8 More Applications of Integrals

It is now time to take a look at some more applications of integrals. As noted the last time we looked at applications of integrals many, although, not all of these new applications in this chapter have a fairly high chance of needing some of the integration techniques from the last chapter.

The first application, Arc Length can be kept to only u-substitutions at the worst, but most of those problems tend to be very simple. Once we start moving into more complicated problems arc length problems they tend to involve trig substitutions.

The next application, Surface Area tends to be u-substitutions but the notation used here is also used in the Arc Length section and so the surface area section is also here because of the shared notation.

Center of Mass and Probability are applications that will, in almost every case, involve integration by parts. In addition, the Probability section has the potential for improper integrals to show up.

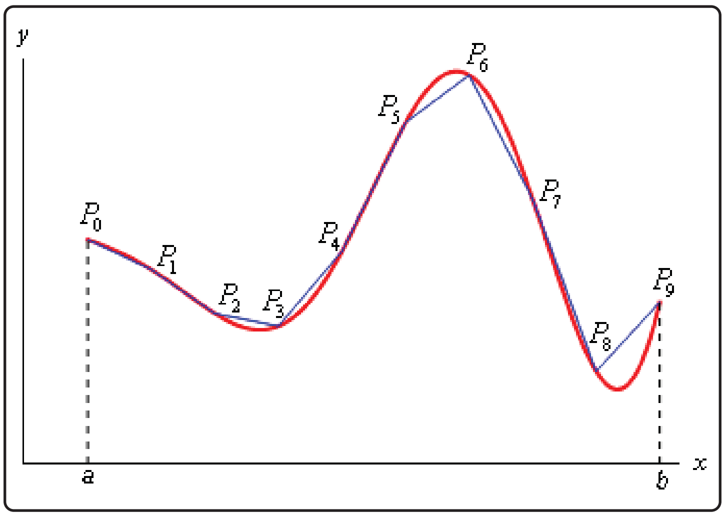
The other application we'll be looking at in this chapter, Hydrostatic Pressure and Force, will typically involve fairly simple integrals that could have been done in the earlier chapter. The reason the topic is here is because we have to derive up the integral using the definition of the definite integral in every problem. In addition, more complicated problems could lead to much more complicated integrals. The integrals in this section are kept simple mostly to keep the derivation work simpler.

8.1 Arc Length

In this section we are going to look at computing the arc length of a function. Because it's easy enough to derive the formulas that we'll use in this section we will derive one of them and leave the other to you to derive.

We want to determine the length of the continuous function $y = f(x)$ on the interval $[a, b]$. We'll also need to assume that the derivative is continuous on $[a, b]$.

Initially we'll need to estimate the length of the curve. We'll do this by dividing the interval up into n equal subintervals each of width Δx and we'll denote the point on the curve at each point by P_i. We can then approximate the curve by a series of straight lines connecting the points. Here is a sketch of this situation for $n = 9$.



Now denote the length of each of these line segments by $|P_{i-1}\ P_i|$ and the length of the curve will then be approximately,

$$L \approx \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

and we can get the exact length by taking n larger and larger. In other words, the exact length will be,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

Now, let's get a better grasp on the length of each of these line segments. First, on each segment let's define $\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$. We can then compute directly the length of the line segments as follows.

$$|P_{i-1}\ P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

By the Mean Value Theorem we know that on the interval $[x_{i-1}, x_i]$ there is a point x_i^* so that,

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= f'(x_i^*)(x_i - x_{i-1}) \\ \Delta y_i &= f'(x_i^*)\,\Delta x \end{aligned}$$

Therefore, the length can now be written as,

$$\begin{aligned} |P_{i-1}\ P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x^2 + \left[f'(x_i^*)\right]^2 \Delta x^2} \\ &= \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \Delta x \end{aligned}$$

The exact length of the curve is then,

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \Delta x \end{aligned}$$

However, using the definition of the definite integral, this is nothing more than,

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

A slightly more convenient notation (in our opinion anyway) is the following.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

In a similar fashion we can also derive a formula for $x = h(y)$ on $[c, d]$. This formula is,

$$L = \int_c^d \sqrt{1 + \left[h'(y)\right]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Again, the second form is probably a little more convenient.

Note the difference in the derivative under the square root! Don't get too confused. With one we differentiate with respect to x and with the other we differentiate with respect to y. One way to keep the two straight is to notice that the differential in the "denominator" of the derivative will match up with the differential in the integral. This is one of the reasons why the second form is a little more convenient.

Before we work any examples we need to make a small change in notation. Instead of having two formulas for the arc length of a function we are going to reduce it, in part, to a single formula.

From this point on we are going to use the following formula for the length of the curve.

Arc Length Formula(s)

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

Note that no limits were put on the integral as the limits will depend upon the ds that we're using. Using the first ds will require x limits of integration and using the second ds will require y limits of integration.

Thinking of the arc length formula as a single integral with different ways to define ds will be convenient when we run across arc lengths in future sections. Also, this ds notation will be a nice notation for the next section as well.

Now that we've derived the arc length formula let's work some examples.

Example 1

Determine the length of $y = \ln\left(\sec(x)\right)$ between $0 \le x \le \frac{\pi}{4}$.

Solution

In this case we'll need to use the first ds since the function is in the form $y = f(x)$. So, let's get the derivative out of the way.

$$\frac{dy}{dx} = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan(x) \qquad \left(\frac{dy}{dx}\right)^2 = \tan^2(x)$$

Let's also get the root out of the way since there is often simplification that can be done and there's no reason to do that inside the integral.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2(x)} = \sqrt{\sec^2(x)} = |\sec(x)| = \sec(x)$$

Note that we could drop the absolute value bars here since secant is positive in the range given.

The arc length is then,

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sec(x)\, dx \\ &= \ln\left|\sec(x) + \tan(x)\right|\Big|_0^{\frac{\pi}{4}} \\ &= \ln\left(\sqrt{2}+1\right) \end{aligned}$$

Example 2

Determine the length of $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ between $1 \le y \le 4$.

Solution

There is a very common mistake that students make in problems of this type. Many students see that the function is in the form $x = h(y)$ and they immediately decide that it will be too difficult to work with it in that form so they solve for y to get the function into the form $y = f(x)$. While that can be done here it will lead to a messier integral for us to deal with.

Sometimes it's just easier to work with functions in the form $x = h(y)$. In fact, if you can work with functions in the form $y = f(x)$ then you can work with functions in the form $x = h(y)$. There really isn't a difference between the two so don't get excited about functions in the form $x = h(y)$.

Let's compute the derivative and the root.

$$\frac{dx}{dy} = (y-1)^{\frac{1}{2}} \qquad \Rightarrow \qquad \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+y-1} = \sqrt{y}$$

As you can see keeping the function in the form $x = h(y)$ is going to lead to a very easy integral. To see what would happen if we tried to work with the function in the form $y = f(x)$ see the next example.

Let's get the length.

$$\begin{aligned} L &= \int_1^4 \sqrt{y}\, dy \\ &= \frac{2}{3} y^{\frac{3}{2}}\Big|_1^4 \\ &= \frac{14}{3} \end{aligned}$$

As noted in the last example we really do have a choice as to which ds we use. Provided we can get the function in the form required for a particular ds we can use it. However, as also noted above, there will often be a significant difference in difficulty in the resulting integrals. Let's take a quick look at what would happen in the previous example if we did put the function into the form $y = f(x)$.

Example 3

Redo the previous example using the function in the form $y = f(x)$ instead.

Solution

In this case the function and its derivative would be,

$$y = \left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1 \qquad\qquad \frac{dy}{dx} = \left(\frac{3x}{2}\right)^{-\frac{1}{3}}$$

The root in the arc length formula would then be.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{\left(\frac{3x}{2}\right)^{\frac{2}{3}}}} = \sqrt{\frac{\left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1}{\left(\frac{3x}{2}\right)^{\frac{2}{3}}}} = \frac{\sqrt{\left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1}}{\left(\frac{3x}{2}\right)^{\frac{1}{3}}}$$

All the simplification work above was just to put the root into a form that will allow us to do the integral.

Now, before we write down the integral we'll also need to determine the limits. This particular ds requires x limits of integration and we've got y limits. They are easy enough to get however. Since we know x as a function of y all we need to do is plug in the original y limits of integration and get the x limits of integration. Doing this gives,

$$0 \le x \le \frac{2}{3}(3)^{\frac{3}{2}}$$

Not easy limits to deal with, but there they are.

Let's now write down the integral that will give the length.

$$L = \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \frac{\sqrt{\left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1}}{\left(\frac{3x}{2}\right)^{\frac{1}{3}}}\, dx$$

That's a really unpleasant looking integral. It can be evaluated however using the following substitution.

$$u = \left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1 \qquad\qquad du = \left(\frac{3x}{2}\right)^{-\frac{1}{3}} dx$$

$$x = 0 \qquad \Rightarrow \qquad u = 1$$
$$x = \frac{2}{3}(3)^{\frac{3}{2}} \qquad \Rightarrow \qquad u = 4$$

Using this substitution the integral becomes,

$$\begin{aligned} L &= \int_1^4 \sqrt{u}\, du \\ &= \left.\frac{2}{3}u^{\frac{3}{2}}\right|_1^4 \\ &= \frac{14}{3} \end{aligned}$$

So, we got the same answer as in the previous example. Although that shouldn't really be all that surprising since we were dealing with the same curve.

From a technical standpoint the integral in the previous example was not that difficult. It was just a Calculus I substitution. However, from a practical standpoint the integral was significantly more difficult than the integral we evaluated in Example 2. So, the moral of the story here is that we can use either formula (provided we can get the function in the correct form of course) however one will often be significantly easier to actually evaluate.

Okay, let's work one more example.

Example 4

Determine the length of $x = \frac{1}{2}y^2$ for $0 \le x \le \frac{1}{2}$. Assume that y is positive.

Solution

We'll use the second ds for this one as the function is already in the correct form for that one. Also, the other ds would again lead to a particularly difficult integral. The derivative and root will then be,

$$\frac{dx}{dy} = y \qquad \Rightarrow \qquad \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + y^2}$$

Before writing down the length notice that we were given x limits and we will need y limits for this ds. With the assumption that y is positive these are easy enough to get. All we need to do is plug x into our equation and solve for y. Doing this gives,

$$0 \le y \le 1$$

The integral for the arc length is then,

$$L = \int_0^1 \sqrt{1 + y^2}\, dy$$

This integral will require the following trig substitution.

$$y = \tan(\theta) \qquad\qquad dy = \sec^2(\theta)\, d\theta$$

$$\begin{aligned} y &= 0 & \Rightarrow \quad & 0 = \tan(\theta) & \Rightarrow \quad & \theta = 0 \\ y &= 1 & \Rightarrow \quad & 1 = \tan(\theta) & \Rightarrow \quad & \theta = \frac{\pi}{4} \end{aligned}$$

$$\sqrt{1 + y^2} = \sqrt{1 + \tan^2(\theta)} = \sqrt{\sec^2(\theta)} = |\sec(\theta)| = \sec(\theta)$$

The length is then,

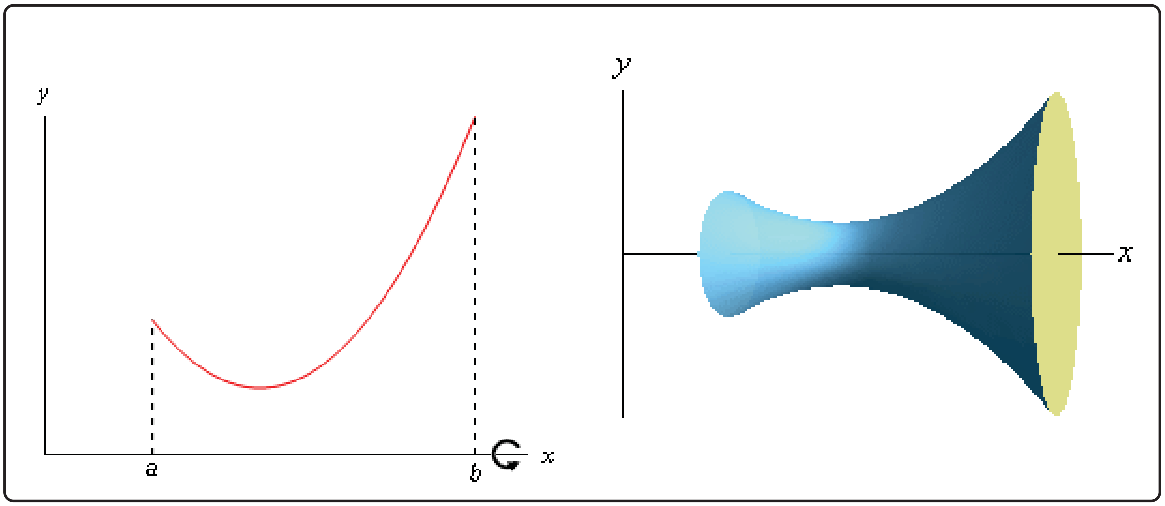
$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sec^3(\theta)\, d\theta \\ &= \frac{1}{2}\left(\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right) \end{aligned}$$

The first couple of examples ended up being fairly simple Calculus I substitutions. However, as this last example had shown we can end up with trig substitutions as well for these integrals.

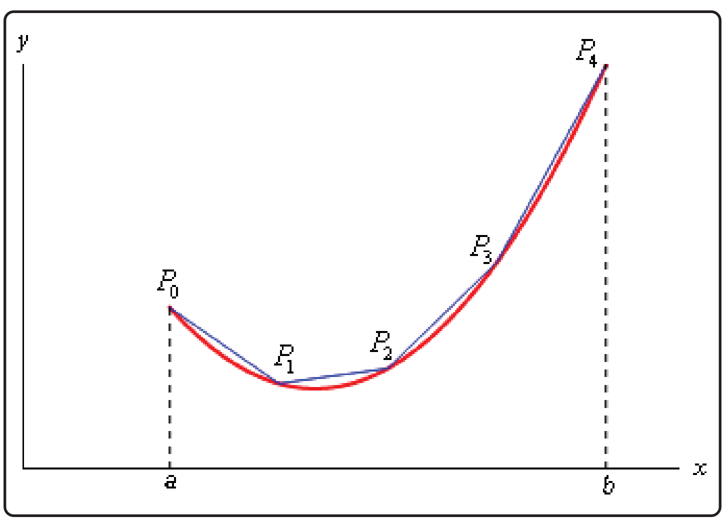
8.2 Surface Area

In this section we are going to look once again at solids of revolution. We first looked at them back in Calculus I when we found the volume of the solid of revolution. In this section we want to find the surface area of this region.

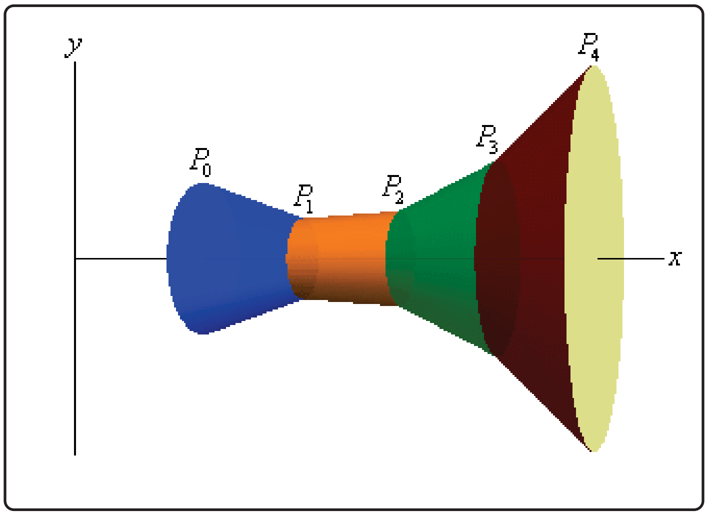
So, for the purposes of the derivation of the formula, let's look at rotating the continuous function $y = f(x)$ in the interval $[a, b]$ about the x-axis. We'll also need to assume that the derivative is continuous on $[a, b]$. Below is a sketch of a function and the solid of revolution we get by rotating the function about the x-axis.



We can derive a formula for the surface area much as we derived the formula for arc length. We'll start by dividing the interval into n equal subintervals of width Δx. On each subinterval we will approximate the function with a straight line that agrees with the function at the endpoints of each interval. Here is a sketch of that for our representative function using $n = 4$.



Now, rotate the approximations about the x-axis and we get the following solid.



The approximation on each interval gives a distinct portion of the solid and to make this clear each portion is colored differently. Each of these portions are called frustums and we know how to find the surface area of frustums.

The surface area of a frustum is given by,

$$A = 2\pi r l$$

where,

$$r = \frac{1}{2}(r_1 + r_2) \qquad \begin{aligned} r_1 &= \text{radius of right end} \\ r_2 &= \text{radius of left end} \end{aligned}$$

and l is the length of the slant of the frustum.

For the frustum on the interval $[x_{i-1}, x_i]$ we have,

$$\begin{aligned} r_1 &= f(x_i) \\ r_2 &= f(x_{i-1}) \\ l &= |P_{i-1}\ P_i| \qquad (\text{length of the line segment connecting } P_i \text{ and } P_{i-1}) \end{aligned}$$

and we know from the previous section that,

$$|P_{i-1}\ P_i| = \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x \quad \text{where } x_i^* \text{ is some point in } [x_{i-1}, x_i]$$

Before writing down the formula for the surface area we are going to assume that Δx is "small" and since $f(x)$ is continuous we can then assume that,

$$f(x_i) \approx f(x_i^*) \qquad \text{and} \qquad f(x_{i-1}) \approx f(x_i^*)$$

So, the surface area of the frustum on the interval $[x_{i-1}, x_i]$ is approximately,

$$\begin{aligned} A_i &= 2\pi\left(\frac{f(x_i)+f(x_{i-1})}{2}\right)|P_{i-1}\,P_i| \\ &\approx 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\;\Delta x \end{aligned}$$

The surface area of the whole solid is then approximately,

$$S \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\;\Delta x$$

and we can get the exact surface area by taking the limit as n goes to infinity.

$$\begin{aligned} S &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\;\Delta x \\ &= \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx \end{aligned}$$

If we wanted to we could also derive a similar formula for rotating $x = h(y)$ on $[c, d]$ about the y-axis. This would give the following formula.

$$S = \int_c^d 2\pi\, h(y)\sqrt{1+\left[h'(y)\right]^2}\,dy$$

These are not the "standard" formulas however. Notice that the roots in both of these formulas are nothing more than the two ds's we used in the previous section. Also, we will replace $f(x)$ with y and $h(y)$ with x. Doing this gives the following two formulas for the surface area.

Surface Area Formulas

$$S = \int 2\pi y\, ds \qquad \text{rotation about } x-\text{axis}$$

$$S = \int 2\pi x\, ds \qquad \text{rotation about } y-\text{axis}$$

where,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{if } y = f(x),\; a \le x \le b$$

$$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{if } x = h(y),\; c \le y \le d$$

There are a couple of things to note about these formulas. First, notice that the variable in the integral itself is always the opposite variable from the one we're rotating about. Second, we are allowed to use either ds in either formula. This means that there are, in some way, four formulas here. We will choose the ds based upon which is the most convenient for a given function and problem.

Now let's work a couple of examples.

Example 1

Determine the surface area of the solid obtained by rotating $y = \sqrt{9 - x^2}$, $-2 \le x \le 2$ about the x-axis.

Solution

The formula that we'll be using here is,

$$S = \int 2\pi y \, ds$$

since we are rotating about the x-axis and we'll use the first ds in this case because our function is in the correct form for that ds and we won't gain anything by solving it for x.

Let's first get the derivative and the root taken care of.

$$\frac{dy}{dx} = \frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}}(-2x) = -\frac{x}{\left(9 - x^2\right)^{\frac{1}{2}}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{9 - x^2}} = \sqrt{\frac{9}{9 - x^2}} = \frac{3}{\sqrt{9 - x^2}}$$

Here's the integral for the surface area,

$$S = \int_{-2}^{2} 2\pi y \frac{3}{\sqrt{9 - x^2}} \, dx$$

There is a problem however. The dx means that we shouldn't have any y's in the integral. So, before evaluating the integral we'll need to substitute in for y as well.

The surface area is then,

$$\begin{aligned} S &= \int_{-2}^{2} 2\pi \sqrt{9 - x^2} \frac{3}{\sqrt{9 - x^2}} \, dx \\ &= \int_{-2}^{2} 6\pi \, dx \\ &= 24\pi \end{aligned}$$

Previously we made the comment that we could use either ds in the surface area formulas. Let's work an example in which using either ds won't create integrals that are too difficult to evaluate and so we can check both ds's.

Example 2

Determine the surface area of the solid obtained by rotating $y = \sqrt[3]{x}$, $1 \le y \le 2$ about the y-axis. Use both ds's to compute the surface area.

Solution

Note that we've been given the function set up for the first ds and limits that work for the second ds.

Solution 1

This solution will use the first ds listed above. We'll start with the derivative and root.

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{9x^{\frac{4}{3}}}} = \sqrt{\frac{9x^{\frac{4}{3}} + 1}{9x^{\frac{4}{3}}}} = \frac{\sqrt{9x^{\frac{4}{3}} + 1}}{3x^{\frac{2}{3}}}$$

We'll also need to get new limits. That isn't too bad however. All we need to do is plug in the given y's into our equation and solve to get that the range of x's is $1 \le x \le 8$. The integral for the surface area is then,

$$\begin{aligned} S &= \int_1^8 2\pi x \frac{\sqrt{9x^{\frac{4}{3}} + 1}}{3x^{\frac{2}{3}}}\, dx \\ &= \frac{2\pi}{3}\int_1^8 x^{\frac{1}{3}}\sqrt{9x^{\frac{4}{3}} + 1}\, dx \end{aligned}$$

Note that this time we didn't need to substitute in for the x as we did in the previous example. In this case we picked up a dx from the ds and so we don't need to do a substitution for the x. In fact, if we had substituted for x we would have put y's into the integral which would have caused problems.

