1.7 Exponential Functions

In this section we're going to review one of the more common functions in both calculus and the sciences. However, before getting to this function let's take a much more general approach to things.

Let's start with $b > 0$, $b \neq 1$. An exponential function is then a function in the form,

$$f\left(x\right) = b^x$$

Note that we avoid $b = 1$ because that would give the constant function, $f\left(x\right) = 1$. We avoid $b = 0$ since this would also give a constant function and we avoid negative values of b for the following reason.

Let's, for a second, suppose that we did allow b to be negative and look at the following function.

$$g\left(x\right) = \left(-4\right)^x$$

Let's do some evaluation.

$$g\left(2\right) = \left(-4\right)^2 = 16 \qquad g\left(\frac{1}{2}\right) = \left(-4\right)^{\frac{1}{2}} = \sqrt{-4} = 2i$$

So, for some values of x we will get real numbers and for other values of x we will get complex numbers. We want to avoid this so if we require $b > 0$ this will not be a problem.

Let's take a look at a couple of exponential functions.

Example 1

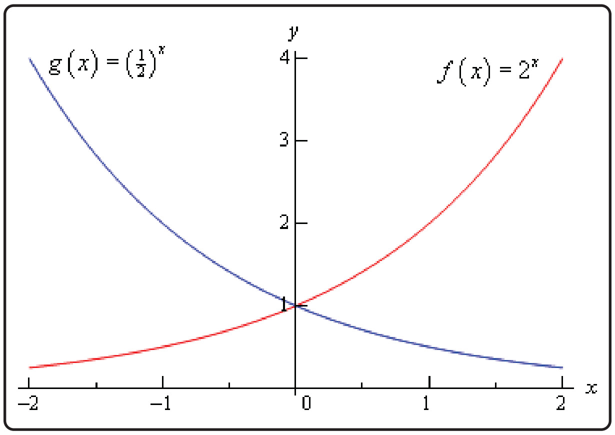
Sketch the graph of $f\left(x\right) = 2^x$ and $g\left(x\right) = \left(\frac{1}{2}\right)^x$.

Solution

Let's first get a table of values for these two functions.

x	$f(x)$	$g(x)$
-2	$f\left(-2\right) = 2^{-2} = \frac{1}{4}$	$g\left(-2\right) = \left(\frac{1}{2}\right)^{-2} = 4$
-1	$f\left(-1\right) = 2^{-1} = \frac{1}{2}$	$g\left(-1\right) = \left(\frac{1}{2}\right)^{-1} = 2$
0	$f\left(0\right) = 2^0 = 1$	$g\left(0\right) = \left(\frac{1}{2}\right)^0 = 1$
1	$f\left(1\right) = 2$	$g\left(1\right) = \frac{1}{2}$
2	$f\left(2\right) = 4$	$g\left(2\right) = \frac{1}{4}$

Here's the sketch of both of these functions.



This graph illustrates some very nice properties about exponential functions in general.

Properties of $f(x) = b^x$

1. $f(0) = 1$. The function will always take the value of 1 at $x = 0$.
2. $f(x) \neq 0$. An exponential function will never be zero.
3. $f(x) > 0$. An exponential function is always positive.
4. The previous two properties can be summarized by saying that the range of an exponential function is $(0, \infty)$.
5. The domain of an exponential function is $(-\infty, \infty)$. In other words, you can plug every x into an exponential function.
6. If $0 < b < 1$ then,
   (a) $f(x) \to 0$ as $x \to \infty$
   (b) $f(x) \to \infty$ as $x \to -\infty$
7. If $b > 1$ then,
   (a) $f(x) \to \infty$ as $x \to \infty$
   (b) $f(x) \to 0$ as $x \to -\infty$

These will all be very useful properties to recall at times as we move throughout this course (and later Calculus courses for that matter...).

There is a very important exponential function that arises naturally in many places. This function is called the **natural exponential function**. However, for most people, this is simply the exponential function.

