and y-axis in the "normal" orientation. The solid vertical line in this sketch is the cross-sectional area. From this we can see that the cross-section occurs at a given x and the top half will have a length of y where the value of y will be the y-coordinate of the point on the circle and so is,

$$y = \sqrt{r^2 - x^2}$$

Also, because the cross-section is an equilateral triangle that is centered on the x-axis the bottom half will also have a length of y. Thus, the base of the cross-section must have a length of $2y$.

The sketch to the right is of one of the cross-sections. As noted above the base of the triangle has a length of $2y$. Also note that because it is an equilateral triangle the angles are all $\frac{\pi}{3}$. If we divide the cross-section in two (as shown with the dashed line) we now have two right triangles and using right triangle trig we can see that the length of the dashed line is,

$$\text{dashed line} = y \tan\left(\frac{\pi}{3}\right) = \sqrt{3}\, y$$

Therefore, the height of the cross-section is $\sqrt{3}\, y$. Because the cross-section is a triangle we know that that it's area must then be,

$$A(x) = \frac{1}{2}(2y)\left(\sqrt{3}\, y\right) = \frac{1}{2}\left(2\sqrt{r^2 - x^2}\right)\left(\sqrt{3}\sqrt{r^2 - x^2}\right) = \sqrt{3}\left(r^2 - x^2\right)$$

Note that we used the cross-sectional area in terms of x because each of the cross-sections is perpendicular to the x-axis and this pretty much forces us to integrate with respect to x.

The volume of the solid is then,

$$V(x) = \int_{-r}^{r} \sqrt{3}\left(r^2 - x^2\right)\, dx = \sqrt{3}\left(r^2 x - \frac{1}{3}x^3\right)\Big|_{-r}^{r} = \frac{4}{\sqrt{3}}r^3$$

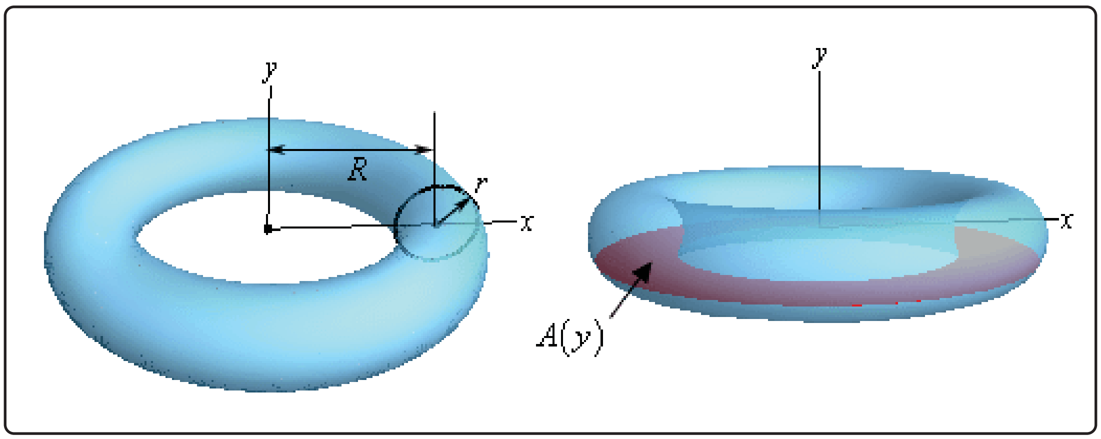
The final example we're going to work here is a little tricky both in seeing how to set it up and in doing the integral.

Example 6

Find the volume of a torus with radii r and R.