Using the substitution

$$u = 9x^{\frac{4}{3}} + 1 \qquad\qquad du = 12x^{\frac{1}{3}}\, dx$$

the integral becomes,

$$\begin{aligned} S &= \frac{\pi}{18}\int_{10}^{145} \sqrt{u}\, du \\ &= \frac{\pi}{27}u^{\frac{3}{2}}\Big|_{10}^{145} \\ &= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48 \end{aligned}$$

Solution 2

This time we'll use the second ds. So, we'll first need to solve the equation for x. We'll also go ahead and get the derivative and root while we're at it.

$$x = y^3 \qquad\qquad \frac{dx}{dy} = 3y^2$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+9y^4}$$

The surface area is then,

$$S = \int_1^2 2\pi x\sqrt{1+9y^4}\,dy$$

We used the original y limits this time because we picked up a dy from the ds. Also note that the presence of the dy means that this time, unlike the first solution, we'll need to substitute in for the x. Doing that gives,

$$\begin{aligned} S &= \int_1^2 2\pi y^3\sqrt{1+9y^4}\,dy \qquad\qquad u = 1+9y^4 \\ &= \frac{\pi}{18}\int_{10}^{145}\sqrt{u}\,du \\ &= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48 \end{aligned}$$

Note that after the substitution the integral was identical to the first solution and so the work was skipped.

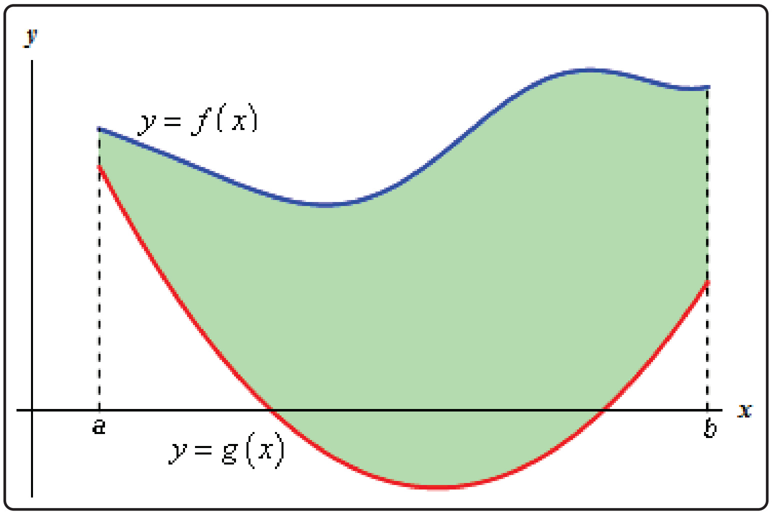
As this example has shown we can use either ds to get the surface area. It is important to point out as well that with one ds we had to do a substitution for the x and with the other we didn't. This will always work out that way.

Note as well that in the case of the last example it was just as easy to use either ds. That often won't be the case. In many examples only one of the ds will be convenient to work with so we'll always need to determine which ds is liable to be the easiest to work with before starting the problem.

8.3 Center Of Mass

In this section we are going to find the **center of mass** or **centroid** of a thin plate with uniform density ρ. The center of mass or centroid of a region is the point in which the region will be perfectly balanced horizontally if suspended from that point.

So, let's suppose that the plate is the region bounded by the two curves $f(x)$ and $g(x)$ on the interval $[a, b]$. So, we want to find the center of mass of the region below.



We'll first need the mass of this plate. The mass is,

$$\begin{aligned} M &= \rho\,(\text{Area of plate}) \\ &= \rho \int_a^b f(x) - g(x)\,dx \end{aligned}$$

Next, we'll need the **moments** of the region. There are two moments, denoted by M_x and M_y. The moments measure the tendency of the region to rotate about the x and y-axis respectively. The moments are given by,

Equations of Moments

$$M_x = \rho \int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)\,dx$$

$$M_y = \rho \int_a^b x\left(f(x) - g(x)\right)\,dx$$

The coordinates of the center of mass, $(\overline{x}, \overline{y})$, are then,

Center of Mass Coordinates

$$\overline{x} = \frac{M_y}{M} = \frac{\int_a^b x\left(f(x) - g(x)\right)\,dx}{\int_a^b f(x) - g(x)\,dx} = \frac{1}{A}\int_a^b x\big(f(x) - g(x)\big)\,dx$$

$$\overline{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)\,dx}{\int_a^b f(x) - g(x)\,dx} = \frac{1}{A}\int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)\,dx$$

where,

$$A = \int_a^b f(x) - g(x)\,dx$$

Note that the density, ρ, of the plate cancels out and so isn't really needed.

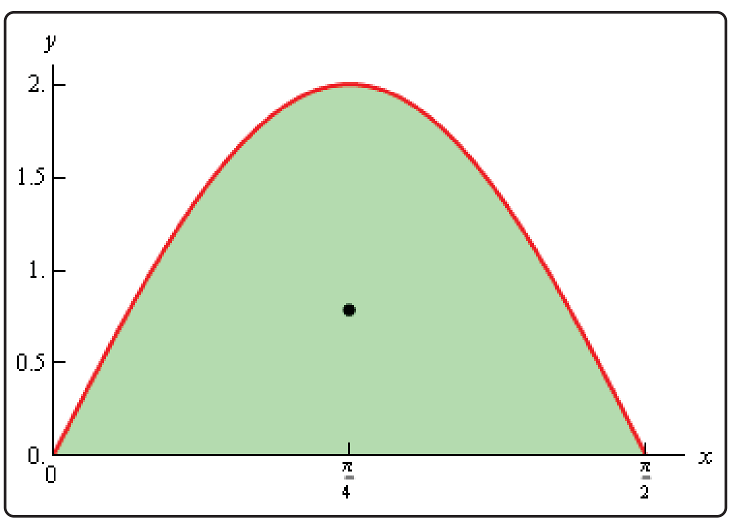
Let's work a couple of examples.

Example 1

Determine the center of mass for the region bounded by $y = 2\sin(2x)$, $y = 0$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Solution

Here is a sketch of the region with the center of mass denoted with a dot.



Let's first get the area of the region.

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} 2\sin(2x)\,dx \\ &= -\cos(2x)\Big|_0^{\frac{\pi}{2}} \\ &= 2 \end{aligned}$$

Now, the moments (without density since it will just drop out) are,

$$\begin{aligned} M_x &= \int_0^{\frac{\pi}{2}} 2\sin^2(2x)\,dx & M_y &= \int_0^{\frac{\pi}{2}} 2x\sin(2x)\,dx \quad \text{integrating by parts...} \\ &= \int_0^{\frac{\pi}{2}} 1-\cos(4x)\,dx & &= -x\cos(2x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos(2x)\,dx \\ &= \left(x - \frac{1}{4}\sin(4x)\right)\Big|_0^{\frac{\pi}{2}} & &= -x\cos(2x)\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\sin(2x)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} & &= \frac{\pi}{2} \end{aligned}$$

The coordinates of the center of mass are then,

$$\begin{aligned} \overline{x} &= \frac{\pi/2}{2} = \frac{\pi}{4} \\ \overline{y} &= \frac{\pi/2}{2} = \frac{\pi}{4} \end{aligned}$$

Again, note that we didn't put in the density since it will cancel out.

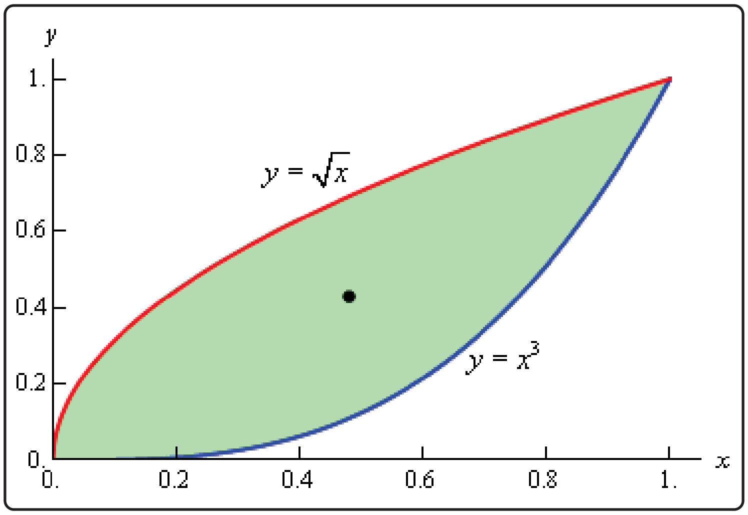
So, the center of mass for this region is $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

Example 2

Determine the center of mass for the region bounded by $y = x^3$ and $y = \sqrt{x}$.

Solution

The two curves intersect at $x = 0$ and $x = 1$ and here is a sketch of the region with the center of mass marked with a box.



We'll first get the area of the region.

$$\begin{aligned} A &= \int_0^1 \sqrt{x} - x^3\,dx \\ &= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^4\right)\Bigg|_0^1 \\ &= \frac{5}{12} \end{aligned}$$

Now the moments, again without density, are

$$\begin{aligned} M_x &= \int_0^1 \frac{1}{2}\left(x - x^6\right)\,dx \\ &= \frac{1}{2}\left(\frac{1}{2}x^2 - \frac{1}{7}x^7\right)\Bigg|_0^1 \\ &= \frac{5}{28} \end{aligned}$$

$$\begin{aligned} M_y &= \int_0^1 x\left(\sqrt{x} - x^3\right)\,dx \\ &= \int_0^1 x^{\frac{3}{2}} - x^4\,dx \\ &= \left(\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{5}x^5\right)\Bigg|_0^1 \\ &= \frac{1}{5} \end{aligned}$$

The coordinates of the center of mass is then,

$$\begin{aligned} \overline{x} &= \frac{1/5}{5/12} = \frac{12}{25} \\ \overline{y} &= \frac{5/28}{5/12} = \frac{3}{7} \end{aligned}$$

The coordinates of the center of mass are then, $\left(\frac{12}{25}, \frac{3}{7}\right)$.

8.4 Hydrostatic Pressure and Force

In this section we are going to submerge a vertical plate in water and we want to know the force that is exerted on the plate due to the pressure of the water. This force is often called the hydrostatic force.

There are two basic formulas that we'll be using here. First, if we are d meters below the surface then the hydrostatic pressure is given by,

$$P = \rho g d$$

where, ρ is the density of the fluid and g is the gravitational acceleration. We are going to assume that the fluid in question is water and since we are going to be using the metric system these quantities become,

$$\rho = 1000 \text{ kg/m}^3 \qquad g = 9.81 \text{ m/s}^2$$

The second formula that we need is the following. Assume that a constant pressure P is acting on a surface with area A. Then the hydrostatic force that acts on the area is,

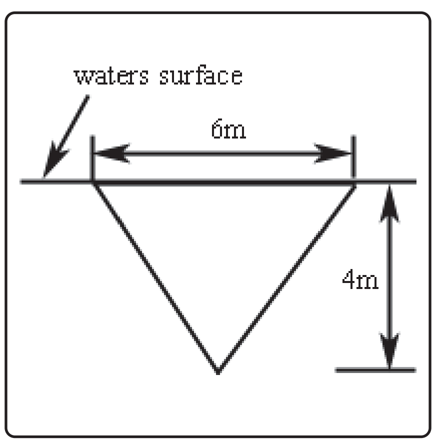
$$F = PA$$

Note that we won't be able to find the hydrostatic force on a vertical plate using this formula since the pressure will vary with depth and hence will not be constant as required by this formula. We will however need this for our work.

The best way to see how these problems work is to do an example or two.

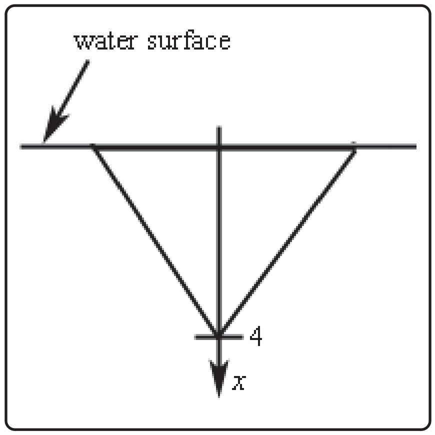
Example 1

Determine the hydrostatic force on the following triangular plate that is submerged in water as shown.



Solution

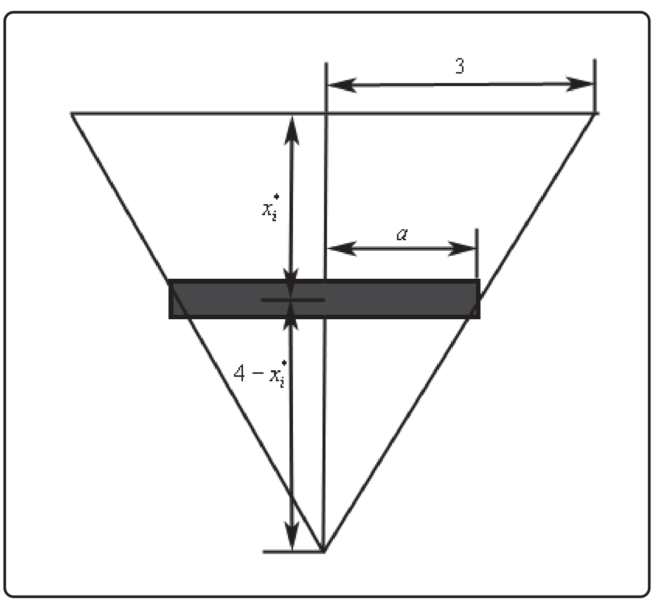
The first thing to do here is set up an axis system. So, let's redo the sketch above with the following axis system added in.



So, we are going to orient the x-axis so that positive x is downward, $x = 0$ corresponds to the water surface and $x = 4$ corresponds to the depth of the tip of the triangle.

Next we break up the triangle into n horizontal strips each of equal width Δx and in each interval $[x_{i-1}, x_i]$ choose any point x_i^*. In order to make the computations easier we are going to make two assumptions about these strips. First, we will ignore the fact that the ends are actually going to be slanted and assume the strips are rectangular. If Δx is sufficiently small this will not affect our computations much. Second, we will assume that Δx is small enough that the hydrostatic pressure on each strip is essentially constant.

Below is a representative strip.



The height of this strip is Δx and the width is $2a$. We can use similar triangles to determine a as follows,

$$\frac{3}{4} = \frac{a}{4 - x_i^*} \qquad \Rightarrow \qquad a = 3 - \frac{3}{4}x_i^*$$

Now, since we are assuming the pressure on this strip is constant, the pressure is given by,

$$P_i = \rho g d = 1000\,(9.81)\,x_i^* = 9810x_i^*$$

and the hydrostatic force on each strip is,

$$F_i = P_i\,A = P_i\,(2a\Delta x) = 9810x_i^*\,(2)\left(3 - \frac{3}{4}x_i^*\right)\Delta x = 19620x_i^*\left(3 - \frac{3}{4}x_i^*\right)\Delta x$$

The approximate hydrostatic force on the plate is then the sum of the forces on all the strips or,

$$F \approx \sum_{i=1}^{n} 19620x_i^*\left(3 - \frac{3}{4}x_i^*\right)\Delta x$$

Taking the limit will get the exact hydrostatic force,

$$F = \lim_{n\to\infty}\sum_{i=1}^{n} 19620x_i^*\left(3 - \frac{3}{4}x_i^*\right)\Delta x$$

Using the definition of the definite integral this is nothing more than,

$$F = \int_0^4 19620\left(3x - \frac{3}{4}x^2\right)dx$$

The hydrostatic force is then,

$$\begin{aligned} F &= \int_0^4 19620\left(3x - \frac{3}{4}x^2\right)dx \\ &= 19620\left(\frac{3}{2}x^2 - \frac{1}{4}x^3\right)\Big|_0^4 \\ &= 156960\,N \end{aligned}$$

Let’s take a look at another example.

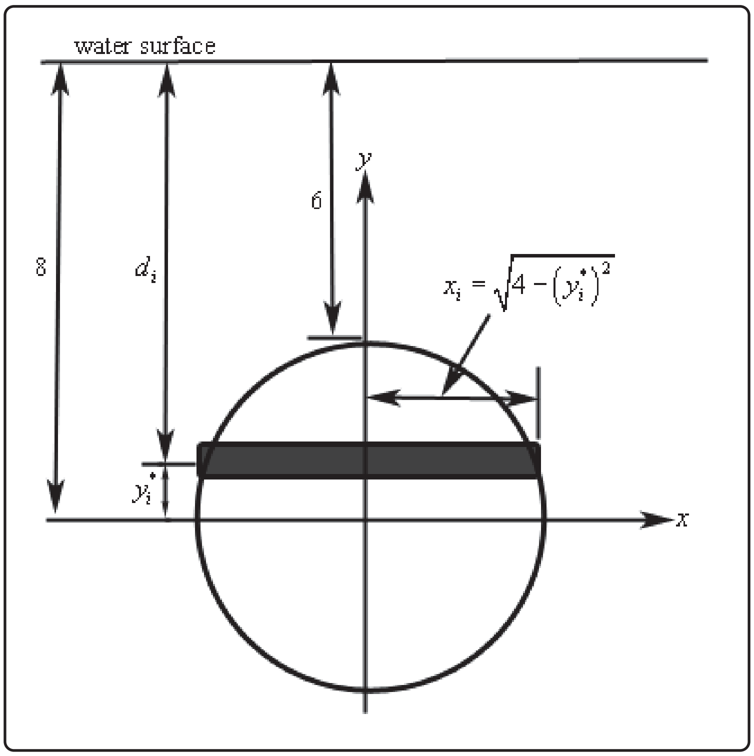
Example 2

Find the hydrostatic force on a circular plate of radius 2 that is submerged 6 meters in the water.

Solution

First, we're going to assume that the top of the circular plate is 6 meters under the water. Next, we will set up the axis system so that the origin of the axis system is at the center of the plate. Setting the axis system up in this way will greatly simplify our work.

Finally, we will again split up the plate into n horizontal strips each of width Δy and we'll choose a point y_i^* from each strip. We'll also assume that the strips are rectangular again to help with the computations. Here is a sketch of the setup.



The depth below the water surface of each strip is,

$$d_i = 8 - y_i^*$$

and that in turn gives us the pressure on the strip,

$$P_i = \rho g d_i = 9810\,(8 - y_i^*)$$

The area of each strip is,

$$A_i = 2\sqrt{4 - (y_i^*)^2}\;\Delta y$$

The hydrostatic force on each strip is,

$$F_i = P_i A_i = 9810\,(8 - y_i^*)\,(2)\,\sqrt{4 - (y_i^*)^2}\;\Delta y$$

The total force on the plate is,

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} 19620\,(8 - y_i^*)\,\sqrt{4 - (y_i^*)^2}\;\Delta y \\ &= 19620 \int_{-2}^{2} (8 - y)\,\sqrt{4 - y^2}\,dy \end{aligned}$$

To do this integral we'll need to split it up into two integrals.

$$F = 19620 \int_{-2}^{2} 8\sqrt{4 - y^2}\,dy - 19620 \int_{-2}^{2} y\sqrt{4 - y^2}\,dy$$

The first integral requires the trig substitution $y = 2\sin(\theta)$ and the second integral needs the substitution $v = 4 - y^2$. After using these substitutions we get,

$$\begin{aligned} F &= 627840 \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\,d\theta + 9810 \int_{0}^{0} \sqrt{v}\,dv \\ &= 313920 \int_{-\pi/2}^{\pi/2} 1 + \cos(2\theta)\;d\theta + 0 \\ &= 313920 \left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= 313920\pi \end{aligned}$$

Note that after the substitution we know the second integral will be zero because the upper and lower limit is the same.

8.5 Probability

In this last application of integrals that we'll be looking at we're going to look at probability. Before actually getting into the applications we need to get a couple of definitions out of the way.

Suppose that we wanted to look at the age of a person, the height of a person, the amount of time spent waiting in line, or maybe the lifetime of a battery. Each of these quantities have values that will range over an interval of real numbers. Because of this these are called **continuous random variables**. Continuous random variables are often represented by X.

Every continuous random variable, X, has a **probability density function**, $f(x)$. Probability density functions satisfy the following conditions.

1. $f(x) \geq 0$ for all x.
2. $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$

Probability density functions can be used to determine the probability that a continuous random variable lies between two values, say a and b. This probability is denoted by $P(a \leq X \leq b)$ and is given by,

Fact

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

Let's take a look at an example of this.

Example 1

Let $f(x) = \frac{x^3}{5000}(10-x)$ for $0 \leq x \leq 10$ and $f(x) = 0$ for all other values of x. Answer each of the following questions about this function.

(a) Show that $f(x)$ is a probability density function.

(b) Find $P(1 \leq X \leq 4)$

(c) Find $P(x \geq 6)$

Solution

(a) Show that $f(x)$ is a probability density function.

First note that in the range $0 \leq x \leq 10$ is clearly positive and outside of this range we've defined it to be zero.