Definition

The **natural exponential function** is $f(x) = \mathbf{e}^x$ where, $\mathbf{e} = 2.71828182845905\ldots$.

So, since $\mathbf{e} > 1$ we also know that $\mathbf{e}^x \to \infty$ as $x \to \infty$ and $\mathbf{e}^x \to 0$ as $x \to -\infty$.

Let's take a quick look at an example.

Example 2

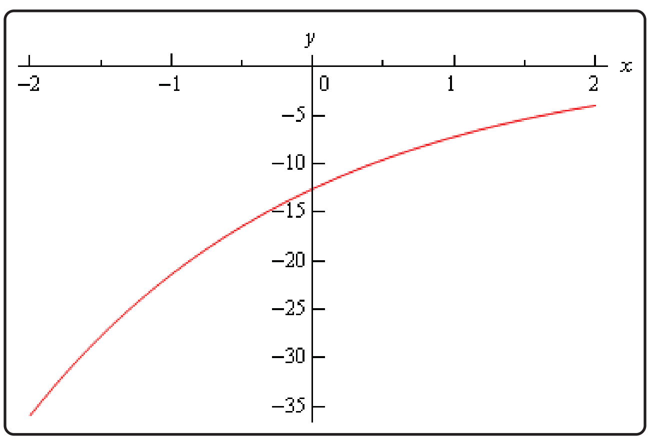
Sketch the graph of $h(t) = 1 - 5\mathbf{e}^{1-\frac{t}{2}}$.

Solution

Let's first get a table of values for this function.

t	-2	-1	0	1	2	3
$h(t)$	-35.9453	-21.4084	-12.5914	-7.2436	-4	-2.0327

Here is the sketch.



The main point behind this problem is to make sure you can do this type of evaluation so make sure that you can get the values that we graphed in this example. You will be asked to do this kind of evaluation on occasion in this class.

You will be seeing exponential functions in pretty much every chapter in this class so make sure that you are comfortable with them.

1.8 Logarithm Functions

In this section we'll take a look at a function that is related to the exponential functions we looked at in the last section. We will look at logarithms in this section. Logarithms are one of the functions that students fear the most. The main reason for this seems to be that they simply have never really had to work with them. Once they start working with them, students come to realize that they aren't as bad as they first thought.

We'll start with $b > 0$, $b \neq 1$ just as we did in the last section. Then we have

$$y = \log_b x \qquad \text{is equivalent to} \qquad x = b^y$$

The first is called logarithmic form and the second is called the exponential form. Remembering this equivalence is the key to evaluating logarithms. The number, b, is called the base.

Let's do some quick evaluations.

Example 1

Without a calculator give the exact value of each of the following logarithms.

(a) $\log_2 16$

(b) $\log_4 16$

(c) $\log_5 625$

(d) $\log_9 \frac{1}{531441}$

(e) $\log_{\frac{1}{6}} 36$

(f) $\log_{\frac{3}{2}} \frac{27}{8}$

Solution

To quickly evaluate logarithms the easiest thing to do is to convert the logarithm to exponential form. So, let's take a look at the first one.

(a) $\log_2 16$

First, let's convert to exponential form.

$$\log_2 16 = ? \qquad \text{is equivalent to} \qquad 2^? = 16$$

So, we're really asking 2 raised to what gives 16. Since 2 raised to 4 is 16 we get,

$$\log_2 16 = 4 \qquad \text{because} \qquad 2^4 = 16$$

We'll not do the remaining parts in quite this detail, but they will all work in this way.

(b) $\log_4 16$

$$\log_4 16 = 2 \qquad \text{because} \qquad 4^2 = 16$$

Note the difference between the first and second logarithm! The base is important! It can completely change the answer.