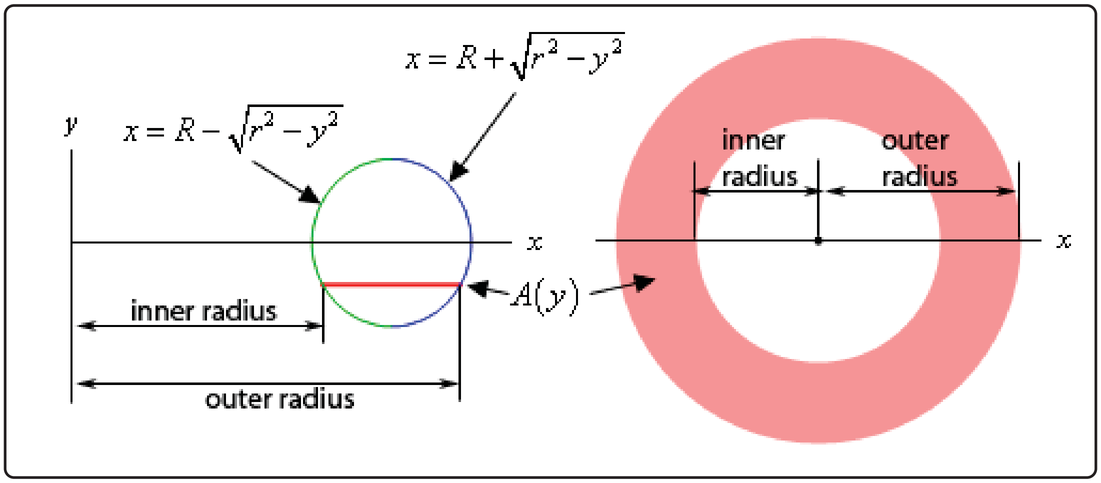
Solution

First, just what is a torus? A torus is a donut shaped solid that is generated by rotating the circle of radius r and centered at $(R, 0)$ about the y-axis. This is shown in the sketch to the left below.



One of the trickiest parts of this problem is seeing what the cross-sectional area needs to be. There is an obvious one. Most people would probably think of using the circle of radius r that we're rotating about the y-axis as the cross-section. It is definitely one of the more obvious choices, however setting up an integral using this is not so easy.

So, what we'll do is use a cross-section as shown in the sketch to the right above. If we cut the torus perpendicular to the y-axis we'll get a cross-section of a ring and finding the area of that shouldn't be too bad. To do that let's take a look at the two sketches below.



The sketch to the left is a sketch of the full cross-section. The sketch to the right is more important however. This is a sketch of the circle that we are rotating about the y-axis. Included is a line representing where the cross-sectional area would be in the torus.

Notice that the inner radius will always be the left portion of the circle and the outer radius will always be the right portion of the circle. Now, we know that the equation of this is,

$$(x - R)^2 + y^2 = r^2$$

and so if we solve for x we can get the equations for the left and right sides as shown above in the sketch. This however means that we also now have equations for the inner and outer radii.

inner radius : $x = R - \sqrt{r^2 - y^2}$ outer radius : $x = R + \sqrt{r^2 - y^2}$

The cross-sectional area is then,

$$\begin{aligned} A(y) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi\left[\left(R + \sqrt{r^2 - y^2}\right)^2 - \left(R - \sqrt{r^2 - y^2}\right)^2\right] \\ &= \pi\left[R^2 + 2R\sqrt{r^2 - y^2} + r^2 - y^2 - \left(R^2 - 2R\sqrt{r^2 - y^2} + r^2 - y^2\right)\right] \\ &= 4\pi R\sqrt{r^2 - y^2} \end{aligned}$$

Next, the lowest cross-section will occur at $y = -r$ and the highest cross-section will occur at $y = r$ and so the limits for the integral will be $-r \le y \le r$.

The integral giving the volume is then,

$$V = \int_{-r}^{r} 4\pi R\sqrt{r^2 - y^2}\, dy = 2\int_{0}^{r} 4\pi R\sqrt{r^2 - y^2}\, dy = 8\pi R \int_{0}^{r} \sqrt{r^2 - y^2}\, dy$$

Note that we used the fact that because the integrand is an even function and we're integrating over $[-r, r]$ we could change the lower limit to zero and double the value of the integral. We saw this fact back in the Computing Definite Integrals section.

We've now reached the second really tricky part of this example. With the knowledge that we've currently got at this point this integral is not possible to do. It requires something called a trig substitution and that is a topic for Calculus II. Luckily enough for us, and this is the tricky part, in this case we can actually determine the integral's value using what we know about integrals.

Just for a second let's think about a different problem. Let's suppose we wanted to use an integral to determine the area under the portion of the circle of radius r and centered at the origin that is in the first quadrant. There are a couple of ways to do this, but to match what we're doing here let's do the following.