So, to show this is a probability density function we'll need to show that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\, dx &= \int_0^{10} \frac{x^3}{5000}(10-x)\, dx \\ &= \left(\frac{x^4}{2000} - \frac{x^5}{25000}\right)\Bigg|_0^{10} \\ &= 1\end{aligned}$$

Note the change in limits on the integral. The function is only non-zero in these ranges and so the integral can be reduced down to only the interval where the function is not zero.

(b) Find $P(1 \leq X \leq 4)$

In this case we need to evaluate the following integral.

$$\begin{aligned}P(1 \leq X \leq 4) &= \int_1^{4} \frac{x^3}{5000}(10-x)\, dx \\ &= \left(\frac{x^4}{2000} - \frac{x^5}{25000}\right)\Bigg|_1^{4} \\ &= 0.08658\end{aligned}$$

So the probability of X being between 1 and 4 is 8.658

(c) Find $P(x \geq 6)$

Note that in this case $P(x \geq 6)$ is equivalent to $P(6 \leq X \leq 10)$ since 10 is the largest value that X can be. So the probability that X is greater than or equal to 6 is,

$$\begin{aligned}P(X \geq 6) &= \int_6^{10} \frac{x^3}{5000}(10-x)\, dx \\ &= \left(\frac{x^4}{2000} - \frac{x^5}{25000}\right)\Bigg|_6^{10} \\ &= 0.66304\end{aligned}$$

This probability is then 66.304%.

Probability density functions can also be used to determine the mean of a continuous random variable. The mean is given by,

Fact

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

Let's work one more example.

Example 2

It has been determined that the probability density function for the wait in line at a counter is given by,

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 0.1\mathbf{e}^{-\frac{t}{10}} & \text{if } t \geq 0 \end{cases}$$

(a) Verify that this is in fact a probability density function.

(b) Determine the probability that a person will wait in line for at least 6 minutes.

(c) Determine the mean wait in line.

Solution

(a) Verify that this is in fact a probability density function.

This function is clearly positive or zero and so there's not much to do here other than compute the integral.

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\, dt &= \int_{0}^{\infty} 0.1\mathbf{e}^{-\frac{t}{10}}\, dt \\ &= \lim_{u \to \infty} \int_{0}^{u} 0.1\mathbf{e}^{-\frac{t}{10}}\, dt \\ &= \lim_{u \to \infty} \left(-\mathbf{e}^{-\frac{t}{10}}\right)\Big|_{0}^{u} \\ &= \lim_{u \to \infty} \left(1 - \mathbf{e}^{-\frac{u}{10}}\right) = 1 \end{aligned}$$

So it is a probability density function.

(b) Determine the probability that a person will wait in line for at least 6 minutes.

The probability that we're looking for here is $P(X \geq 6)$.

$$\begin{aligned} P(X \geq 6) &= \int_6^\infty 0.1\mathbf{e}^{-\frac{t}{10}}\,dt \\ &= \lim_{u\to\infty} \int_6^u 0.1\mathbf{e}^{-\frac{t}{10}}\,dt \\ &= \lim_{u\to\infty} \left(-\mathbf{e}^{-\frac{t}{10}}\right)\Big|_6^u \\ &= \lim_{u\to\infty} \left(\mathbf{e}^{-\frac{6}{10}} - \mathbf{e}^{-\frac{u}{10}}\right) = \mathbf{e}^{-\frac{3}{5}} = 0.548812 \end{aligned}$$

So the probability that a person will wait in line for more than 6 minutes is 54.8811%.

(c) Determine the mean wait in line.

Here's the mean wait time.

$$\begin{aligned} \mu &= \int_{-\infty}^\infty t\,f(t)\,dt \\ &= \int_0^\infty 0.1t\,\mathbf{e}^{-\frac{t}{10}}\,dt \\ &= \lim_{u\to\infty} \int_0^u 0.1t\,\mathbf{e}^{-\frac{t}{10}}\,dt \qquad \text{integrating by parts....} \\ &= \lim_{u\to\infty} \left(-(t+10)\,\mathbf{e}^{-\frac{t}{10}}\right)\Big|_0^u \\ &= \lim_{u\to\infty} \left(10 - (u+10)\,\mathbf{e}^{-\frac{u}{10}}\right) = 10 \end{aligned}$$

So, it looks like the average wait time is 10 minutes.

9 Parametric Equations and Polar Coordinates

We are now going to take a look at a couple of topics that are completely different from anything we've seen to this point. That does not mean, however, that we can just forget everything that we've seen to this point. As we will see before too long we will still need to be able to do a large part of the material (both Calculus I and Calculus II material) that we've looked at to this point.

The first major topic that we'll look at in this chapter will be that of Parametric Equations. Parametric Equations will allow us to work with and perform Calculus operations on equations that cannot be (easily) solved into the form $y = f(x)$ or $x = h(y)$ (assuming we are using x and y as our variables). Also, as we'll see we can write some equations that can be solved for y or x as a set of easier to work with parametric equations.

Once we've got an idea of what parametric equations are and how to sketch graphs of them we will revisit some of the Calculus topics we've looked at to this point. Specifically we'll take a look at how to use only parametric equations to get the equation of tangent lines, where the graph is increasing/decreasing and the concavity of the graph. In addition, we'll revisit the idea of using a definite integral to find the area between the graph of a set of parametric equation and the x-axis. We will close out the Calculus topics by discussing arc length and surface area for a set of parametric equations.

We will then move into the other major topic of this chapter, namely Polar Coordinates. Once we've defined polar coordinates and gotten comfortable with them we will, again, revisit the same Calculus topics we looked at in terms of parametric equations only now we will look at how to work them in terms of polar coordinates.

On the surface it will appear that polar coordinates has nothing in common with parametric equations. We will see however that several topics in Polar Coordinates can be easily done, in some way, if we first set them up in terms of parametric equations.

In addition, we should point out that the purpose of the topics in this chapter is in preparation for multi-variable Calculus (*i.e.* the material that is usually taught in Calculus III). As we will see when we get to that point there are a lot of topics that involve and/or require parametric equations. In addition, polar coordinates will pop up every so often so keep that in mind as we go through this stuff. It is easy sometimes to get the idea that the topics in this chapter don't have a lot of use but once we hit multi-variable Calculus they will start to pop up with some regularity.

9.1 Parametric Equations and Curves

To this point (in both Calculus I and Calculus II) we've looked almost exclusively at functions in the form $y = f(x)$ or $x = h(y)$ and almost all of the formulas that we've developed require that functions be in one of these two forms. The problem is that not all curves or equations that we'd like to look at fall easily into this form.

Take, for example, a circle. It is easy enough to write down the equation of a circle centered at the origin with radius r.

$$x^2 + y^2 = r^2$$

However, we will never be able to write the equation of a circle down as a single equation in either of the forms above. Sure we can solve for x or y as the following two formulas show

$$y = \pm\sqrt{r^2 - x^2} \qquad\qquad x = \pm\sqrt{r^2 - y^2}$$

but there are in fact two functions in each of these. Each formula gives a portion of the circle.

$$\begin{array}{llll} y = \sqrt{r^2 - x^2} & \text{(top)} & x = \sqrt{r^2 - y^2} & \text{(right side)} \\ y = -\sqrt{r^2 - x^2} & \text{(bottom)} & x = -\sqrt{r^2 - y^2} & \text{(left side)} \end{array}$$

Unfortunately, we usually are working on the whole circle, or simply can't say that we're going to be working only on one portion of it. Even if we can narrow things down to only one of these portions the function is still often fairly unpleasant to work with.

There are also a great many curves out there that we can't even write down as a single equation in terms of only x and y. So, to deal with some of these problems we introduce **parametric equations**. Instead of defining y in terms of x ($y = f(x)$) or x in terms of y ($x = h(y)$) we define both x and y in terms of a third variable called a parameter as follows,

$$x = f(t) \qquad\qquad y = g(t)$$

This third variable is usually denoted by t (as we did here) but doesn't have to be of course. Sometimes we will restrict the values of t that we'll use and at other times we won't. This will often be dependent on the problem and just what we are attempting to do.

Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called the **parametric curve**.

To help visualize just what a parametric curve is pretend that we have a big tank of water that is in constant motion and we drop a ping pong ball into the tank. The point $(x, y) = (f(t), g(t))$ will then represent the location of the ping pong ball in the tank at time t and the parametric curve will be a trace of all the locations of the ping pong ball. Note that this is not always a correct analogy but it is useful initially to help visualize just what a parametric curve is.

Sketching a parametric curve is not always an easy thing to do. Let's take a look at an example to see one way of sketching a parametric curve. This example will also illustrate why this method is usually not the best.

Example 1

Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1$$

Solution

At this point our only option for sketching a parametric curve is to pick values of t, plug them into the parametric equations and then plot the points. So, let's plug in some t's.

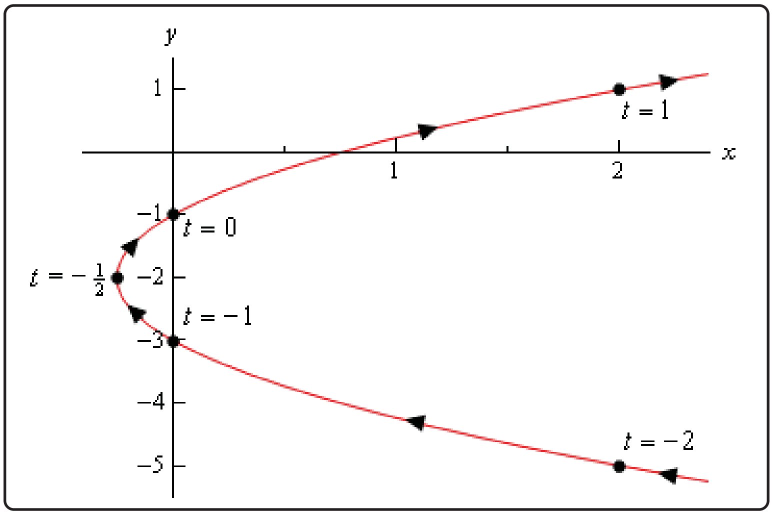
t	x	y
-2	2	-5
-1	0	-3
$-\frac{1}{2}$	$-\frac{1}{4}$	-2
0	0	-1
1	2	1

The first question that should be asked at this point is, how did we know to use the values of t that we did, especially the third choice? Unfortunately, there is no real answer to this question at this point. We simply pick t's until we are fairly confident that we've got a good idea of what the curve looks like. It is this problem with picking "good" values of t that make this method of sketching parametric curves one of the poorer choices. Sometimes we have no choice, but if we do have a choice we should avoid it.

We'll discuss an alternate graphing method in later examples that will help to explain how these values of t were chosen.

We have one more idea to discuss before we actually sketch the curve. Parametric curves have a **direction of motion**. The direction of motion is given by increasing t. So, when plotting parametric curves, we also include arrows that show the direction of motion. We will often give the value of t that gave specific points on the graph as well to make it clear the value of t that gave that particular point.

Here is the sketch of this parametric curve.



So, it looks like we have a parabola that opens to the right.

Before we end this example there is a somewhat important and subtle point that we need to discuss first. Notice that we made sure to include a portion of the sketch to the right of the points corresponding to $t = -2$ and $t = 1$ to indicate that there are portions of the sketch there. Had we simply stopped the sketch at those points we are indicating that there was no portion of the curve to the right of those points and there clearly will be. We just didn't compute any of those points.

This may seem like an unimportant point, but as we'll see in the next example it's more important than we might think.

Before addressing a much easier way to sketch this graph let's first address the issue of limits on the parameter. In the previous example we didn't have any limits on the parameter. Without limits on the parameter the graph will continue in both directions as shown in the sketch above.

We will often have limits on the parameter however and this will affect the sketch of the parametric equations. To see this effect let's look a slight variation of the previous example.

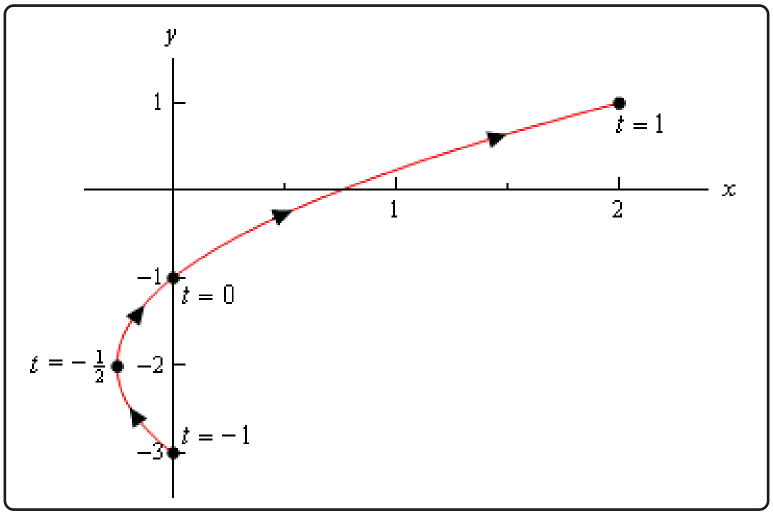
Example 2

Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \qquad -1 \le t \le 1$$

Solution

Note that the only difference here is the presence of the limits on t. All these limits do is tell us that we can't take any value of t outside of this range. Therefore, the parametric curve will only be a portion of the curve above. Here is the parametric curve for this example.



Notice that with this sketch we started and stopped the sketch right on the points originating from the end points of the range of t's. Contrast this with the sketch in the previous example where we had a portion of the sketch to the right of the "start" and "end" points that we computed.

In this case the curve starts at $t = -1$ and ends at $t = 1$, whereas in the previous example the curve didn't really start at the right most points that we computed. We need to be clear in our sketches if the curve starts/ends right at a point, or if that point was simply the first/last one that we computed.

It is now time to take a look at an easier method of sketching this parametric curve. This method uses the fact that in many, but not all, cases we can actually eliminate the parameter from the parametric equations and get a function involving only x and y. We will sometimes call this the *algebraic equation* to differentiate it from the original parametric equations. There will be two small problems with this method, but it will be easy to address those problems. It is important to note however that we won't always be able to do this.

Just how we eliminate the parameter will depend upon the parametric equations that we've got. Let's see how to eliminate the parameter for the set of parametric equations that we've been working with to this point.

Example 3

Eliminate the parameter from the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1$$

Solution

One of the easiest ways to eliminate the parameter is to simply solve one of the equations for the parameter (t, in this case) and substitute that into the other equation. Note that while this may be the easiest to eliminate the parameter, it's usually not the best way as we'll see soon enough.

In this case we can easily solve y for t.

$$t = \frac{1}{2}(y+1)$$

Plugging this into the equation for x gives the following algebraic equation,

$$x = \left(\frac{1}{2}(y+1)\right)^2 + \frac{1}{2}(y+1) = \frac{1}{4}y^2 + y + \frac{3}{4}$$

Sure enough from our Algebra knowledge we can see that this is a parabola that opens to the right and will have a vertex at $\left(-\frac{1}{4}, -2\right)$.

We won't bother with a sketch for this one as we've already sketched this once and the point here was more to eliminate the parameter anyway.

Before we leave this example let's address one quick issue.

In the first example we just, seemingly randomly, picked values of t to use in our table, especially the third value. There really was no apparent reason for choosing $t = -\frac{1}{2}$. It is however probably the most important choice of t as it is the one that gives the vertex.

The reality is that when writing this material up we actually did this problem first then went back and did the first problem. Plotting points is generally the way most people first learn how to construct graphs and it does illustrate some important concepts, such as direction, so it made sense to do that first in the notes. In practice however, this example is often done first.

So, how did we get those values of t? Well let's start off with the vertex as that is probably the most important point on the graph. We have the x and y coordinates of the vertex and we also have x and y parametric equations for those coordinates. So, plug in the coordinates for the vertex into the parametric equations and solve for t. Doing this gives,

$$\begin{array}{l} -\frac{1}{4} = t^2 + t \\ -2 = 2t - 1 \end{array} \quad \Rightarrow \quad \begin{array}{l} t = -\frac{1}{2} \quad \text{(double root)} \\ t = -\frac{1}{2} \end{array}$$

So, as we can see, the value of t that will give both of these coordinates is $t = -\frac{1}{2}$. Note that the x parametric equation gave a double root and this will often not happen. Often we would have gotten two distinct roots from that equation. In fact, it won't be unusual to get multiple values of t from each of the equations.

However, what we can say is that there will be a value(s) of t that occurs in both sets of solutions and that is the t that we want for that point. We'll eventually see an example where this happens in a later section.

Now, from this work we can see that if we use $t = -\frac{1}{2}$ we will get the vertex and so we included that value of t in the table in Example 1. Once we had that value of t we chose two integer values of t on either side to finish out the table.

As we will see in later examples in this section determining values of t that will give specific points is something that we'll need to do on a fairly regular basis. It is fairly simple however as this example has shown. All we need to be able to do is solve a (usually) fairly basic equation which by this point in time shouldn't be too difficult.

Getting a sketch of the parametric curve once we've eliminated the parameter seems fairly simple. All we need to do is graph the equation that we found by eliminating the parameter. As noted already however, there are two small problems with this method. The first is direction of motion. The equation involving only x and y will NOT give the direction of motion of the parametric curve. This is generally an easy problem to fix however. Let's take a quick look at the derivatives of the parametric equations from the last example. They are,

$$\frac{dx}{dt} = 2t + 1$$
$$\frac{dy}{dt} = 2$$

Now, all we need to do is recall our Calculus I knowledge. The derivative of y with respect to t is clearly always positive. Recalling that one of the interpretations of the first derivative is rate of change we now know that as t increases y must also increase. Therefore, we must be moving up the curve from bottom to top as t increases as that is the only direction that will always give an increasing y as t increases.

Note that the x derivative isn't as useful for this analysis as it will be both positive and negative and hence x will be both increasing and decreasing depending on the value of t. That doesn't help with direction much as following the curve in either direction will exhibit both increasing and decreasing x.

In some cases, only one of the equations, such as this example, will give the direction while in other cases either one could be used. It is also possible that, in some cases, both derivatives would be needed to determine direction. It will always be dependent on the individual set of parametric equations.

The second problem with eliminating the parameter is best illustrated in an example as we'll be running into this problem in the remaining examples.

Example 4

Sketch the parametric curve for the following set of parametric equations. Clearly indicate direction of motion.

$$x = 5\cos(t) \qquad y = 2\sin(t) \qquad 0 \le t \le 2\pi$$

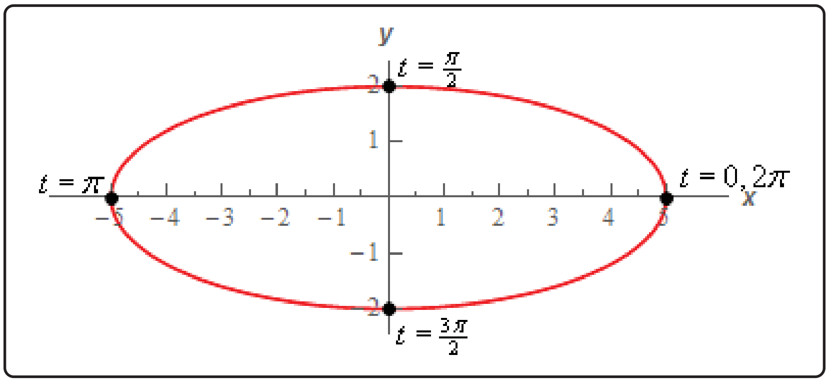
Solution

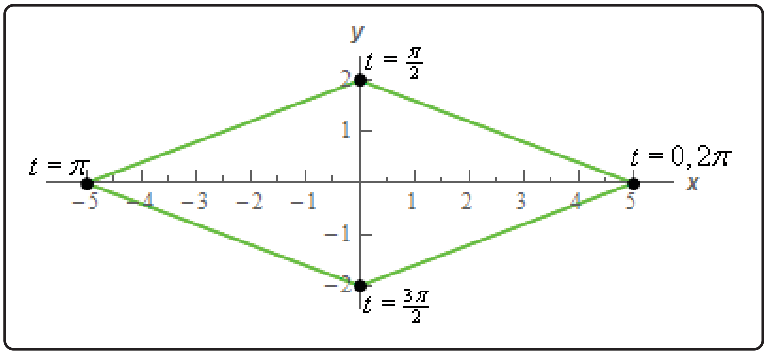
Before we proceed with eliminating the parameter for this problem let's first address again why just picking t's and plotting points is not really a good idea.

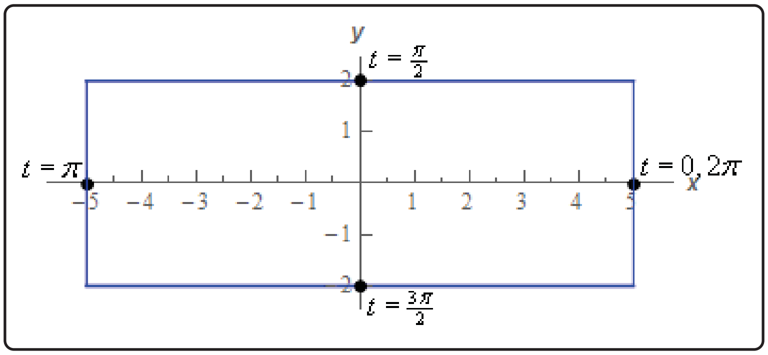
Given the range of t's in the problem statement let's use the following set of t's.

t	x	y
0	5	0
$\frac{\pi}{2}$	0	2
π	-5	0
$\frac{3\pi}{2}$	0	-2
2π	5	0

The question that we need to ask now is do we have enough points to accurately sketch the graph of this set of parametric equations? Below are some sketches of some possible graphs of the parametric equation based only on these five points.




y
x
$t = \frac{\pi}{2}$
$t = \pi$
$t = 0, 2\pi$
$t = \frac{3\pi}{2}$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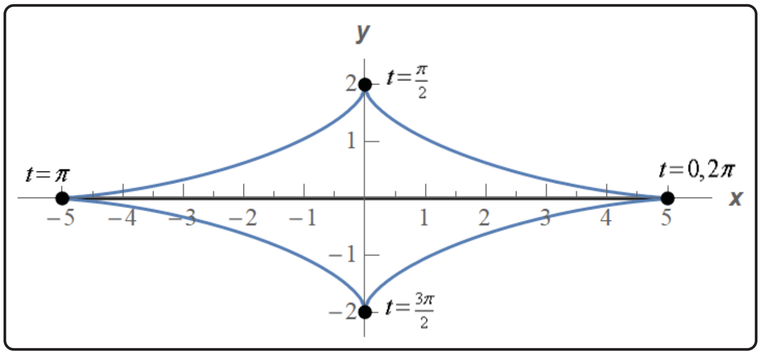
y
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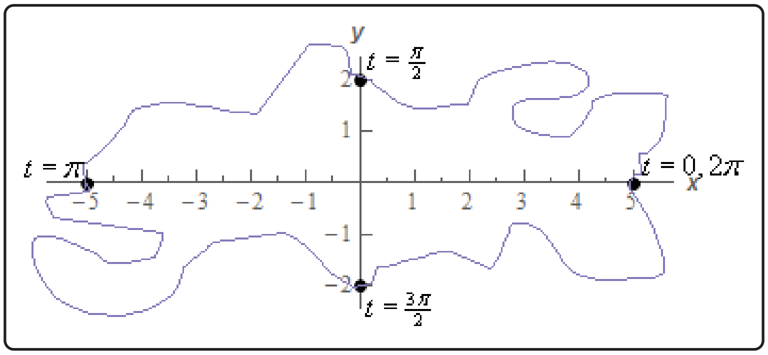
$t=\frac{\pi}{2}$
$t=\pi$
$t=0,2\pi$
$t=\frac{3\pi}{2}$


y
x
$t = \frac{\pi}{2}$
$t = \pi$
$t = 0, 2\pi$
$t = \frac{3\pi}{2}$

Given the nature of sine/cosine you might be tempted to eliminate the diamond and the square but there is no denying that they are graphs that go through the given points. The first and fourth graphs both have some curvature to them and so you might be tempted to assume that one of those is the correct one given the sine/cosine in the equations. The last graph is also a little silly but it does show a graph going through the given points.

Again, given the nature of sine/cosine you are probably guessing that the correct graph is the the first or third graph. However, that is all that would be at this point. A guess. Nothing actually says unequivocally that the parametric curve is an will be one of those two just from those five points. That is the danger of sketching parametric curves based on a handful of points. Unless we know what the graph will be ahead of time we are really just making a guess.

It is important to note at this point that it is very easy to construct a set of parametric equations both containing sines and/or cosines and yet have the graph not have any curvature at all. You can often make some guesses as to the shape of the curve from the parametric equations but you won't always guess correctly unfortunately. Care must be taken when graphing parametric equations to not take the behavior of the individual parametric equations and just assume that behavior will translate to the curve of the set of parametric equations.

Also, in general, we should avoid plotting points to sketch parametric curves as that will, on occasion, lead to incorrect graphs. The best method, provided it can be done, is to eliminate the parameter. As noted just prior to starting this example there is still a potential problem with eliminating the parameter that we'll need to deal with. We will eventually discuss this issue. For now, let's just proceed with eliminating the parameter.

We'll start by eliminating the parameter as we did in the previous section. We'll solve one of the of the equations for t and plug this into the other equation. For example, we could do the following,

$$t = \cos^{-1}\left(\frac{x}{5}\right) \qquad \Rightarrow \qquad y = 2\sin\left(\cos^{-1}\left(\frac{x}{5}\right)\right)$$

Can you see the problem with doing this? This is definitely easy to do but we have a greater chance of correctly graphing the original parametric equations by plotting points than we do graphing this!

There are many ways to eliminate the parameter from the parametric equations and solving for t is usually not the best way to do it. While it is often easy to do we will, in most cases, end up with an equation that is almost impossible to deal with.

So, how can we eliminate the parameter here? In this case all we need to do is recall a very nice trig identity and the equation of an ellipse. Recall,

$$\cos^2(t) + \sin^2(t) = 1$$

Then from the parametric equations we get,

$$\cos(t) = \frac{x}{5} \qquad \sin(t) = \frac{y}{2}$$

Then, using the trig identity from above and these equations we get,

$$1 = \cos^2(t) + \sin^2(t) = \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = \frac{x^2}{25} + \frac{y^2}{4}$$

So we now know that we will have an ellipse.

Now, let's continue on with the example. We've identified that the parametric equations describe an ellipse, but we can't just sketch an ellipse and be done with it.

First, just because the algebraic equation was an ellipse doesn't actually mean that the parametric curve is the full ellipse. It is always possible that the parametric curve is only a portion of the ellipse. In order to identify just how much of the ellipse the parametric curve will cover let's go back to the parametric equations and see what they tell us about any limits on x and y. Based on our knowledge of sine and cosine we have the following,

$$\begin{array}{lllll} -1 \le \cos(t) \le 1 & \Rightarrow & -5 \le 5\cos(t) \le 5 & \Rightarrow & -5 \le x \le 5 \\ -1 \le \sin(t) \le 1 & \Rightarrow & -2 \le 2\sin(t) \le 2 & \Rightarrow & -2 \le y \le 2 \end{array}$$

So, by starting with sine/cosine and "building up" the equation for x and y using basic algebraic manipulations we get that the parametric equations enforce the above limits on x and y. In this case, these also happen to be the full limits on x and y we get by graphing the full ellipse.

This is the second potential issue alluded to above. The parametric curve may not always trace out the full graph of the algebraic curve. We should always find limits on x and y enforced upon us by the parametric curve to determine just how much of the algebraic curve is actually sketched out by the parametric equations.

Therefore, in this case, we now know that we get a full ellipse from the parametric equations. Before we proceed with the rest of the example be careful to not always just assume we will get the full graph of the algebraic equation. There are definitely times when we will not get the full graph and we'll need to do a similar analysis to determine just how much of the graph we actually get. We'll see an example of this later.

Note as well that any limits on t given in the problem statement can also affect how much of the graph of the algebraic equation we get. In this case however, based on the table of values we computed at the start of the problem we can see that we do indeed get the full ellipse in the range $0 \le t \le 2\pi$. That won't always be the case however, so pay attention to any restrictions on t that might exist!

Next, we need to determine a direction of motion for the parametric curve. Recall that all parametric curves have a direction of motion and the equation of the ellipse simply tells us nothing about the direction of motion.

To get the direction of motion it is tempting to just use the table of values we computed above to get the direction of motion. In this case, we would guess (and yes that is all it is - a guess) that the curve traces out in a counter-clockwise direction. We'd be correct. In this case, we'd be correct! The problem is that tables of values can be misleading when determining a direction of motion as we'll see in the next example.

Therefore, it is best to not use a table of values to determine the direction of motion. To correctly determine the direction of motion we'll use the same method of determining the direction that we discussed after Example 3. In other words, we'll take the derivative of the parametric equations and use our knowledge of Calculus I and trig to determine the direction of motion.

The derivatives of the parametric equations are,

$$\frac{dx}{dt} = -5\sin(t) \qquad\qquad \frac{dy}{dt} = 2\cos(t)$$

Now, at $t = 0$ we are at the point $(5, 0)$ and let's see what happens if we start increasing t. Let's increase t from $t = 0$ to $t = \frac{\pi}{2}$. In this range of t's we know that sine is always positive and so from the derivative of the x equation we can see that x must be decreasing in this range of t's.

This, however, doesn't really help us determine a direction for the parametric curve. Starting at $(5, 0)$ no matter if we move in a clockwise or counter-clockwise direction x will have to decrease so we haven't really learned anything from the x derivative.

The derivative from the y parametric equation on the other hand will help us. Again, as we increase t from $t = 0$ to $t = \frac{\pi}{2}$ we know that cosine will be positive and so y must be increasing in this range. That however, can only happen if we are moving in a counter-clockwise direction. If we were moving in a clockwise direction from the point $(5, 0)$ we can see that y would have to decrease!

Therefore, in the first quadrant we must be moving in a counter-clockwise direction. Let's move on to the second quadrant.

So, we are now at the point $(0, 2)$ and we will increase t from $t = \frac{\pi}{2}$ to $t = \pi$. In this range of t we know that cosine will be negative and sine will be positive. Therefore, from the derivatives of the parametric equations we can see that x is still decreasing and y will now be decreasing as well.

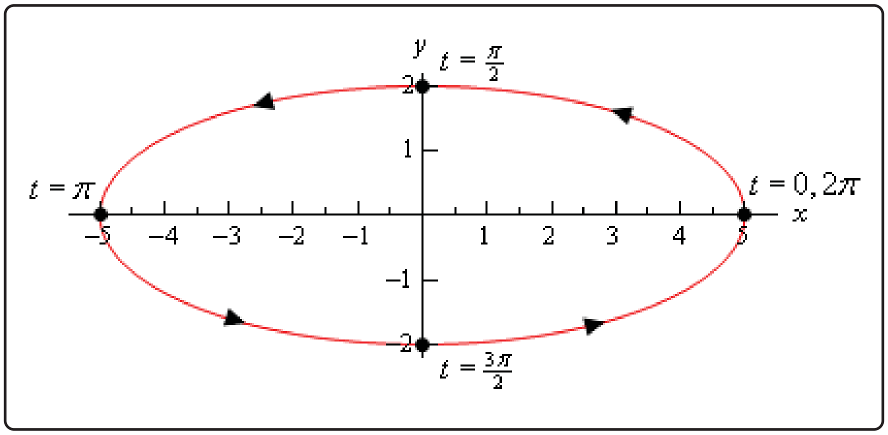
In this quadrant the y derivative tells us nothing as y simply must decrease to move from $(0, 2)$. However, in order for x to decrease, as we know it does in this quadrant, the direction must still be moving a counter-clockwise rotation.

We are now at $(-5, 0)$ and we will increase t from $t = \pi$ to $t = \frac{3\pi}{2}$. In this range of t we know that cosine is negative (and hence y will be decreasing) and sine is also negative (and hence x will be increasing). Therefore, we will continue to move in a counter-clockwise motion.

For the 4^{th} quadrant we will start at $(0, -2)$ and increase t from $t = \frac{3\pi}{2}$ to $t = 2\pi$. In this range of t we know that cosine is positive (and hence y will be increasing) and sine is negative (and hence x will be increasing). So, as in the previous three quadrants, we continue to move in a counter-clockwise motion.

At this point we covered the range of t's we were given in the problem statement and during the full range the motion was in a counter-clockwise direction.

We can now fully sketch the parametric curve so, here is the sketch.



Okay, that was a really long example. Most of these types of problems aren't as long. We just had a lot to discuss in this one so we could get a couple of important ideas out of the way. The rest of the examples in this section shouldn't take as long to go through.

Now, let's take a look at another example that will illustrate an important idea about parametric equations.

Example 5

Sketch the parametric curve for the following set of parametric equations. Clearly indicate direction of motion.

$$x = 5\cos(3t) \qquad y = 2\sin(3t) \qquad 0 \le t \le 2\pi$$

Solution

Note that the only difference in between these parametric equations and those in Example 4 is that we replaced the t with 3 t. We can eliminate the parameter here in the same manner as we did in the previous example.

$$\cos(3t) = \frac{x}{5} \qquad \sin(3t) = \frac{y}{2}$$

We then get,

$$1 = \cos^2(3t) + \sin^2(3t) = \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = \frac{x^2}{25} + \frac{y^2}{4}$$

So, we get the same ellipse that we did in the previous example. Also note that we can do the same analysis on the parametric equations to determine that we have exactly the same limits on x and y. Namely,

$$-5 \le x \le 5 \qquad\qquad -2 \le y \le 2$$

It's starting to look like changing the t into a $3t$ in the trig equations will not change the parametric curve in any way. That is not correct however. The curve does change in a small but important way which we will be discussing shortly.

Before discussing that small change the $3t$ brings to the curve let's discuss the direction of motion for this curve. Despite the fact that we said in the last example that picking values of t and plugging in to the equations to find points to plot is a bad idea let's do it any way.

Given the range of t's from the problem statement the following set looks like a good choice of t's to use.

t	x	y
0	5	0
$\frac{\pi}{2}$	0	-2
π	-5	0
$\frac{3\pi}{2}$	0	2
2π	5	0

So, the only change to this table of values/points from the last example is all the nonzero y values changed sign. From a quick glance at the values in this table it would look like the curve, in this case, is moving in a clockwise direction. But is that correct? Recall we said that these tables of values can be misleading when used to determine direction and that's why we don't use them.

Let's see if our first impression is correct. We can check our first impression by doing the derivative work to get the correct direction. Let's work with just the y parametric equation as the x will have the same issue that it had in the previous example. The derivative of the y parametric equation is,

$$\frac{dy}{dt} = 6\cos(3t)$$

Now, if we start at $t = 0$ as we did in the previous example and start increasing t. At $t = 0$ the derivative is clearly positive and so increasing t (at least initially) will force y to also be increasing. The only way for this to happen is if the curve is in fact tracing out in a counter-clockwise direction initially.

Now, we could continue to look at what happens as we further increase t, but when dealing with a parametric curve that is a full ellipse (as this one is) and the argument of the trig functions is of the form *nt* for any constant n the direction will not change so once we know the initial direction we know that it will always move in that direction. Note that this is only true for parametric equations in the form that we have here. We'll see in later examples that for different kinds of parametric equations this may no longer be true.

Okay, from this analysis we can see that the curve must be traced out in a counter-clockwise direction. This is directly counter to our guess from the tables of values above and so we can see that, in this case, the table would probably have led us to the wrong direction. So, once again, tables are generally not very reliable for getting pretty much any real information about a parametric curve other than a few points that must be on the curve. Outside of that the tables are rarely useful and will generally not be dealt with in further examples.

So, why did our table give an incorrect impression about the direction? Well recall that we mentioned earlier that the $3t$ will lead to a small but important change to the curve versus just a t? Let's take a look at just what that change is as it will also answer what "went wrong" with our table of values.

Let's start by looking at $t = 0$. At $t = 0$ we are at the point $(5, 0)$ and let's ask ourselves what values of t put us back at this point. We saw in Example 3 how to determine value(s) of t that put us at certain points and the same process will work here with a minor modification.

Instead of looking at both the x and y equations as we did in that example let's just look at the x equation. The reason for this is that we'll note that there are two points on the ellipse that will have a y coordinate of zero, $(5, 0)$ and $(-5, 0)$. If we set the y coordinate equal to zero we'll find all the t's that are at both of these points when we only want the values of t that are at $(5, 0)$.

So, because the x coordinate of five will only occur at this point we can simply use the x parametric equation to determine the values of t that will put us at this point. Doing this gives the following equation and solution,

$$\begin{aligned} 5 &= 5\cos(3t) \\ 3t &= \cos^{-1}(1) = 0 + 2\pi n \qquad \to \qquad t = \frac{2}{3}\pi n \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \end{aligned}$$

Don't forget that when solving a trig equation we need to add on the "$+2\pi n$" where n represents the number of full revolutions in the counter-clockwise direction (positive n) and clockwise direction (negative n) that we rotate from the first solution to get all possible solutions to the equation.

Now, let's plug in a few values of n starting at $n = 0$. We don't need negative n in this case since all of those would result in negative t and those fall outside of the range of t's we were given in the problem statement. The first few values of t are then,

$$\begin{aligned} n = 0 \quad &: \quad t = 0 \\ n = 1 \quad &: \quad t = \frac{2\pi}{3} \\ n = 2 \quad &: \quad t = \frac{4\pi}{3} \\ n = 3 \quad &: \quad t = \frac{6\pi}{3} = 2\pi \end{aligned}$$

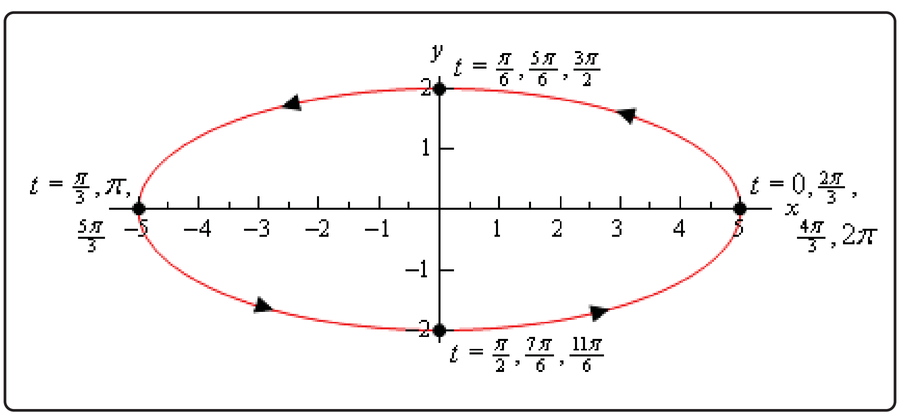
We can stop here as all further values of t will be outside the range of t's given in this problem.

So, what is this telling us? Well back in Example 4 when the argument was just t the ellipse was traced out exactly once in the range $0 \leq t \leq 2\pi$. However, when we change the argument to 3 t (and recalling that the curve will always be traced out in a counter-clockwise direction for this problem) we are going through the "starting" point of $(5, 0)$ two more times than we did in the previous example.

In fact, this curve is tracing out three separate times. The first trace is completed in the range $0 \leq t \leq \frac{2\pi}{3}$. The second trace is completed in the range $\frac{2\pi}{3} \leq t \leq \frac{4\pi}{3}$ and the third and final trace is completed in the range $\frac{4\pi}{3} \leq t \leq 2\pi$. In other words, changing the argument from t to 3 t increase the speed of the trace and the curve will now trace out three times in the range $0 \leq t \leq 2\pi$!

This is why the table gives the wrong impression. The speed of the tracing has increased leading to an incorrect impression from the points in the table. The table seems to suggest that between each pair of values of t a quarter of the ellipse is traced out in the clockwise direction when in reality it is tracing out three quarters of the ellipse in the counter-clockwise direction.

Here's a final sketch of the curve and note that it really isn't all that different from the previous sketch. The only differences are the values of t and the various points we included. We did include a few more values of t at various points just to illustrate where the curve is at for various values of t but in general these really aren't needed.



So, we saw in the last two examples two sets of parametric equations that in some way gave the same graph. Yet, because they traced out the graph a different number of times we really do need to think of them as different parametric curves at least in some manner. This may seem like a difference that we don't need to worry about, but as we will see in later sections this can be a very important difference. In some of the later sections we are going to need a curve that is traced out exactly once.

Before we move on to other problems let's briefly acknowledge what happens by changing the t to an nt in these kinds of parametric equations. When we are dealing with parametric equations involving only sines and cosines and they both have the same argument if we change the argument from t to nt we simply change the speed with which the curve is traced out. If $n > 1$ we will increase the speed and if $n < 1$ we will decrease the speed.

Let's take a look at a couple more examples.

Example 6

Sketch the parametric curve for the following set of parametric equations. Clearly identify the direction of motion. If the curve is traced out more than once give a range of the parameter for which the curve will trace out exactly once.

$$x = \sin^2(t) \qquad y = 2\cos(t)$$

Solution

We can eliminate the parameter much as we did in the previous two examples. However, we'll need to note that the x already contains a $\sin^2(t)$ and so we won't need to square the x. We will however, need to square the y as we need in the previous two examples.

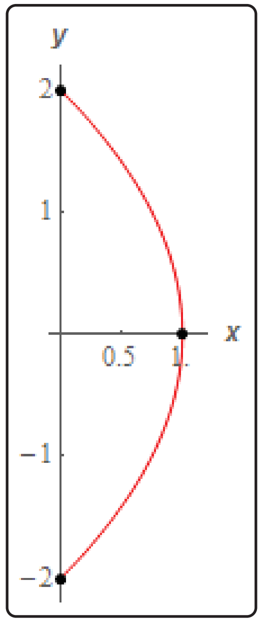
$$x + \frac{y^2}{4} = \sin^2(t) + \cos^2(t) = 1 \qquad \Rightarrow \qquad x = 1 - \frac{y^2}{4}$$

In this case the algebraic equation is a parabola that opens to the left.

We will need to be very, very careful however in sketching this parametric curve. We will NOT get the whole parabola. A sketch of the algebraic form parabola will exist for all possible values of y. However, the parametric equations have defined both x and y in terms of sine and cosine and we know that the ranges of these are limited and so we won't get all possible values of x and y here. So, first let's get limits on x and y as we did in previous examples. Doing this gives,

$$\begin{array}{lclcl} -1 \le \sin(t) \le 1 & \Rightarrow & 0 \le \sin^2(t) \le 1 & \Rightarrow & 0 \le x \le 1 \\ -1 \le \cos(t) \le 1 & \Rightarrow & -2 \le 2\cos(t) \le 2 & \Rightarrow & -2 \le y \le 2 \end{array}$$

So, it is clear from this that we will only get a portion of the parabola that is defined by the algebraic equation. Below is a quick sketch of the portion of the parabola that the parametric curve will cover.



To finish the sketch of the parametric curve we also need the direction of motion for the curve. Before we get to that however, let's jump forward and determine the range of t's for one trace. To do this we'll need to know the t's that put us at each end point and we can follow the same procedure we used in the previous example. The only difference is this time let's use the y parametric equation instead of the x because the y coordinates of the two end points of the curve are different whereas the x coordinates are the same.