We know that the equation of the circle is $x^2 + y^2 = r^2$ and if we solve for x the equation of the circle in the first (and fourth for that matter) quadrant is,

$$x = \sqrt{r^2 - y^2}$$

If we want an integral for the area in the first quadrant we can think of this area as the region between the curve $x = \sqrt{r^2 - y^2}$ and the y-axis for $0 \le y \le r$ and this is,

$$A = \int_0^r \sqrt{r^2 - y^2}\, dy$$

In other words, this integral represents one quarter of the area of a circle of radius r and from basic geometric formulas we now know that this integral must have the value,

$$A = \int_0^r \sqrt{r^2 - y^2}\, dy = \frac{1}{4}\pi r^2$$

So, putting all this together the volume of the torus is then,

$$V = 8R\pi \int_0^r \sqrt{r^2 - y^2}\, dy = 8\pi R\left(\frac{1}{4}\pi r^2\right) = 2R\pi^2 r^2$$

6.6 Work

This is the final application of integral that we'll be looking at in this course. In this section we will be looking at the amount of work that is done by a force in moving an object.

In a first course in Physics you typically look at the work that a constant force, F, does when moving an object over a distance of d. In these cases the work is,

$$W = Fd$$

However, most forces are not constant and will depend upon where exactly the force is acting. So, let's suppose that the force at any x is given by $F(x)$. Then the work done by the force in moving an object from $x = a$ to $x = b$ is given by,

$$W = \int_a^b F(x)\, dx$$

To see a justification of this formula see the Proof of Various Integral Properties section of the Extras appendix.

Notice that if the force is constant we get the correct formula for a constant force.

$$\begin{aligned} W &= \int_a^b F\,dx \\ &= Fx\Big|_a^b \\ &= F(b-a) \end{aligned}$$

where $b - a$ is simply the distance moved, or d.

So, let's take a look at a couple of examples of non-constant forces.

Example 1

A spring has a natural length of 20 cm. A 40 N force is required to stretch (and hold the spring) to a length of 30 cm. How much work is done in stretching the spring from 35 cm to 38 cm?

Solution

This example will require Hooke's Law to determine the force. Hooke's Law tells us that the force required to stretch a spring a distance of x meters from its natural length is,

$$F(x) = k\,x$$

where $k > 0$ is called the spring constant. It is important to remember that the x in this formula is the distance the spring is stretched from its natural length and not the actual length of the spring.

So, the first thing that we need to do is determine the spring constant for this spring. We can do that using the initial information. A force of 40 N is required to stretch the spring

$$30 \text{ cm} - 20 \text{ cm} = 10 \text{ cm} = 0.1 \text{ m}$$

from its natural length. Using Hooke's Law we have,

$$40 = 0.10k \qquad \Rightarrow \qquad k = 400$$

So, according to Hooke's Law the force required to hold this spring x meters from its natural length is,

$$F(x) = 400x$$

We want to know the work required to stretch the spring from 35cm to 38cm. First, we need to convert these into distances from the natural length in meters. Doing that gives us x's of 0.15m and 0.18m.

The work is then,

$$\begin{aligned} W &= \int_{0.15}^{0.18} 400x \, dx \\ &= 200x^2 \Big|_{0.15}^{0.18} \\ &= 1.98 \text{ J} \end{aligned}$$

Example 2

We have a cable that weighs 2 lbs/ft attached to a bucket filled with coal that weighs 800 lbs. The bucket is initially at the bottom of a 500 ft mine shaft. Answer each of the following about this.

(a) Determine the amount of work required to lift the bucket to the midpoint of the shaft.

(b) Determine the amount of work required to lift the bucket from the midpoint of the shaft to the top of the shaft.

(c) Determine the amount of work required to lift the bucket all the way up the shaft.

Solution

Before answering either part we first need to determine the force. In this case the force will be the weight of the bucket and cable at any point in the shaft.

To determine a formula for this we will first need to set a convention for x. For this problem we will set x to be the amount of cable that has been pulled up. So at the bottom of the shaft $x = 0$, at the midpoint of the shaft $x = 250$ and at the top of the shaft $x = 500$. Also, at any point in the shaft there is $500 - x$ feet of cable still in the shaft.

The force then for any x is then nothing more than the weight of the cable and bucket at that point. This is,

$$\begin{aligned} F(x) &= \text{weight of cable} + \text{ weight of bucket/coal} \\ &= 2(500 - x) + 800 \\ &= 1800 - 2x \end{aligned}$$

We can now answer the questions.

(a) Determine the amount of work required to lift the bucket to the midpoint of the shaft.