So, for the top point we have,

$$\begin{aligned} 2 &= 2\cos(t) \\ t &= \cos^{-1}(1) = 0 + 2\pi n = 2\pi n, \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots \end{aligned}$$

For, plugging in some values of n we get that the curve will be at the top point at,

$$t = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$$

Similarly, for the bottom point we have,

$$\begin{aligned} -2 &= 2\cos(t) \\ t &= \cos^{-1}(-1) = \pi + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots \end{aligned}$$

So, we see that we will be at the bottom point at,

$$t = \ldots, -3\pi, -\pi, \pi, 3\pi, \ldots$$

So, if we start at say, $t = 0$, we are at the top point and we increase t we have to move along the curve downwards until we reach $t = \pi$ at which point we are now at the bottom point. This means that we will trace out the curve exactly once in the range $0 \le t \le \pi$.

This is not the only range that will trace out the curve however. Note that if we further increase t from $t = \pi$ we will now have to travel back up the curve until we reach $t = 2\pi$ and we are now back at the top point. Increasing t again until we reach $t = 3\pi$ will take us back down the curve until we reach the bottom point again, *etc*. From this analysis we can get two more ranges of t for one trace,

$$\pi \le t \le 2\pi \qquad\qquad 2\pi \le t \le 3\pi$$

As you can probably see there are an infinite number of ranges of t we could use for one trace of the curve. Any of them would be acceptable answers for this problem.

Note that in the process of determining a range of t's for one trace we also managed to determine the direction of motion for this curve. In the range $0 \le t \le \pi$ we had to travel downwards along the curve to get from the top point at $t = 0$ to the bottom point at $t = \pi$. However, at $t = 2\pi$ we are back at the top point on the curve and to get there we must travel along the path. We can't just jump back up to the top point or take a different path to get there. All travel must be done on the path sketched out. This means that we had to go back up the path. Further increasing t takes us back down the path, then up the path again *etc*.

In other words, this path is sketched out in both directions because we are not putting any restrictions on the t's and so we have to assume we are using all possible values of t. If we had put restrictions on which t's to use we might really have ended up only moving in one direction. That however would be a result only of the range of t's we are using and not the parametric equations themselves.

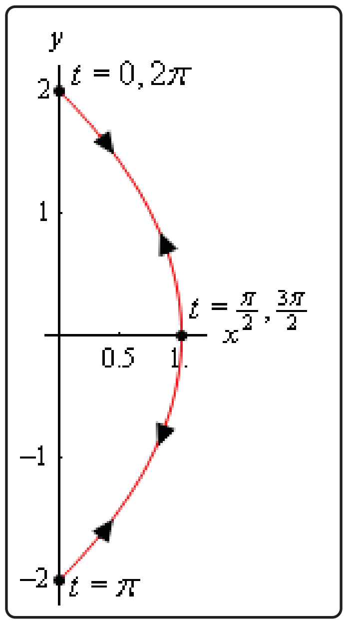
Note that we didn't really need to do the above work to determine that the curve traces out in both directions.in this case. Both the x and y parametric equations involve sine or cosine and we know both of those functions oscillate. This, in turn means that both x and y will oscillate as well. The only way for that to happen on this particular this curve will be for the curve to be traced out in both directions.

Be careful with the above reasoning that the oscillatory nature of sine/cosine forces the curve to be traced out in both directions. It can only be used in this example because the "starting" point and "ending" point of the curves are in different places. The only way to get from one of the "end" points on the curve to the other is to travel back along the curve in the opposite direction.

Contrast this with the ellipse in Example 4. In that case we had sine/cosine in the parametric equations as well. However, the curve only traced out in one direction, not in both directions. In Example 4 we were graphing the full ellipse and so no matter where we start sketching the graph we will eventually get back to the "starting" point without ever retracing any portion of the graph. In Example 4 as we trace out the full ellipse both x and y do in fact oscillate between their two "endpoints" but the curve itself does not trace out in both directions for this to happen.

Basically, we can only use the oscillatory nature of sine/cosine to determine that the curve traces out in both directions if the curve starts and ends at different points. If the starting/ending point is the same then we generally need to go through the full derivative argument to determine the actual direction of motion.

So, to finish this problem out, below is a sketch of the parametric curve. Note that we put direction arrows in both directions to clearly indicate that it would be traced out in both directions. We also put in a few values of t just to help illustrate the direction of motion.



To this point we've seen examples that would trace out the complete graph that we got by eliminating the parameter if we took a large enough range of t's. However, in the previous example we've now seen that this will not always be the case. It is more than possible to have a set of parametric equations which will continuously trace out just a portion of the curve. We can usually determine if this will happen by looking for limits on x and y that are imposed up us by the parametric equation.

We will often use parametric equations to describe the path of an object or particle. Let's take a look at an example of that.

Example 7

The path of a particle is given by the following set of parametric equations.

$$x = 3\cos(2t) \qquad y = 1 + \cos^2(2t)$$

Completely describe the path of this particle. Do this by sketching the path, determining limits on x and y and giving a range of t's for which the path will be traced out exactly once (provide it traces out more than once of course).

Solution

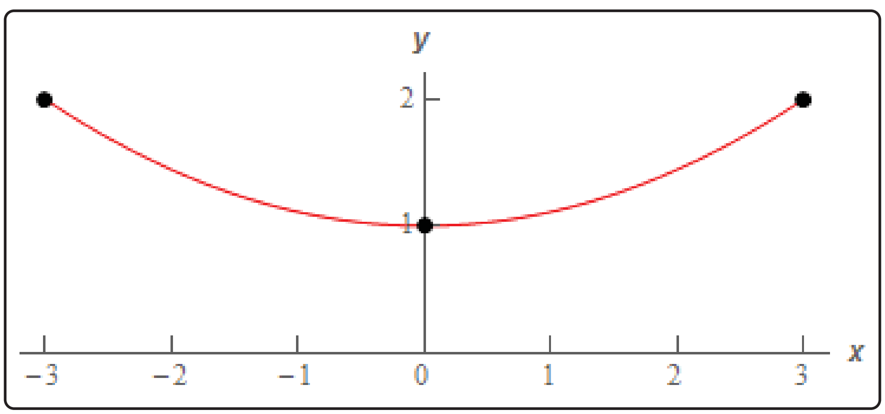
Eliminating the parameter this time will be a little different. We only have cosines this time and we'll use that to our advantage. We can solve the x equation for cosine and plug that into the equation for y. This gives,

$$\cos(2t) = \frac{x}{3} \qquad y = 1 + \left(\frac{x}{3}\right)^2 = 1 + \frac{x^2}{9}$$

This time the algebraic equation is a parabola that opens upward. We also have the following limits on x and y.

$$\begin{array}{ccc} -1 \le \cos(2t) \le 1 & -3 \le 3\cos(2t) \le 3 & -3 \le x \le 3 \\ 0 \le \cos^2(2t) \le 1 & 1 \le 1 + \cos^2(2t) \le 2 & 1 \le y \le 2 \end{array}$$

So, again we only trace out a portion of the curve. Here is a quick sketch of the portion of the parabola that the parametric curve will cover.



Now, as we discussed in the previous example because both the x and y parametric equations involve cosine we know that both x and y must oscillate and because the "start" and "end" points of the curve are not the same the only way x and y can oscillate is for the curve to trace out in both directions.

To finish the problem then all we need to do is determine a range of t's for one trace. Because the "end" points on the curve have the same y value and different x values we can use the x parametric equation to determine these values. Here is that work.

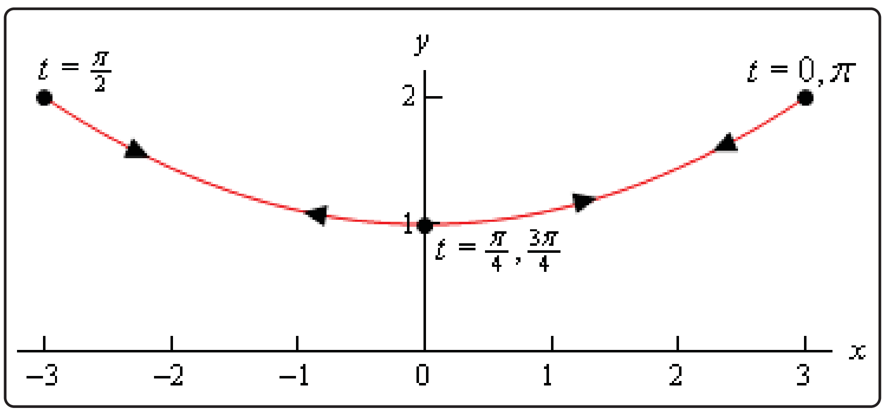
$$\begin{aligned} x = 3: \quad & 3 = 3\cos(2t) \\ & 1 = \cos(2t) \\ & 2t = 0 + 2\pi n \qquad \to \qquad t = \pi n \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots \end{aligned}$$

$$\begin{aligned} x = -3: \quad & -3 = 3\cos(2t) \\ & -1 = \cos(2t) \\ & 2t = \pi + 2\pi n \qquad \to \qquad t = \frac{1}{2}\pi + \pi n \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots \end{aligned}$$

So, we will be at the right end point at $t = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$ and we'll be at the left end point at $t = \ldots, -\frac{3}{2}\pi, -\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi, \ldots$. So, in this case there are an infinite number of ranges of t's for one trace. Here are a few of them.

$$-\frac{1}{2}\pi \le t \le 0 \qquad 0 \le t \le \frac{1}{2}\pi \qquad \frac{1}{2}\pi \le t \le \pi$$

Here is a final sketch of the particle's path with a few values of t on it.



We should give a small warning at this point. Because of the ideas involved in them we concentrated on parametric curves that retraced portions of the curve more than once. Do not, however, get too locked into the idea that this will always happen. Many, if not most parametric curves will only trace out once. The first one we looked at is a good example of this. That parametric curve will never repeat any portion of itself.

There is one final topic to be discussed in this section before moving on. So far we've started with parametric equations and eliminated the parameter to determine the parametric curve.

However, there are times in which we want to go the other way. Given a function or equation we might want to write down a set of parametric equations for it. In these cases we say that we **parameterize** the function.

If we take Examples 4 and 5 as examples we can do this for ellipses (and hence circles). Given the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

a set of parametric equations for it would be,

$$x = a\cos(t) \qquad y = b\sin(t)$$

This set of parametric equations will trace out the ellipse starting at the point $(a, 0)$ and will trace in a counter-clockwise direction and will trace out exactly once in the range $0 \le t \le 2\pi$. This is a fairly important set of parametric equations as it used continually in some subjects with dealing with ellipses and/or circles.

Every curve can be parameterized in more than one way. Any of the following will also parameterize the same ellipse.

$$\begin{aligned} x &= a\cos(\omega\, t) & y &= b\sin(\omega\, t) \\ x &= a\sin(\omega\, t) & y &= b\cos(\omega\, t) \\ x &= a\cos(\omega\, t) & y &= -b\sin(\omega\, t) \end{aligned}$$

The presence of the ω will change the speed that the ellipse rotates as we saw in Example 5. Note as well that the last two will trace out ellipses with a clockwise direction of motion (you might want to verify this). Also note that they won't all start at the same place (if we think of $t = 0$ as the starting point that is).

There are many more parameterizations of an ellipse of course, but you get the idea. It is important to remember that each parameterization will trace out the curve once with a potentially different range of t's. Each parameterization may rotate with different directions of motion and may start at different points.

You may find that you need a parameterization of an ellipse that starts at a particular place and has a particular direction of motion and so you now know that with some work you can write down a set of parametric equations that will give you the behavior that you're after.

Now, let's write down a couple of other important parameterizations and all the comments about direction of motion, starting point, and range of t's for one trace (if applicable) are still true.

First, because a circle is nothing more than a special case of an ellipse we can use the parameterization of an ellipse to get the parametric equations for a circle centered at the origin of radius r as well. One possible way to parameterize a circle is,

$$x = r\cos(t) \qquad y = r\sin(t)$$

Finally, even though there may not seem to be any reason to, we can also parameterize functions in the form $y = f(x)$ or $x = h(y)$. In these cases we parameterize them in the following way,

$$\begin{aligned} x &= t & \qquad x &= h(t) \\ y &= f(t) & \qquad y &= t \end{aligned}$$

At this point it may not seem all that useful to do a parameterization of a function like this, but there are many instances where it will actually be easier, or it may even be required, to work with the parameterization instead of the function itself. Unfortunately, almost all of these instances occur in a Calculus III course.

9.2 Tangents with Parametric Equations

In this section we want to find the tangent lines to the parametric equations given by,

$$x = f(t) \qquad\qquad y = g(t)$$

To do this let's first recall how to find the tangent line to $y = F(x)$ at $x = a$. Here the tangent line is given by,

$$y = F(a) + m(x-a), \text{ where } m = \left.\frac{dy}{dx}\right|_{x=a} = F'(a)$$

Now, notice that if we could figure out how to get the derivative $\frac{dy}{dx}$ from the parametric equations we could simply reuse this formula since we will be able to use the parametric equations to find the x and y coordinates of the point.

So, just for a second let's suppose that we were able to eliminate the parameter from the parametric form and write the parametric equations in the form $y = F(x)$. Now, plug the parametric equations in for x and y. Yes, it seems silly to eliminate the parameter, then immediately put it back in, but it's what we need to do in order to get our hands on the derivative. Doing this gives,

$$g(t) = F\big(f(t)\big)$$

Now, differentiate with respect to t and notice that we'll need to use the Chain Rule on the right-hand side.

$$g'(t) = F'\big(f(t)\big)\, f'(t)$$

Let's do another change in notation. We need to be careful with our derivatives here. Derivatives of the lower case function are with respect to t while derivatives of upper case functions are with respect to x. So, to make sure that we keep this straight let's rewrite things as follows.

$$\frac{dy}{dt} = F'(x)\frac{dx}{dt}$$

At this point we should remind ourselves just what we are after. We needed a formula for $\frac{dy}{dx}$ or $F'(x)$ that is in terms of the parametric formulas. Notice however that we can get that from the above equation.

Derivative for Parametric Equations, $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ provided } \frac{dx}{dt} \neq 0$$

Notice as well that this will be a function of t and not x.

As an aside, notice that we could also get the following formula with a similar derivation if we needed to,

Derivative for Parametric Equations, $\frac{dx}{dy}$

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}}, \text{ provided } \frac{dy}{dt} \neq 0$$

Why would we want to do this? Well, recall that in the arc length section of the Applications of Integral section we actually needed this derivative on occasion.

So, let's find a tangent line.

Example 1

Find the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \qquad y = t^2$$

at $(0, 4)$.

Solution

Note that there is apparently the potential for more than one tangent line here! We will look into this more after we're done with the example.

The first thing that we should do is find the derivative so we can get the slope of the tangent line.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

At this point we've got a small problem. The derivative is in terms of t and all we've got is an *x*-*y* coordinate pair. The next step then is to determine the value(s) of t which will give this point. We find these by plugging the x and y values into the parametric equations and solving for t.

$$\begin{aligned} 0 &= t^5 - 4t^3 = t^3\left(t^2 - 4\right) & \Rightarrow \quad & t = 0, \pm 2 \\ 4 &= t^2 & \Rightarrow \quad & t = \pm 2 \end{aligned}$$

Any value of t which appears in both lists will give the point. So, since there are two values of t that give the point we will in fact get two tangent lines. That's definitely not something that happened back in Calculus I and we're going to need to look into this a little more. However, before we do that let's actually get the tangent lines.

$\underline{t = -2:}$
Since we already know the x and y-coordinates of the point all that we need to do is find the slope of the tangent line.

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{1}{8}$$

The tangent line (at $t = -2$) is then,

$$y = 4 - \frac{1}{8}x$$

$\underline{t = 2:}$
Again, all we need is the slope.

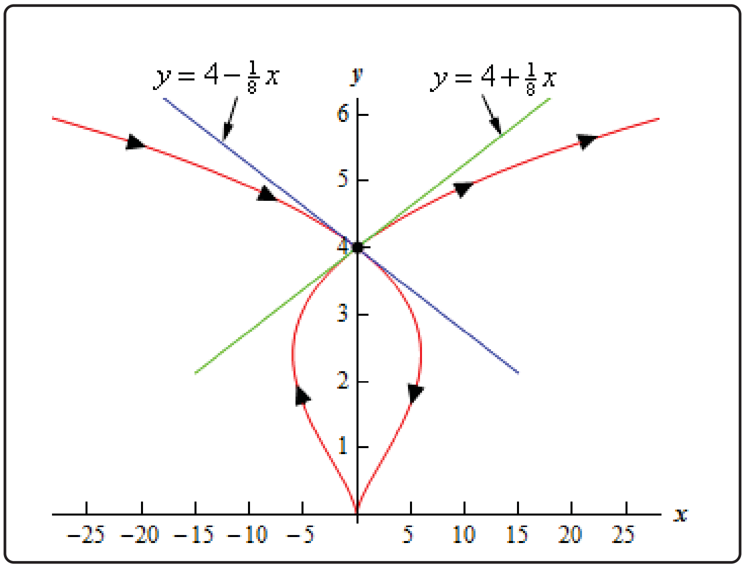
$$m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{8}$$

The tangent line (at $t = 2$) is then,

$$y = 4 + \frac{1}{8}x$$

Before we leave this example let's take a look at just how we could possibly get two tangents lines at a point. This was definitely not possible back in Calculus I where we first ran across tangent lines.

A quick graph of the parametric curve will explain what is going on here.



So, the parametric curve crosses itself! That explains how there can be more than one tangent line. There is one tangent line for each instance that the curve goes through the point.

The next topic that we need to discuss in this section is that of horizontal and vertical tangents. We can easily identify where these will occur (or at least the t's that will give them) by looking at the derivative formula.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Horizontal tangents will occur where the derivative is zero and that means that we'll get horizontal tangent at values of t for which we have,

Horizontal Tangent for Parametric Equations

$$\frac{dy}{dt} = 0, \text{ provided } \frac{dx}{dt} \neq 0$$

Vertical tangents will occur where the derivative is not defined and so we'll get vertical tangents at values of t for which we have,

Vertical Tangent for Parametric Equations

$$\frac{dx}{dt} = 0, \text{ provided } \frac{dy}{dt} \neq 0$$

Let's take a quick look at an example of this.

Example 2

Determine the x-y coordinates of the points where the following parametric equations will have horizontal or vertical tangents.

$$x = t^3 - 3t \qquad y = 3t^2 - 9$$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt} = 3t^2 - 3 = 3\left(t^2 - 1\right) \qquad \frac{dy}{dt} = 6t$$

Horizontal Tangents

We'll have horizontal tangents where,

$$6t = 0 \qquad \Rightarrow \qquad t = 0$$

Now, this is the value of t which gives the horizontal tangents and we were asked to find the *x*-*y* coordinates of the point. To get these we just need to plug t into the parametric equations. Therefore, the only horizontal tangent will occur at the point $(0, -9)$.

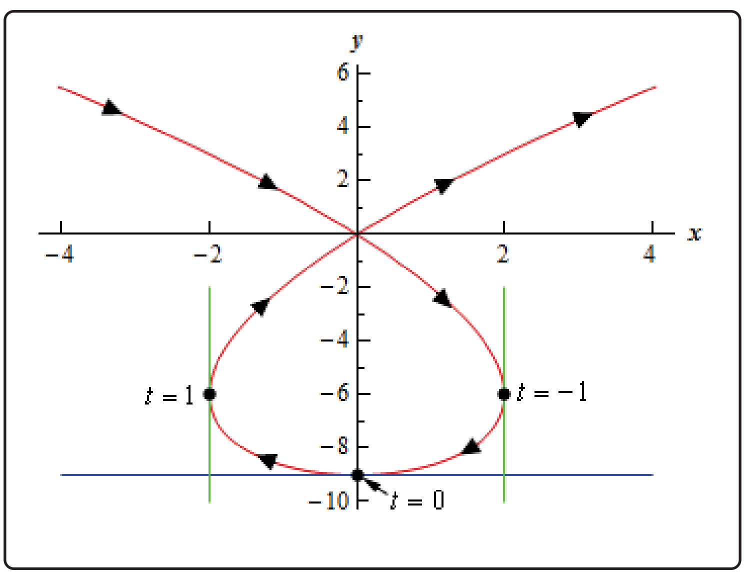
Vertical Tangents

In this case we need to solve,

$$3\left(t^2 - 1\right) = 0 \qquad \Rightarrow \qquad t = \pm 1$$

The two vertical tangents will occur at the points $(2, -6)$ and $(-2, -6)$.

For the sake of completeness and at least partial verification here is the sketch of the parametric curve.



The final topic that we need to discuss in this section really isn't related to tangent lines but does fit in nicely with the derivation of the derivative that we needed to get the slope of the tangent line.

Before moving into the new topic let's first remind ourselves of the formula for the first derivative and in the process rewrite it slightly.

$$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{\frac{d}{dt}(y)}{\frac{dx}{dt}}$$

Written in this way we can see that the formula actually tells us how to differentiate a function y (as a function of t) with respect to x (when x is also a function of t) when we are using parametric equations.

Now let's move onto the final topic of this section. We would also like to know how to get the second derivative of y with respect to x.

$$\frac{d^2y}{dx^2}$$

Getting a formula for this is fairly simple if we remember the rewritten formula for the first derivative above.

Second Derivative for Parametric Equations, $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

It is important to note that,

$$\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$$

Let's work a quick example.