In this case we want to know the work required to move the cable and bucket/coal from $x = 0$ to $x = 250$. The work required is,

$$\begin{aligned} W &= \int_0^{250} F(x)\, dx \\ &= \int_0^{250} 1800 - 2x\, dx \\ &= \left(1800x - x^2\right)\Big|_0^{250} \\ &= 387500 \text{ ft - lb} \end{aligned}$$

(b) Determine the amount of work required to lift the bucket from the midpoint of the shaft to the top of the shaft.

In this case we want to move the cable and bucket/coal from $x = 250$ to $x = 500$. The work required is,

$$\begin{aligned} W &= \int_{250}^{500} F(x)\, dx \\ &= \int_{250}^{500} 1800 - 2x\, dx \\ &= \left(1800x - x^2\right)\Big|_{250}^{500} \\ &= 262500 \text{ ft - lb} \end{aligned}$$

(c) Determine the amount of work required to lift the bucket all the way up the shaft.

In this case the work is,

$$\begin{aligned} W &= \int_0^{500} F(x)\, dx \\ &= \int_0^{500} 1800 - 2x\, dx \\ &= \left(1800x - x^2\right)\Big|_0^{500} \\ &= 650000 \text{ ft - lb} \end{aligned}$$

Note that we could have instead just added the results from the first two parts and we would have gotten the same answer to the third part.

Example 3

A 20 ft cable weighs 80 lbs and hangs from the ceiling of a building without touching the floor. Determine the work that must be done to lift the bottom end of the chain all the way up until it touches the ceiling.

Solution

First, we need to determine the weight per foot of the cable. This is easy enough to get,

$$\frac{80 \text{ lbs}}{20 \text{ ft}} = 4 \text{ lb/ft}$$

Next, let x be the distance from the ceiling to any point on the cable. Using this convention we can see that the portion of the cable in the range $10 < x \leq 20$ will actually be lifted. The portion of the cable in the range $0 \leq x \leq 10$ will not be lifted at all since once the bottom of the cable has been lifted up to the ceiling the cable will be doubled up and each portion will have a length of 10 ft. So, the upper 10 foot portion of the cable will never be lifted while the lower 10 ft portion will be lifted.

Now, the very bottom of the cable, $x = 20$, will be lifted 10 feet to get to the midpoint and then a further 10 feet to get to the ceiling. A point 2 feet from the bottom of the cable, $x = 18$ will lift 8 feet to get to the midpoint and then a further 8 feet until it reaches its final position (if it is 2 feet from the bottom then its final position will be 2 feet from the ceiling). Continuing on in this fashion we can see that for any point on the lower half of the cable, *i.e.* $10 \leq x \leq 20$ it will be lifted a total of $2(x - 10)$.

As with the previous example the force required to lift any point of the cable in this range is simply the distance that point will be lifted times the weight/foot of the cable. So, the force is then,

$$\begin{aligned} F(x) &= (\text{distance lifted})\,(\text{weight per foot of cable}) \\ &= 2(x-10)(4) \\ &= 8(x-10) \end{aligned}$$

The work required is now,

$$\begin{aligned} W &= \int_{10}^{20} 8(x-10)\,dx \\ &= \left(4x^2 - 80x\right)\Big|_{10}^{20} \\ &= 400 \text{ ft - lb} \end{aligned}$$

Provided we can find the force, $F(x)$, for a given problem then using the above method for determining the work is (generally) pretty simple. However, there are some problems where this approach won't easily work. Let's take a look at one of those kinds of problems.

Example 4

A tank in the shape of an inverted cone has a height of 15 meters and a base radius of 4 meters and is filled with water to a depth of 12 meters. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the density of the water is 1000 kg/m^3.

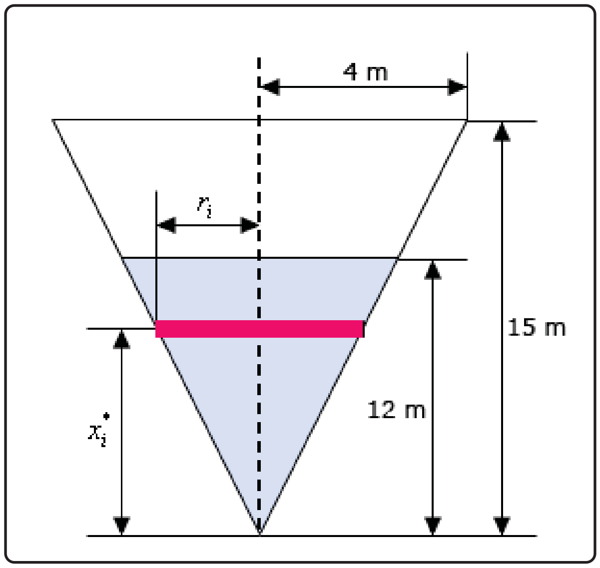
Solution

Okay, in this case we cannot just determine a force function, $F(x)$ that will work for us. So, we are going to need to approach this from a different standpoint.