Example 3

Find the second derivative for the following set of parametric equations.

$$x = t^5 - 4t^3 \qquad y = t^2$$

Solution

This is the set of parametric equations that we used in the first example and so we already have the following computations completed.

$$\frac{dy}{dt} = 2t \qquad \frac{dx}{dt} = 5t^4 - 12t^2 \qquad \frac{dy}{dx} = \frac{2}{5t^3 - 12t}$$

We will first need the following,

$$\frac{d}{dt}\left(\frac{2}{5t^3 - 12t}\right) = \frac{-2\left(15t^2 - 12\right)}{\left(5t^3 - 12t\right)^2} = \frac{24 - 30t^2}{\left(5t^3 - 12t\right)^2}$$

The second derivative is then,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}\\ &= \frac{\frac{24-30t^2}{\left(5t^3-12t\right)^2}}{5t^4 - 12t^2}\\ &= \frac{24 - 30t^2}{\left(5t^4 - 12t^2\right)\left(5t^3 - 12t\right)^2}\\ &= \frac{24 - 30t^2}{t\left(5t^3 - 12t\right)^3}\end{aligned}$$

So, why would we want the second derivative? Well, recall from your Calculus I class that with the second derivative we can determine where a curve is concave up and concave down. We could do the same thing with parametric equations if we wanted to.

Example 4

Determine the values of t for which the parametric curve given by the following set of parametric equations is concave up and concave down.

$$x = 1 - t^2 \qquad y = t^7 + t^5$$

Solution

To compute the second derivative we'll first need the following.

$$\frac{dy}{dt} = 7t^6 + 5t^4 \qquad \frac{dx}{dt} = -2t \qquad \frac{dy}{dx} = \frac{7t^6 + 5t^4}{-2t} = -\frac{1}{2}\left(7t^5 + 5t^3\right)$$

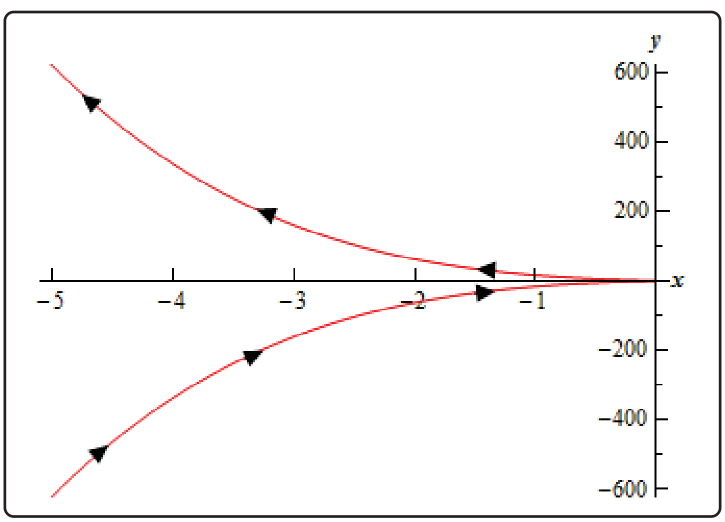
Note that we can also use the first derivative above to get some information about the increasing/decreasing nature of the curve as well. In this case it looks like the parametric curve will be increasing if $t < 0$ and decreasing if $t > 0$.

Now let's move on to the second derivative.

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{2}\left(35t^4 + 15t^2\right)}{-2t} = \frac{1}{4}\left(35t^3 + 15t\right)$$

It's clear, hopefully, that the second derivative will only be zero at $t = 0$. Using this we can see that the second derivative will be negative if $t < 0$ and positive if $t > 0$. So the parametric curve will be concave down for $t < 0$ and concave up for $t > 0$.

Here is a sketch of the curve for completeness sake.



9.3 Area with Parametric Equations

In this section we will find a formula for determining the area under a parametric curve given by the parametric equations,

$$x = f(t) \qquad y = g(t)$$

We will also need to further add in the assumption that the curve is traced out exactly once as t increases from α to β.

We will do this in much the same way that we found the first derivative in the previous section. We will first recall how to find the area under $y = F(x)$ on $a \le x \le b$.

$$A = \int_a^b F(x)\, dx$$

We will now think of the parametric equation $x = f(t)$ as a substitution in the integral. We will also assume that $a = f(\alpha)$ and $b = f(\beta)$ for the purposes of this formula. There is actually no reason to assume that this will always be the case and so we'll give a corresponding formula later if it's the opposite case ($b = f(\alpha)$ and $a = f(\beta)$).

So, if this is going to be a substitution we'll need,

$$dx = f'(t)\, dt$$

Plugging this into the area formula above and making sure to change the limits to their corresponding t values gives us,

$$A = \int_\alpha^\beta F(f(t))\, f'(t)\, dt$$

Since we don't know what $F(x)$ is we'll use the fact that

$$y = F(x) = F(f(t)) = g(t)$$

and we arrive at the formula that we want.

Area Under Parametric Curve, Formula I

$$A = \int_\alpha^\beta g(t)\, f'(t)\, dt$$

Now, if we should happen to have $b = f(\alpha)$ and $a = f(\beta)$ the formula would be,

Area Under Parametric Curve, Formula II

$$A = \int_\beta^\alpha g(t)\, f'(t)\, dt$$

Let's work an example.

Example 1

Determine the area under the parametric curve given by the following parametric equations.

$$x = 6\left(\theta - \sin(\theta)\right) \qquad y = 6\left(1 - \cos(\theta)\right) \qquad 0 \le \theta \le 2\pi$$

Solution

First, notice that we've switched the parameter to θ for this problem. This is to make sure that we don't get too locked into always having t as the parameter.

Now, we could graph this to verify that the curve is traced out exactly once for the given range if we wanted to. We are going to be looking at this curve in more detail after this example so we won't sketch its graph here.

There really isn't too much to this example other than plugging the parametric equations into the formula. We'll first need the derivative of the parametric equation for x however.

$$\frac{dx}{d\theta} = 6\left(1 - \cos(\theta)\right)$$

The area is then,

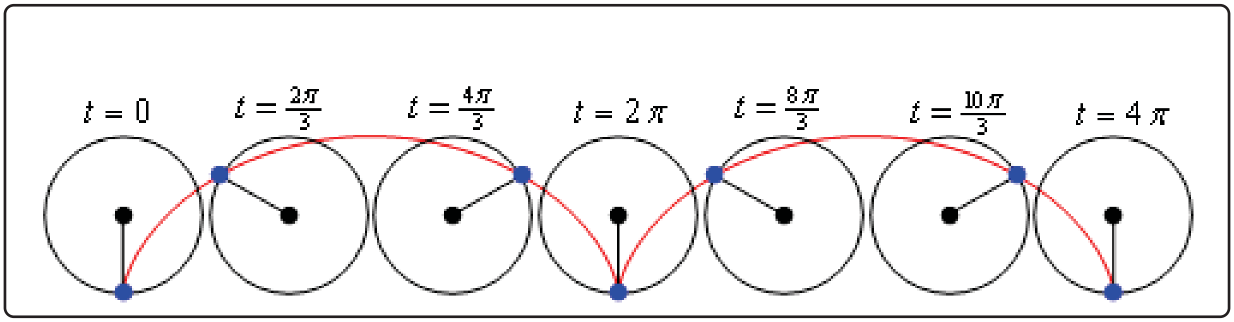
$$\begin{aligned} A &= \int_0^{2\pi} 36\left(1 - \cos(\theta)\right)^2 d\theta \\ &= 36\int_0^{2\pi} 1 - 2\cos(\theta) + \cos^2(\theta)\, d\theta \\ &= 36\int_0^{2\pi} \frac{3}{2} - 2\cos(\theta) + \frac{1}{2}\cos(2\theta)\, d\theta \\ &= 36\left(\frac{3}{2}\theta - 2\sin(\theta) + \frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{2\pi} \\ &= 108\pi \end{aligned}$$

The parametric curve (without the limits) we used in the previous example is called a **cycloid**. In its general form the cycloid is,

$$x = r\left(\theta - \sin(\theta)\right) \qquad y = r\left(1 - \cos(\theta)\right)$$

The cycloid represents the following situation. Consider a wheel of radius r. Let the point where the wheel touches the ground initially be called P. Then start rolling the wheel to the right. As the wheel rolls to the right trace out the path of the point P. The path that the point P traces out is called a cycloid and is given by the equations above. In these equations we can think of θ as the angle through which the point P has rotated.

Here is a cycloid sketched out with the wheel shown at various places. The blue dot is the point P on the wheel that we're using to trace out the curve.



From this sketch we can see that one arch of the cycloid is traced out in the range $0 \le \theta \le 2\pi$. This makes sense when you consider that the point P will be back on the ground after it has rotated through an angle of 2π.

9.4 Arc Length with Parametric Equations

In the previous two sections we've looked at a couple of Calculus I topics in terms of parametric equations. We now need to look at a couple of Calculus II topics in terms of parametric equations.

In this section we will look at the arc length of the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

We will also be assuming that the curve is traced out exactly once as t increases from α to β. We will also need to assume that the curve is traced out from left to right as t increases. This is equivalent to saying,

$$\frac{dx}{dt} \ge 0 \qquad \text{for } \alpha \le t \le \beta$$

So, let's start out the derivation by recalling the arc length formula as we first derived it in the arc length section of the Applications of Integrals chapter.

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{if } y = f(x),\ a \le x \le b$$
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{if } x = h(y),\ c \le y \le d$$

We will use the first ds above because we have a nice formula for the derivative in terms of the parametric equations (see the Tangents with Parametric Equations section). To use this we'll also need to know that,

$$dx = f'(t)\,dt = \frac{dx}{dt}\,dt$$

The arc length formula then becomes,

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\;\frac{dx}{dt}dt = \int_{\alpha}^{\beta} \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\;\frac{dx}{dt}dt$$

This is a particularly unpleasant formula. However, if we factor out the denominator from the square root we arrive at,

$$L = \int_{\alpha}^{\beta} \frac{1}{\left|\frac{dx}{dt}\right|}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;\frac{dx}{dt}dt$$

Now, making use of our assumption that the curve is being traced out from left to right we can drop the absolute value bars on the derivative which will allow us to cancel the two derivatives that are outside the square root and this gives,

Arc Length for Parametric Equations

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Notice that we could have used the second formula for ds above if we had assumed instead that

$$\frac{dy}{dt} \geq 0 \qquad \text{for } \alpha \leq t \leq \beta$$

If we had gone this route in the derivation we would have gotten the same formula.

Let's take a look at an example.

Example 1

Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(t) \qquad y = 3\cos(t) \qquad 0 \leq t \leq 2\pi$$

Solution

We know that this is a circle of radius 3 centered at the origin from our prior discussion about graphing parametric curves. We also know from this discussion that it will be traced out exactly once in this range.

So, we can use the formula we derived above. We'll first need the following,

$$\frac{dx}{dt} = 3\cos(t) \qquad \frac{dy}{dt} = -3\sin(t)$$

The length is then,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)}\, dt \\ &= \int_0^{2\pi} 3\sqrt{\sin^2(t) + \cos^2(t)}\, dt \\ &= 3\int_0^{2\pi} dt \\ &= 6\pi \end{aligned}$$

Since this is a circle we could have just used the fact that the length of the circle is just the circumference of the circle. This is a nice way, in this case, to verify our result.

Let's take a look at one possible consequence if a curve is traced out more than once and we try to find the length of the curve without taking this into account.

Example 2

Use the arc length formula for the following parametric equations.

$$x = 3\sin(3t) \qquad y = 3\cos(3t) \qquad 0 \le t \le 2\pi$$

Solution

Notice that this is the identical circle that we had in the previous example and so the length is still 6 p. However, for the range given we know it will trace out the curve three times instead once as required for the formula. Despite that restriction let's use the formula anyway and see what happens.

In this case the derivatives are,

$$\frac{dx}{dt} = 9\cos(3t) \qquad \frac{dy}{dt} = -9\sin(3t)$$

and the length formula gives,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{81\sin^2(3t) + 81\cos^2(3t)}\, dt \\ &= \int_0^{2\pi} 9\, dt \\ &= 18\pi \end{aligned}$$

The answer we got form the arc length formula in this example was 3 times the actual length. Recalling that we also determined that this circle would trace out three times in the range given, the answer should make some sense.

If we had wanted to determine the length of the circle for this set of parametric equations we would need to determine a range of t for which this circle is traced out exactly once. This is, $0 \le t \le \frac{2\pi}{3}$. Using this range of t we get the following for the length.

$$\begin{aligned} L &= \int_0^{\frac{2\pi}{3}} \sqrt{81\sin^2(3t) + 81\cos^2(3t)}\, dt \\ &= \int_0^{\frac{2\pi}{3}} 9\, dt \\ &= 6\pi \end{aligned}$$

which is the correct answer.

Be careful to not make the assumption that this is always what will happen if the curve is traced out more than once. Just because the curve traces out n times does not mean that the arc length formula will give us n times the actual length of the curve!

Before moving on to the next section let's notice that we can put the arc length formula derived in this section into the same form that we had when we first looked at arc length. The only difference is that we will add in a definition for ds when we have parametric equations.

The arc length formula can be summarized as,

$$L = \int ds$$

where,

$$\begin{aligned} ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx & \text{if } y = f(x),\ a \le x \le b \\ ds &= \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy & \text{if } x = h(y),\ c \le y \le d \\ ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt & \text{if } x = f(t), y = g(t),\ \alpha \le t \le \beta \end{aligned}$$

9.5 Surface Area with Parametric Equations

In this final section of looking at calculus applications with parametric equations we will take a look at determining the surface area of a region obtained by rotating a parametric curve about the x or y-axis.

We will rotate the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

about the x or y-axis. We are going to assume that the curve is traced out exactly once as t increases from α to β. At this point there actually isn't all that much to do. We know that the surface area can be found by using one of the following two formulas depending on the axis of rotation (recall the Surface Area section of the Applications of Integrals chapter).

$$S = \int 2\pi y \, ds \qquad \text{rotation about } x - \text{axis}$$
$$S = \int 2\pi x \, ds \qquad \text{rotation about } y - \text{axis}$$

All that we need is a formula for ds to use and from the previous section we have,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \qquad \text{if } x = f(t), y = g(t), \ \alpha \le t \le \beta$$

which is exactly what we need.

We will need to be careful with the x or y that is in the original surface area formula. Back when we first looked at surface area we saw that sometimes we had to substitute for the variable in the integral and at other times we didn't. This was dependent upon the ds that we used. In this case however, we will always have to substitute for the variable. The ds that we use for parametric equations introduces a dt into the integral and that means that everything needs to be in terms of t. Therefore, we will need to substitute the appropriate parametric equation for x or y depending on the axis of rotation.

Let's take a quick look at an example.

Example 1

Determine the surface area of the solid obtained by rotating the following parametric curve about the x-axis.

$$x = \cos^3(\theta) \qquad y = \sin^3(\theta) \qquad 0 \le \theta \le \frac{\pi}{2}$$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{d\theta} = -3\cos^2(\theta)\sin(\theta) \qquad\qquad \frac{dy}{d\theta} = 3\sin^2(\theta)\cos(\theta)$$

Before plugging into the surface area formula let's get the ds out of the way.

$$\begin{aligned} ds &= \sqrt{9\cos^4(\theta)\sin^2(\theta) + 9\sin^4(\theta)\cos^2(\theta)}\, d\theta \\ &= 3\left|\cos(\theta)\sin(\theta)\right| \sqrt{\cos^2(\theta) + \sin^2(\theta)}\, d\theta \\ &= 3\cos(\theta)\sin(\theta)\, d\theta \end{aligned}$$

Notice that we could drop the absolute value bars since both sine and cosine are positive in this range of q given.

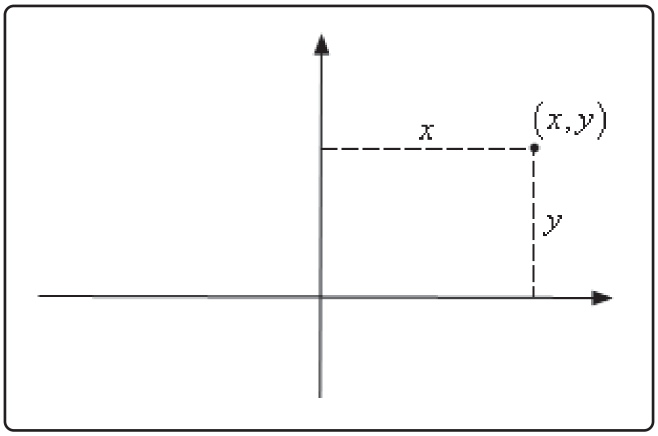
Now let's get the surface area and don't forget to also plug in for the y.

$$\begin{aligned} S &= \int 2\pi y\, ds \\ &= 2\pi \int_0^{\frac{\pi}{2}} \sin^3(\theta)\left(3\cos(\theta)\sin(\theta)\right)\, d\theta \\ &= 6\pi \int_0^{\frac{\pi}{2}} \sin^4(\theta)\cos(\theta)\, d\theta \qquad\qquad u = \sin(\theta) \\ &= 6\pi \int_0^1 u^4\, du \\ &= \frac{6\pi}{5} \end{aligned}$$

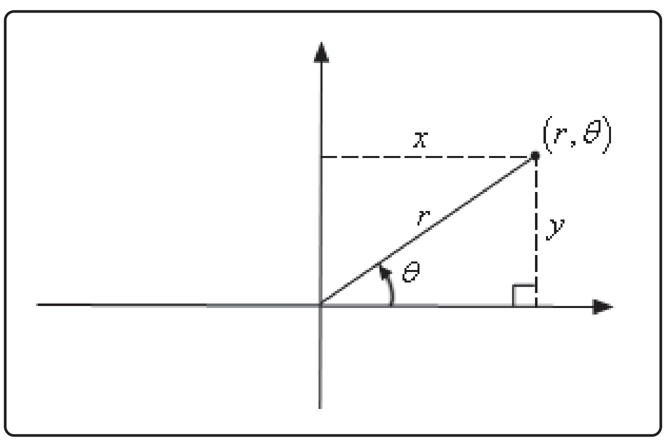
9.6 Polar Coordinates

Up to this point we've dealt exclusively with the Cartesian (or Rectangular, or *x*-*y*) coordinate system. However, as we will see, this is not always the easiest coordinate system to work in. So, in this section we will start looking at the polar coordinate system.

Coordinate systems are really nothing more than a way to define a point in space. For instance in the Cartesian coordinate system at point is given the coordinates (x, y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically. This is shown in the sketch below.

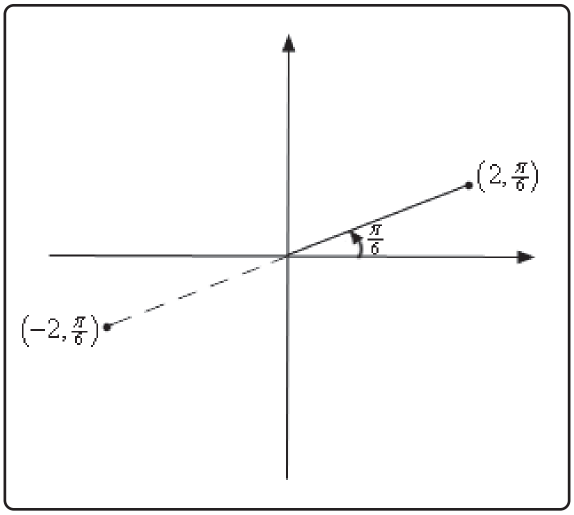


This is not, however, the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive x-axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point. This is shown in the sketch below.



Coordinates in this form are called **polar coordinates**.

The above discussion may lead one to think that r must be a positive number. However, we also allow r to be negative. Below is a sketch of the two points $\left(2, \frac{\pi}{6}\right)$ and $\left(-2, \frac{\pi}{6}\right)$.

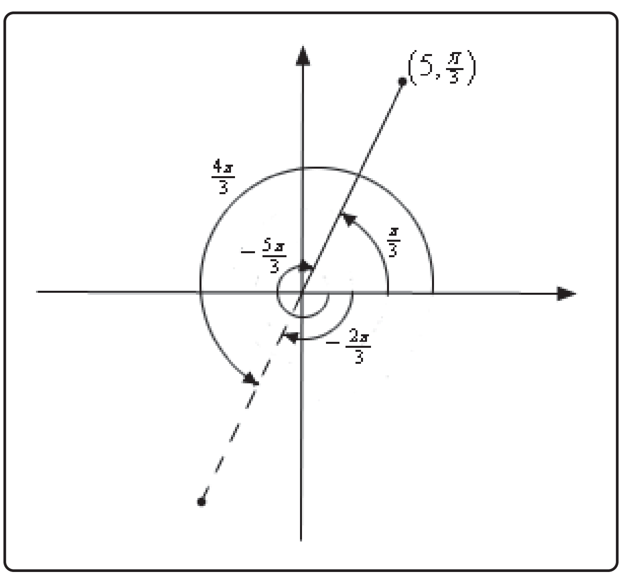


From this sketch we can see that if r is positive the point will be in the same quadrant as θ. On the other hand if r is negative the point will end up in the quadrant exactly opposite θ. Notice as well that the coordinates $\left(-2, \frac{\pi}{6}\right)$ describe the same point as the coordinates $\left(2, \frac{7\pi}{6}\right)$ do. The coordinates $\left(2, \frac{7\pi}{6}\right)$ tells us to rotate an angle of $\frac{7\pi}{6}$ from the positive x-axis, this would put us on the dashed line in the sketch above, and then move out a distance of 2.

This leads to an important difference between Cartesian coordinates and polar coordinates. In Cartesian coordinates there is exactly one set of coordinates for any given point. With polar coordinates this isn't true. In polar coordinates there is literally an infinite number of coordinates for a given point. For instance, the following four points are all coordinates for the same point.

$$\left(5, \frac{\pi}{3}\right) = \left(5, -\frac{5\pi}{3}\right) = \left(-5, \frac{4\pi}{3}\right) = \left(-5, -\frac{2\pi}{3}\right)$$

Here is a sketch of the angles used in these four sets of coordinates.



In the second coordinate pair we rotated in a clock-wise direction to get to the point. We shouldn't forget about rotating in the clock-wise direction. Sometimes it's what we have to do.

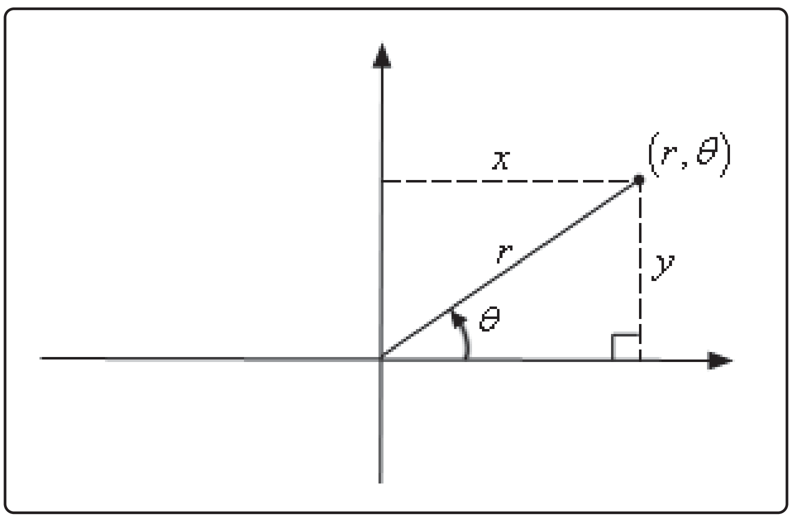
The last two coordinate pairs use the fact that if we end up in the opposite quadrant from the point we can use a negative r to get back to the point and of course there is both a counter clock-wise and a clock-wise rotation to get to the angle.

These four points only represent the coordinates of the point without rotating around the system more than once. If we allow the angle to make as many complete rotations about the axis system as we want then there are an infinite number of coordinates for the same point. In fact, the point (r,θ) can be represented by any of the following coordinate pairs.

$$(r,\theta+2\pi n) \qquad (-r,\theta+(2n+1)\,\pi)\,, \qquad \text{where } n \text{ is any integer.}$$

Next, we should talk about the origin of the coordinate system. In polar coordinates the origin is often called the **pole**. Because we aren't actually moving away from the origin/pole we know that $r=0$. However, we can still rotate around the system by any angle we want and so the coordinates of the origin/pole are $(0,\theta)$.

Now that we've got a grasp on polar coordinates we need to think about converting between the two coordinate systems. Well start out with the following sketch reminding us how both coordinate systems work.



Note that we've got a right triangle above and with that we can get the following equations that will convert polar coordinates into Cartesian coordinates.

Polar to Cartesian Conversion Formulas

$$x=r\cos(\theta) \qquad y=r\sin(\theta)$$

Converting from Cartesian is almost as easy. Let's first notice the following.

$$\begin{aligned} x^2 + y^2 &= \left(r\cos(\theta)\right)^2 + \left(r\sin(\theta)\right)^2 \\ &= r^2\cos^2(\theta) + r^2\sin^2(\theta) \\ &= r^2\left(\cos^2(\theta) + \sin^2(\theta)\right) = r^2 \end{aligned}$$

This is a very useful formula that we should remember, however we are after an equation for r so let's take the square root of both sides. This gives,

$$r = \sqrt{x^2 + y^2}$$

Note that technically we should have a plus or minus in front of the root since we know that r can be either positive or negative. We will run with the convention of positive r here.

Getting an equation for θ is almost as simple. We'll start with,

$$\frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \tan(\theta)$$

Taking the inverse tangent of both sides gives,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

We will need to be careful with this because inverse tangents only return values in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Recall that there is a second possible angle and that the second angle is given by $\theta + \pi$.

Summarizing then gives the following formulas for converting from Cartesian coordinates to polar coordinates.

Cartesian to Polar Conversion Formulas

$$\begin{aligned} &r^2 = x^2 + y^2 && r = \sqrt{x^2 + y^2} \\ &\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{OR} && \theta_2 = \theta_1 + \pi \end{aligned}$$

Let's work a quick example.

Example 1

Convert each of the following points into the given coordinate system.

(a) Convert $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates.

(b) Convert $(-1, -1)$ into polar coordinates.

Solution

(a) Convert $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates.

This conversion is easy enough. All we need to do is plug the points into the formulas.

$$x = -4\cos\left(\frac{2\pi}{3}\right) = -4\left(-\frac{1}{2}\right) = 2$$
$$y = -4\sin\left(\frac{2\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

So, in Cartesian coordinates this point is $\left(2, -2\sqrt{3}\right)$.

(b) Convert $(-1, -1)$ into polar coordinates.

Let's first get r.

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

Now, let's get θ.

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

This is not the correct angle however. This value of θ is in the first quadrant and the point we've been given is in the third quadrant. As noted above we can get the correct angle by adding p onto this. Therefore, the actual angle is,

$$\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

So, in polar coordinates the point is $\left(\sqrt{2}, \frac{5\pi}{4}\right)$. Note as well that we could have used the first θ that we got by using a negative r. In this case the point could also be written in polar coordinates as $\left(-\sqrt{2}, \frac{\pi}{4}\right)$.

We can also use the above formulas to convert equations from one coordinate system to the other.

Example 2

Convert each of the following into an equation in the given coordinate system.

(a) Convert $2x - 5x^3 = 1 + xy$ into polar coordinates.

(b) Convert $r = -8\cos(\theta)$ into Cartesian coordinates.

Solution

(a) Convert $2x - 5x^3 = 1 + xy$ into polar coordinates.

In this case there really isn't much to do other than plugging in the formulas for x and y (*i.e.* the Cartesian coordinates) in terms of r and θ (*i.e.* the polar coordinates).

$$\begin{gathered} 2\left(r\cos(\theta)\right) - 5\left(r\cos(\theta)\right)^3 = 1 + \left(r\cos(\theta)\right)\left(r\sin(\theta)\right) \\ 2r\cos(\theta) - 5r^3\cos^3(\theta) = 1 + r^2\cos(\theta)\sin(\theta) \end{gathered}$$

(b) Convert $r = -8\cos(\theta)$ into Cartesian coordinates.

This one is a little trickier, but not by much. First notice that we could substitute straight for the r. However, there is no straight substitution for the cosine that will give us only Cartesian coordinates. If we had an r on the right along with the cosine then we could do a direct substitution. So, if an r on the right side would be convenient let's put one there, just don't forget to put one on the left side as well.

$$r^2 = -8r\cos(\theta)$$

We can now make some substitutions that will convert this into Cartesian coordinates.

$$x^2 + y^2 = -8x$$

Before moving on to the next subject let's do a little more work on the second part of the previous example.

The equation given in the second part is actually a fairly well known graph; it just isn't in a form that most people will quickly recognize. To identify it let's take the Cartesian coordinate equation and do a little rearranging.

$$x^2 + 8x + y^2 = 0$$

Now, complete the square on the x portion of the equation.

$$\begin{aligned} x^2+8x+16+y^2 &= 16 \\ (x+4)^2+y^2 &= 16 \end{aligned}$$

So, this was a circle of radius 4 and center $(-4,0)$.

This leads us into the final topic of this section.

Common Polar Coordinate Graphs

Let's identify a few of the more common graphs in polar coordinates. We'll also take a look at a couple of special polar graphs.

Lines

Some lines have fairly simple equations in polar coordinates.

1. $\theta = \beta$.

   We can see that this is a line by converting to Cartesian coordinates as follows

$$\begin{aligned} \theta &= \beta \\ \tan^{-1}\left(\frac{y}{x}\right) &= \beta \\ \frac{y}{x} &= \tan\beta \\ y &= (\tan\beta)\,x \end{aligned}$$

   This is a line that goes through the origin and makes an angle of β with the positive x-axis. Or, in other words it is a line through the origin with slope of $\tan\beta$.

2. $r\cos(\theta) = a$

   This is easy enough to convert to Cartesian coordinates to $x = a$. So, this is a vertical line.

3. $r\sin(\theta) = b$

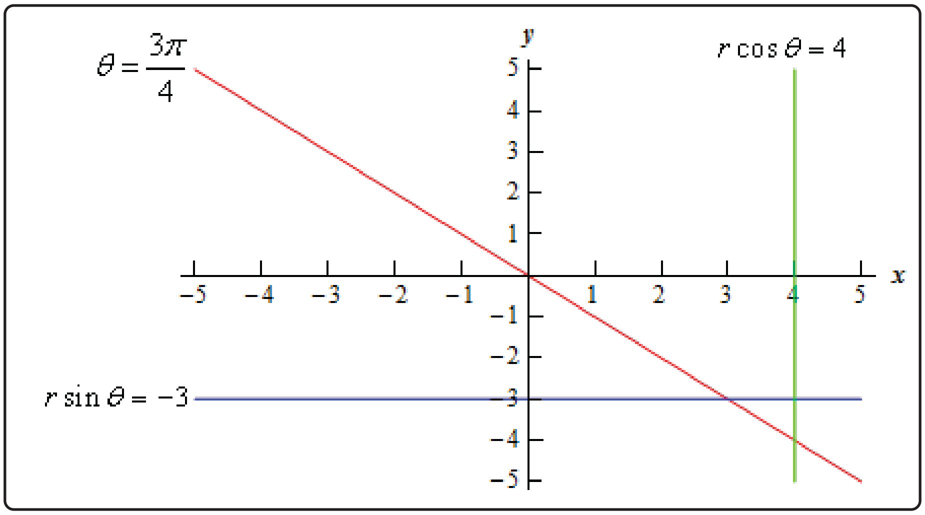
   Likewise, this converts to $y = b$ and so is a horizontal line.

Example 3

Graph $\theta = \frac{3\pi}{4}$, $r\cos(\theta) = 4$ and $r\sin(\theta) = -3$ on the same axis system.

Solution

There really isn't too much to this one other than doing the graph so here it is.



Circles

Let's take a look at the equations of circles in polar coordinates.

1. $r = a$.

   This equation is saying that no matter what angle we've got the distance from the origin must be a. If you think about it that is exactly the definition of a circle of radius a centered at the origin. So, this is a circle of radius a centered at the origin. This is also one of the reasons why we might want to work in polar coordinates. The equation of a circle centered at the origin has a very nice equation, unlike the corresponding equation in Cartesian coordinates.

2. $r = 2a\cos(\theta)$.

   We looked at a specific example of one of these when we were converting equations to Cartesian coordinates.

   This is a circle of radius $|a|$ and center $(a, 0)$. Note that a might be negative (as it was in our example above) and so the absolute value bars are required on the radius. They should not be used however on the center.

3. $r = 2b\sin(\theta)$.

   This is similar to the previous one. It is a circle of radius $|b|$ and center $(0, b)$.

4. $r = 2a\cos(\theta) + 2b\sin(\theta)$.

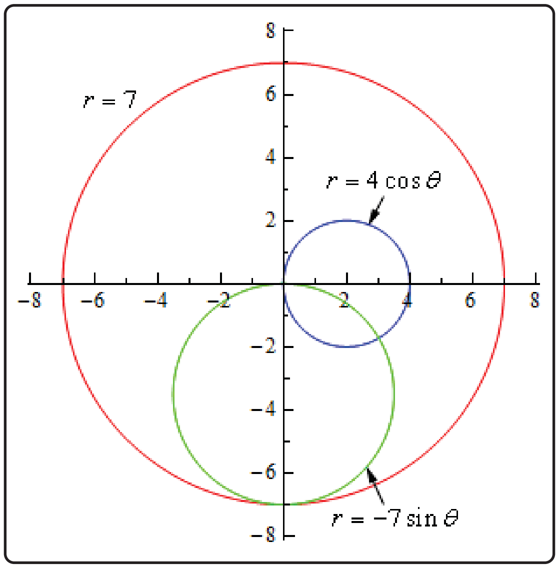
   This is a combination of the previous two and by completing the square twice it can be shown that this is a circle of radius $\sqrt{a^2 + b^2}$ and center (a, b). In other words, this is the general equation of a circle that isn't centered at the origin.

Example 4

Graph $r = 7$, $r = 4\cos(\theta)$, and $r = -7\sin(\theta)$ on the same axis system.

Solution

The first one is a circle of radius 7 centered at the origin. The second is a circle of radius 2 centered at $(2, 0)$. The third is a circle of radius $\frac{7}{2}$ centered at $\left(0, -\frac{7}{2}\right)$. Here is the graph of the three equations.



Note that it takes a range of $0 \leq \theta \leq 2\pi$ for a complete graph of $r = a$ and it only takes a range of $0 \leq \theta \leq \pi$ to graph the other circles given here. You can verify this with a quick table of values if you'd like to.

Cardioids and Limacons

These can be broken up into the following three cases.

1. Cardioids : $r = a \pm a\cos(\theta)$ and $r = a \pm a\sin(\theta)$.
   These have a graph that is vaguely heart shaped and always contain the origin.

2. Limacons with an inner loop : $r = a \pm b\cos(\theta)$ and $r = a \pm b\sin(\theta)$ with $a < b$.

   These will have an inner loop and will always contain the origin.

3. Limacons without an inner loop : $r = a \pm b\cos(\theta)$ and $r = a \pm b\sin(\theta)$ with $a > b$.

   These do not have an inner loop and do not contain the origin.

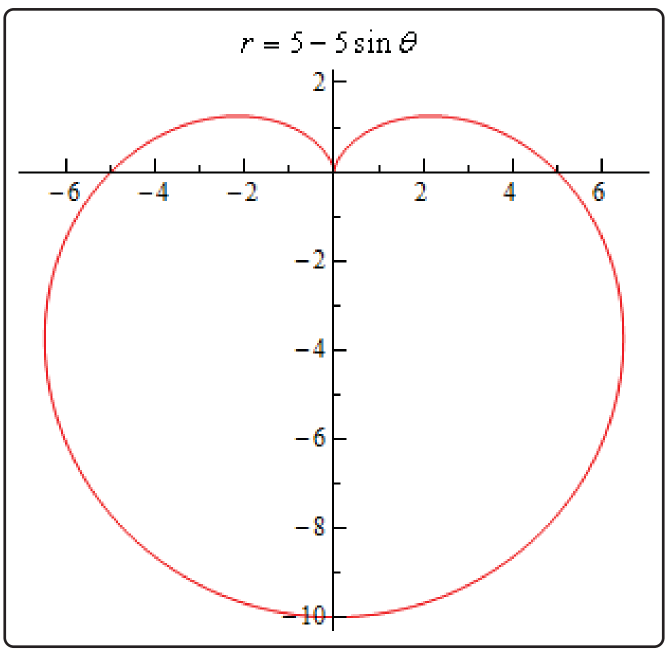
Example 5

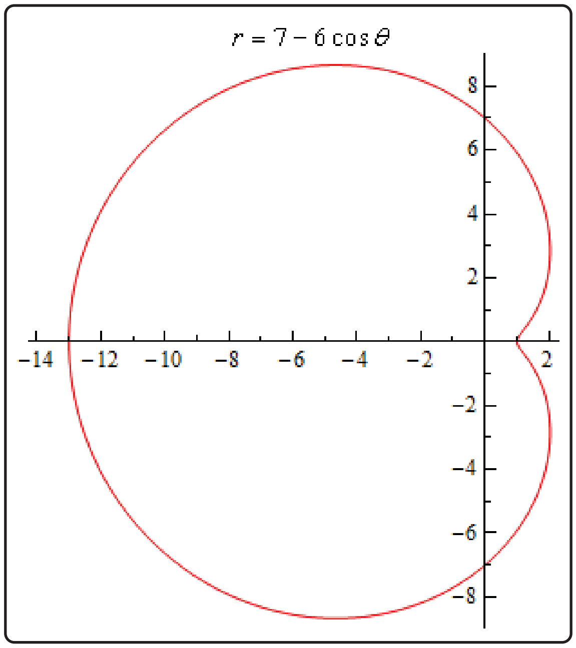
Graph $r = 5 - 5\sin(\theta)$, $r = 7 - 6\cos(\theta)$, and $r = 2 + 4\cos(\theta)$.

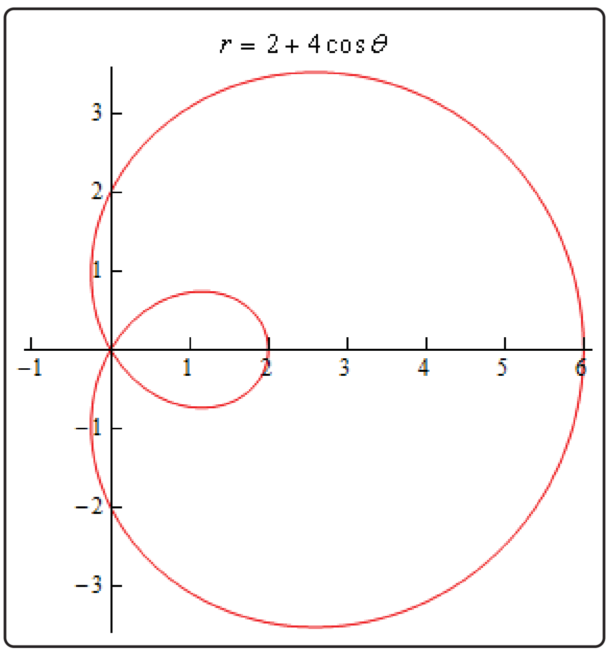
Solution

These will all graph out once in the range $0 \le \theta \le 2\pi$. Here is a table of values for each followed by graphs of each.

θ	$r = 5 - 5\sin(\theta)$	$r = 7 - 6\cos(\theta)$	$r = 2 + 4\cos(\theta)$
0	5	1	6
$\frac{\pi}{2}$	0	7	2
π	5	13	-2
$\frac{3\pi}{2}$	10	7	2
2π	5	1	6







There is one final thing that we need to do in this section. In the third graph in the previous example we had an inner loop. We will, on occasion, need to know the value of θ for which the graph will pass through the origin. To find these all we need to do is set the equation equal to zero and solve as follows,

$$0 = 2 + 4\cos(\theta) \qquad \Rightarrow \qquad \cos(\theta) = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

9.7 Tangents with Polar Coordinates

We now need to discuss some calculus topics in terms of polar coordinates.

We will start with finding tangent lines to polar curves. In this case we are going to assume that the equation is in the form $r = f(\theta)$. With the equation in this form we can actually use the equation for the derivative $\frac{dy}{dx}$ we derived when we looked at tangent lines with parametric equations. To do this however requires us to come up with a set of parametric equations to represent the curve. This is actually pretty easy to do.

From our work in the previous section we have the following set of conversion equations for going from polar coordinates to Cartesian coordinates.

$$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta)$$

Now, we'll use the fact that we're assuming that the equation is in the form $r = f(\theta)$. Substituting this into these equations gives the following set of parametric equations (with θ as the parameter) for the curve.

$$x = f(\theta)\cos(\theta) \qquad\qquad y = f(\theta)\sin(\theta)$$

Now, we will need the following derivatives.

$$\begin{aligned}\frac{dx}{d\theta} &= f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)\\ &= \frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)\end{aligned} \qquad\qquad \begin{aligned}\frac{dy}{d\theta} &= f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)\\ &= \frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)\end{aligned}$$

The derivative $\frac{dy}{dx}$ is then,

Derivative with Polar Coordinates, $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

Note that rather than trying to remember this formula it would probably be easier to remember how we derived it and just remember the formula for parametric equations.

Let's work a quick example with this.

Example 1

Determine the equation of the tangent line to $r = 3 + 8\sin(\theta)$ at $\theta = \dfrac{\pi}{6}$.

Solution

We'll first need the following derivative.

$$\frac{dr}{d\theta} = 8\cos(\theta)$$

The formula for the derivative $\frac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos(\theta)\sin(\theta) + (3+8\sin(\theta))\cos(\theta)}{8\cos^2(\theta) - (3+8\sin(\theta))\sin(\theta)} = \frac{16\cos(\theta)\sin(\theta) + 3\cos(\theta)}{8\cos^2(\theta) - 3\sin(\theta) - 8\sin^2(\theta)}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

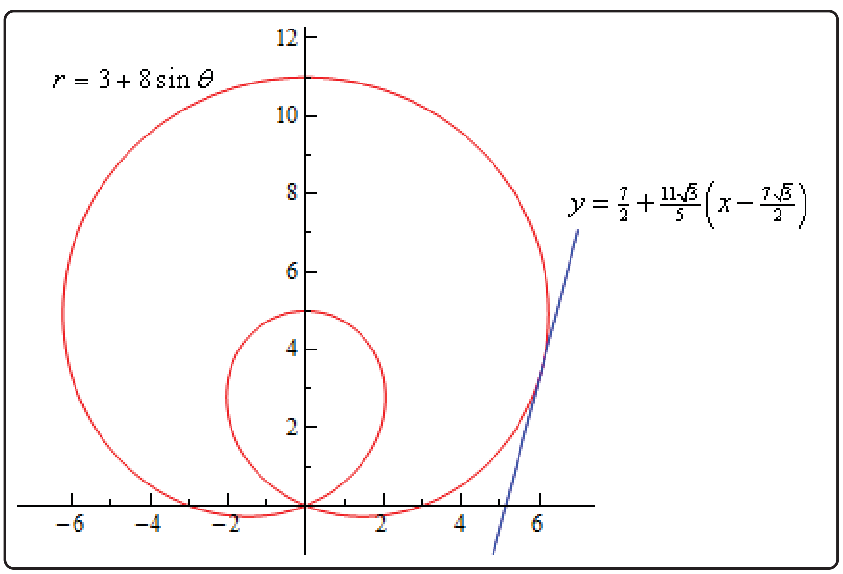
Now, at $\theta = \frac{\pi}{6}$ we have $r = 7$. We'll need to get the corresponding x-y coordinates so we can get the tangent line.