Let's first set $x = 0$ to be the lower end of the tank/cone and $x = 15$ to be the top of the tank/cone. With this *definition* of our x's we can now see that the water in the tank will correspond to the interval $[0, 12]$.

So, let's start off by dividing $[0, 12]$ into n subintervals each of width Δx and let's also let x_i^* be any point from the i^{th} subinterval where $i = 1, 2, \ldots n$. Now, for each subinterval we will approximate the water in the tank corresponding to that interval as a cylinder of radius r_i and height Δx.

Here is a quick sketch of the tank. Note that the sketch really isn't to scale and we are looking at the tank from directly in front so we can see all the various quantities that we need to work with.



The red strip in the sketch represents the “cylinder” of water in the i^{th} subinterval. A quick application of similar triangles will allow us to relate r_i to x_i^* (which we'll need in a bit) as follows.

$$\frac{r_i}{x_i^*} = \frac{4}{15} \qquad \Rightarrow \qquad r_i = \frac{4}{15} x_i^*$$

Okay, the mass, m_i, of the volume of water, V_i, for the i^{th} subinterval is simply,

$$m_i = \text{density} \times V_i$$

We know the density of the water (it was given in the problem statement) and because we are approximating the water in the i^{th} subinterval as a cylinder we can easily approximate the volume as,

$$V_i \approx \pi (\text{radius})^2 \, (\text{height})$$

We can now approximate the mass of water in the i^{th} subinterval,

$$m_i \approx (1000) \left[\pi r_i^2 \Delta x \right] = 1000\pi \left(\frac{4}{15} x_i^* \right)^2 \Delta x = \frac{640}{9} \pi (x_i^*)^2 \Delta x$$

To raise this volume of water we need to overcome the force of gravity that is acting on the volume and that is, $F = m_i g$, where $g = 9.8 \, m/s^2$ is the gravitational acceleration. The force to raise the volume of water in the i^{th} subinterval is then approximately,

$$F_i = m_i g \approx (9.8) \frac{640}{9} \pi (x_i^*)^2 \Delta x$$

Next, in order to reach to the top of the tank the water in the i^{th} subinterval will need to travel approximately $15 - x_i^*$ to reach the top of the tank. Because the volume of the water in the

i^{th} subinterval is constant the force needed to raise the water through any distance is also a constant force.

Therefore, the work to move the volume of water in the i^{th} subinterval to the top of the tank, *i.e.* raise it a distance of $15 - x_i^*$, is then approximately,

$$W_i \approx F_i\,(15 - x_i^*) = (9.8)\,\frac{640}{9}\pi (x_i^*)^2\,(15 - x_i^*)\,\Delta x$$

The total amount of work required to raise all the water to the top of the tank is then approximately the sum of each of the W_i for $i = 1, 2, \ldots n$. Or,

$$W \approx \sum_{i=1}^{n} (9.8)\,\frac{640}{9}\pi (x_i^*)^2\,(15 - x_i^*)\,\Delta x$$

To get the actual amount of work we simply need to take $n \to \infty$. *I.e.* compute the following limit,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} (9.8)\,\frac{640}{9}\pi (x_i^*)^2\,(15 - x_i^*)\,\Delta x$$

This limit of a summation should look somewhat familiar to you. It's probably been some time, but recalling the definition of the definite integral we can see that this is nothing more than the following definite integral,

$$\begin{aligned} W &= \int_0^{12} (9.8)\,\frac{640}{9}\pi x^2\,(15 - x)\;dx = (9.8)\,\frac{640}{9}\pi \int_0^{12} 15x^2 - x^3\,dx \\ &= (9.8)\,\frac{640}{9}\pi \left(5x^3 - \frac{1}{4}x^4\right)\Bigg|_0^{12} = 7{,}566{,}362.543\ \text{J} \end{aligned}$$

As we've seen in the previous example we sometimes need to compute "incremental" work and then use that to determine the actual integral we need to compute. This idea does arise on occasion and we shouldn't forget it!