$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad\qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

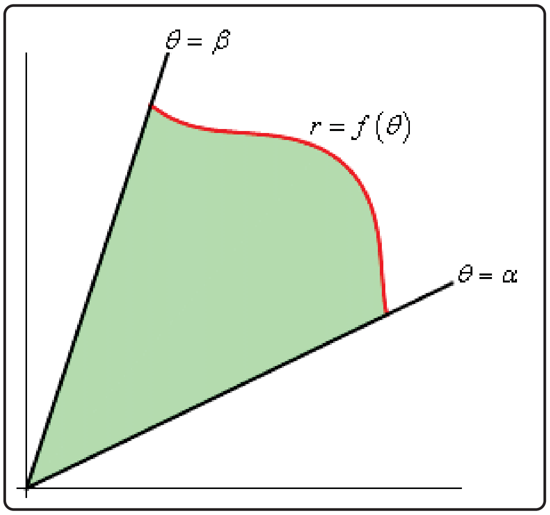
$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

For the sake of completeness here is a graph of the curve and the tangent line.



9.8 Area with Polar Coordinates

In this section we are going to look at areas enclosed by polar curves. Note as well that we said “enclosed by” instead of “under” as we typically have in these problems. These problems work a little differently in polar coordinates. Here is a sketch of what the area that we’ll be finding in this section looks like.



We’ll be looking for the shaded area in the sketch above. The formula for finding this area is,

Area Enclosed by Curve

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

Notice that we use r in the integral instead of $f(\theta)$ so make sure and substitute accordingly when doing the integral.

Let’s take a look at an example.

Example 1

Determine the area of the inner loop of $r = 2 + 4\cos(\theta)$.

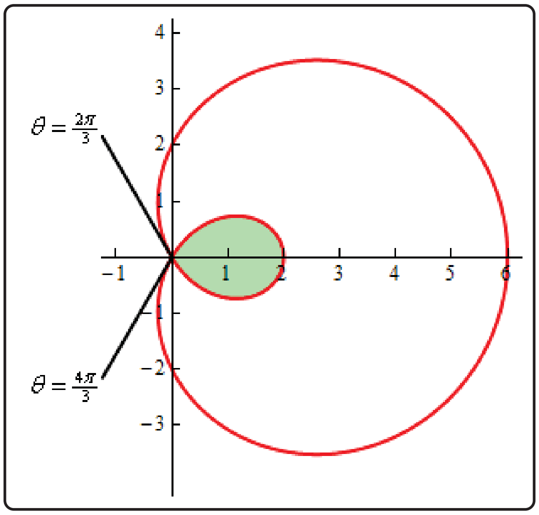
Solution

We graphed this function back when we first started looking at polar coordinates. For this problem we’ll also need to know the values of θ where the curve goes through the origin.

We can get these by setting the equation equal to zero and solving.

$$0 = 2 + 4\cos(\theta)$$
$$\cos(\theta) = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Here is the sketch of this curve with the inner loop shaded in.



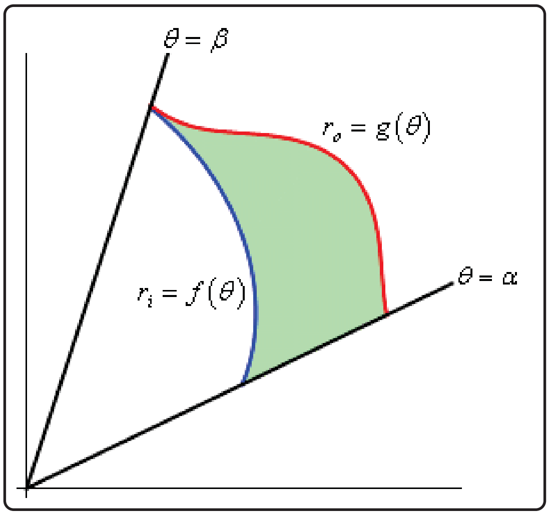
Can you see why we needed to know the values of θ where the curve goes through the origin? These points define where the inner loop starts and ends and hence are also the limits of integration in the formula.

So, the area is then,

$$\begin{aligned} A &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(2 + 4\cos(\theta)\right)^2 d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4 + 16\cos(\theta) + 16\cos^2(\theta)\right) d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2 + 8\cos(\theta) + 4\left(1 + \cos(2\theta)\right) d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 6 + 8\cos(\theta) + 4\cos(2\theta)\ d\theta \\ &= \left(6\theta + 8\sin(\theta) + 2\sin(2\theta)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \\ &= 4\pi - 6\sqrt{3} = 2.174 \end{aligned}$$

You did follow the work done in this integral didn't you? You'll run into quite a few integrals of trig functions in this section so if you need to you should go back to the Integrals Involving Trig Functions sections and do a quick review.

So, that's how we determine areas that are enclosed by a single curve, but what about situations like the following sketch where we want to find the area between two curves.



In this case we can use the above formula to find the area enclosed by both and then the actual area is the difference between the two. The formula for this is,

Area Between Curves

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_o^2 - r_i^2\right)\,d\theta$$

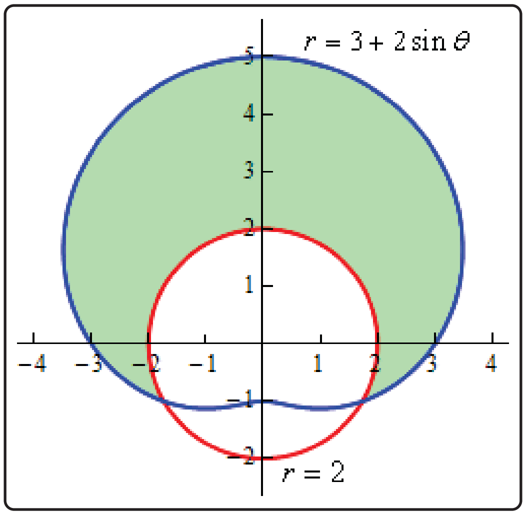
Let's take a look at an example of this.

Example 2

Determine the area that lies inside $r = 3 + 2\sin(\theta)$ and outside $r = 2$.

Solution

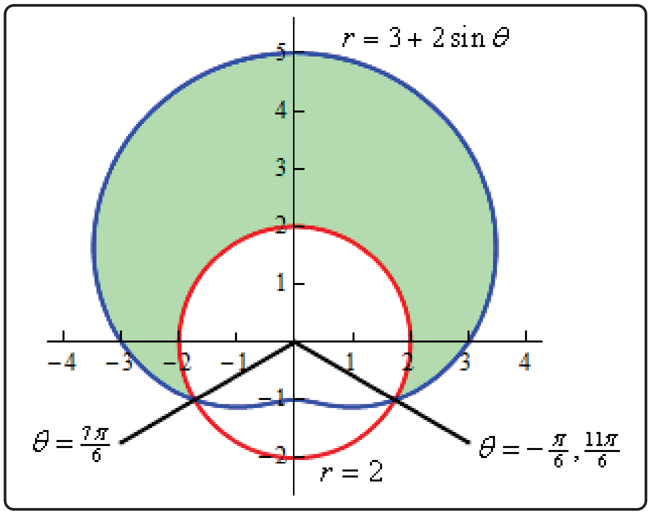
Here is a sketch of the region that we are after.



To determine this area, we'll need to know the values of θ for which the two curves intersect. We can determine these points by setting the two equations and solving.

$$\begin{aligned} 3 + 2\sin(\theta) &= 2 \\ \sin(\theta) &= -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

Here is a sketch of the figure with these angles added.



Note as well here that we also acknowledged that another representation for the angle $\frac{11\pi}{6}$ is $-\frac{\pi}{6}$. This is important for this problem. In order to use the formula above the area must be enclosed as we increase from the smaller to larger angle. So, if we use $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we will not enclose the shaded area, instead we will enclose the bottom most of the three regions. However, if we use the angles $-\frac{\pi}{6}$ to $\frac{7\pi}{6}$ we will enclose the area that we're after.

So, the area is then,

$$\begin{aligned} A &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left((3+2\sin(\theta))^2 - (2)^2\right) d\theta \\ &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(5+12\sin(\theta)+4\sin^2(\theta)\right) d\theta \\ &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(7+12\sin(\theta)-2\cos(2\theta)\right) d\theta \\ &= \frac{1}{2}\left(7\theta - 12\cos(\theta) - \sin(2\theta)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} = \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187 \end{aligned}$$

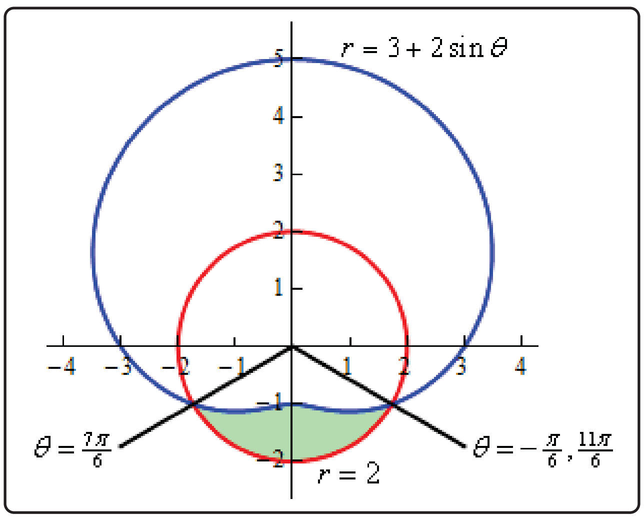
Let's work a slight modification of the previous example.

Example 3

Determine the area of the region outside $r = 3 + 2\sin(\theta)$ and inside $r = 2$.

Solution

This time we're looking for the following region.



So, this is the region that we get by using the limits $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$. The area for this region is,

$$\begin{aligned} A &= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left((2)^2 - (3+2\sin(\theta))^2\right) d\theta \\ &= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-5 - 12\sin(\theta) - 4\sin^2(\theta)\right) d\theta \\ &= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-7 - 12\sin(\theta) + 2\cos(2\theta)\right) d\theta \\ &= \frac{1}{2}\left(-7\theta + 12\cos(\theta) + \sin(2\theta)\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} = \frac{11\sqrt{3}}{2} - \frac{7\pi}{3} = 2.196 \end{aligned}$$

Notice that for this area the "outer" and "inner" function were opposite!

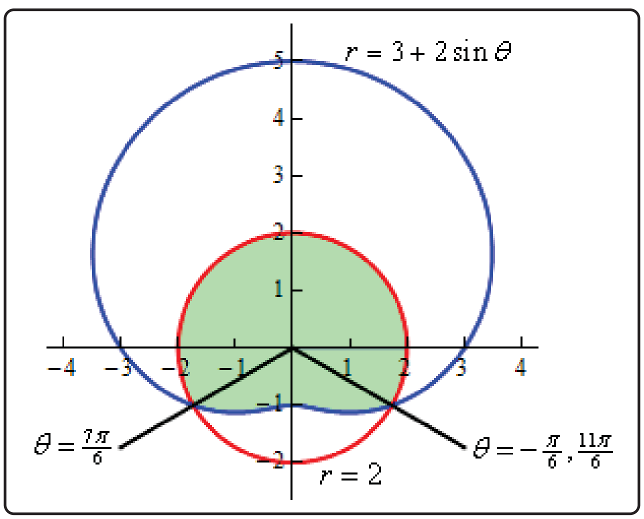
Let's do one final modification of this example.

Example 4

Determine the area that is inside both $r = 3 + 2\sin(\theta)$ and $r = 2$.

Solution

Here is the sketch for this example.



We are not going to be able to do this problem in the same fashion that we did the previous two. There is no set of limits that will allow us to enclose this area as we increase from one to the other. Remember that as we increase θ the area we're after must be enclosed.

However, the only two ranges for θ that we can work with enclose the area from the previous two examples and not this region.

In this case however, that is not a major problem. There are two ways to do get the area in this problem. We'll take a look at both of them.

Solution 1

In this case let's notice that the circle is divided up into two portions and we're after the upper portion. Also notice that we found the area of the lower portion in Example 3. Therefore, the area is,

$$\begin{aligned}\text{Area} &= \text{Area of Circle} - \text{Area from Example 3}\\ &= \pi(2)^2 - 2.196\\ &= 10.370\end{aligned}$$

Solution 2

In this case we do pretty much the same thing except this time we'll think of the area as the other portion of the limacon than the portion that we were dealing with in Example 2. We'll also need to actually compute the area of the limacon in this case.

So, the area using this approach is then,

$$\begin{aligned}\text{Area} &= \text{Area of Limacon} - \text{Area from Example 2}\\ &= \int_0^{2\pi} \frac{1}{2}\left(3 + 2\sin(\theta)\right)^2 d\theta - 24.187\\ &= \int_0^{2\pi} \frac{1}{2}\left(9 + 12\sin(\theta) + 4\sin^2(\theta)\right) d\theta - 24.187\\ &= \int_0^{2\pi} \frac{1}{2}\left(11 + 12\sin(\theta) - 2\cos(2\theta)\right) d\theta - 24.187\\ &= \frac{1}{2}\left(11\theta - 12\cos(\theta) - \sin(2\theta)\right)\Big|_0^{2\pi} - 24.187\\ &= 11\pi - 24.187\\ &= 10.370\end{aligned}$$

A slightly longer approach, but sometimes we are forced to take this longer approach.

As this last example has shown we will not be able to get all areas in polar coordinates straight from an integral.

9.9 Arc Length with Polar Coordinates

We now need to move into the Calculus II applications of integrals and how we do them in terms of polar coordinates. In this section we'll look at the arc length of the curve given by,

$$r = f(\theta) \qquad \alpha \le \theta \le \beta$$

where we also assume that the curve is traced out exactly once. Just as we did with the tangent lines in polar coordinates we'll first write the curve in terms of a set of parametric equations,

$$\begin{aligned} x &= r\cos(\theta) & y &= r\sin(\theta) \\ &= f(\theta)\cos(\theta) & &= f(\theta)\sin(\theta) \end{aligned}$$

and we can now use the parametric formula for finding the arc length.

We'll need the following derivatives for these computations.

$$\begin{aligned} \frac{dx}{d\theta} &= f'(\theta)\cos(\theta) - f(\theta)\sin(\theta) & \frac{dy}{d\theta} &= f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) \\ &= \frac{dr}{d\theta}\cos(\theta) - r\sin(\theta) & &= \frac{dr}{d\theta}\sin(\theta) + r\cos(\theta) \end{aligned}$$

We'll need the following for our ds.

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)\right)^2 + \left(\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)\right)^2 \\ &= \left(\frac{dr}{d\theta}\right)^2\cos^2(\theta) - 2r\frac{dr}{d\theta}\cos(\theta)\sin(\theta) + r^2\sin^2(\theta) \\ &\qquad + \left(\frac{dr}{d\theta}\right)^2\sin^2(\theta) + 2r\frac{dr}{d\theta}\cos(\theta)\sin(\theta) + r^2\cos^2(\theta) \\ &= \left(\frac{dr}{d\theta}\right)^2\left(\cos^2(\theta) + \sin^2(\theta)\right) + r^2\left(\cos^2(\theta) + \sin^2(\theta)\right) \\ &= r^2 + \left(\frac{dr}{d\theta}\right)^2 \end{aligned}$$

The arc length formula for polar coordinates is then,

Arc Length with Polar Coordinates

$$L = \int ds$$

where,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Let's work a quick example of this.

Example 1

Determine the length of $r = \theta$, $0 \le \theta \le 1$.

Solution

Okay, let's just jump straight into the formula since this is a fairly simple function.

$$L = \int_0^1 \sqrt{\theta^2 + 1}\, d\theta$$

We'll need to use a trig substitution here.

$$\theta = \tan(x) \qquad d\theta = \sec^2(x)\, dx$$

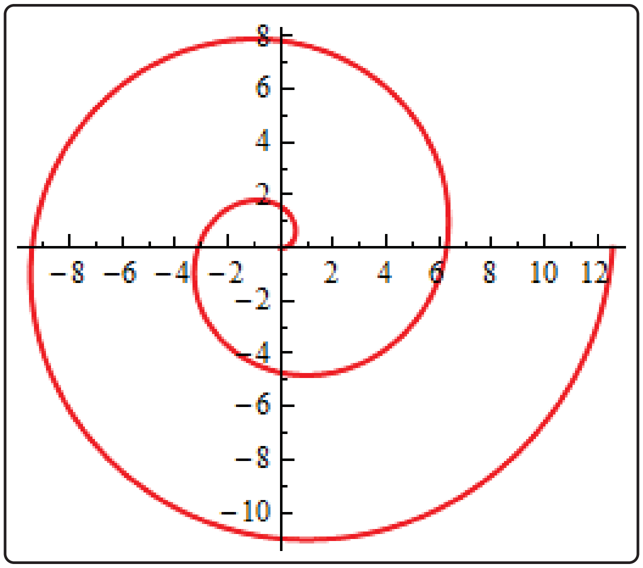
$$\begin{array}{lll} \theta = 0 & 0 = \tan(x) & x = 0 \\ \theta = 1 & 1 = \tan(x) & x = \dfrac{\pi}{4} \end{array}$$

$$\sqrt{\theta^2 + 1} = \sqrt{\tan^2(x) + 1} = \sqrt{\sec^2(x)} = |\sec(x)| = \sec(x)$$

The arc length is then,

$$\begin{aligned} L &= \int_0^1 \sqrt{\theta^2 + 1}\, d\theta \\ &= \int_0^{\frac{\pi}{4}} \sec^3(x)\, dx \\ &= \frac{1}{2}\Big(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|\Big)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right) \end{aligned}$$

Just as an aside before we leave this chapter. The polar equation $r = \theta$ is the equation of a spiral. Here is a quick sketch of $r = \theta$ for $0 \le \theta \le 4\pi$.


8
6
4
2
−2
−4
−6
−8
−10
−8 −6 −4 −2 2 4 6 8 10 12

9.10 Surface Area with Polar Coordinates

We will be looking at surface area in polar coordinates in this section. Note however that all we're going to do is give the formulas for the surface area since most of these integrals tend to be fairly difficult.

We want to find the surface area of the region found by rotating,

$$r = f(\theta) \qquad \alpha \le \theta \le \beta$$

about the x or y-axis.

As we did in the tangent and arc length sections we'll write the curve in terms of a set of parametric equations.

$$\begin{aligned} x &= r\cos(\theta) & \qquad y &= r\sin(\theta) \\ &= f(\theta)\cos(\theta) & &= f(\theta)\sin(\theta) \end{aligned}$$

If we now use the parametric formula for finding the surface area we'll get,

Surface Area with Polar Coordinates

$$\begin{aligned} S &= \int 2\pi y\, ds && \text{rotation about } x-\text{axis} \\ S &= \int 2\pi x\, ds && \text{rotation about } y-\text{axis} \end{aligned}$$

where,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad r = f(\theta), \quad \alpha \le \theta \le \beta$$

Note that because we will pick up a $d\theta$ from the ds we'll need to substitute one of the parametric equations in for x or y depending on the axis of rotation. This will often mean that the integrals will be somewhat unpleasant and so we will not be doing an example in this section.

9.11 Arc Length and Surface Area Revisited

We won't be working any examples in this section. This section is here solely for the purpose of summarizing up all the arc length and surface area problems.

Over the course of the last two chapters the topic of arc length and surface area has arisen many times and each time we got a new formula out of the mix. Students often get a little overwhelmed with all the formulas.

However, there really aren't as many formulas as it might seem at first glance. There is exactly one arc length formula and exactly two surface area formulas. These are,

Arc Length and Surface Area Formulas

$$L = \int ds$$

$$S = \int 2\pi y\, ds \qquad \text{rotation about } x - \text{axis}$$

$$S = \int 2\pi x\, ds \qquad \text{rotation about } y - \text{axis}$$

The problems arise because we have quite a few ds's that we can use. Again, students often have trouble deciding which one to use. The examples/problems usually suggest the correct one to use however. Here is a complete listing of all the ds's that we've seen and when they are used.

Various Formulas for ds

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f\left(x\right),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h\left(y\right),\ c \le y \le d$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{if } x = f\left(t\right), y = g\left(t\right),\ \alpha \le t \le \beta$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \text{if } r = f\left(\theta\right),\ \alpha \le \theta \le \beta$$

Depending on the form of the function we can quickly tell which ds to use.

There is only one other thing to worry about in terms of the surface area formula. The ds will introduce a new differential to the integral. Before integrating make sure all the variables are in terms of this new differential. For example, if we have parametric equations we'll use the third ds and then we'll need to make sure and substitute for the x or y depending on which axis we rotate about to get everything in terms of t.

Likewise, if we have a function in the form $x = h(y)$ then we'll use the second ds and if the rotation is about the y-axis we'll need to substitute for the x in the integral. On the other hand, if we rotate about the x-axis we won't need to do a substitution for the y.

Keep these rules in mind and you'll always be able to determine which formula to use and how to correctly do the integral.