(c) $\log_5 625$

$$\log_5 625 = 4 \qquad \text{because} \qquad 5^4 = 625$$

(d) $\log_9 \frac{1}{531441}$

$$\log_9 \frac{1}{531441} = -6 \qquad \text{because} \qquad 9^{-6} = \frac{1}{9^6} = \frac{1}{531441}$$

(e) $\log_{\frac{1}{6}} 36$

$$\log_{\frac{1}{6}} 36 = -2 \qquad \text{because} \qquad \left(\frac{1}{6}\right)^{-2} = 6^2 = 36$$

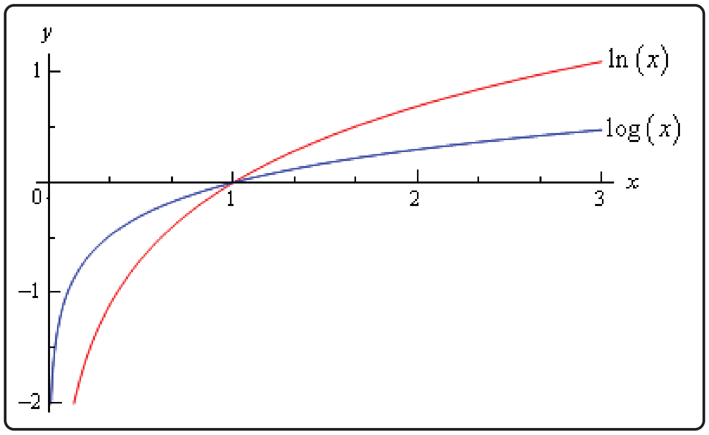
(f) $\log_{\frac{3}{2}} \frac{27}{8}$

$$\log_{\frac{3}{2}} \frac{27}{8} = 3 \qquad \text{because} \qquad \left(\frac{3}{2}\right)^{3} = \frac{27}{8}$$

There are a couple of special logarithms that arise in many places. These are,

$$\ln(x) = \log_{\mathbf{e}}(x) \qquad \text{This log is called the natural logarithm}$$
$$\log(x) = \log_{10}(x) \qquad \text{This log is called the common logarithm}$$

In the natural logarithm the base **e** is the same number as in the natural exponential logarithm that we saw in the last section. Here is a sketch of both of these logarithms.



From this graph we can get a couple of very nice properties about the natural logarithm that we will use many times in this and later Calculus courses.

$$\ln(x) \to \infty \text{ as } x \to \infty$$
$$\ln(x) \to -\infty \text{ as } x \to 0, \ x > 0$$

Let's take a look at a couple of more logarithm evaluations. Some of which deal with the natural or common logarithm and some of which don't.

Example 2

Without a calculator give the exact value of each of the following logarithms.

(a) $\ln \sqrt[3]{\mathbf{e}}$

(b) $\log 1000$

(c) $\log_{16} 16$

(d) $\log_{23} 1$

(e) $\log_2 \sqrt[7]{32}$

Solution

These work exactly the same as previous example so we won't put in too many details.

(a) $\ln \sqrt[3]{\mathbf{e}}$

$$\ln \sqrt[3]{\mathbf{e}} = \frac{1}{3} \qquad \text{because} \qquad \mathbf{e}^{\frac{1}{3}} = \sqrt[3]{\mathbf{e}}$$

(b) $\log 1000$

$$\log 1000 = 3 \qquad \text{because} \qquad 10^3 = 1000$$

(c) $\log_{16} 16$

$$\log_{16} 16 = 1 \qquad \text{because} \qquad 16^1 = 16$$

(d) $\log_{23} 1$

$$\log_{23} 1 = 0 \qquad \text{because} \qquad 23^0 = 1$$

(e) $\log_2 \sqrt[7]{32}$

$$\log_2 \sqrt[7]{32} = \frac{5}{7} \qquad \text{because} \qquad \sqrt[7]{32} = 32^{\frac{1}{7}} = \left(2^5\right)^{\frac{1}{7}} = 2^{\frac{5}{7}}$$

This last set of examples leads us to some of the basic properties of logarithms.

Properties

1. The domain of the logarithm function is $(0, \infty)$. In other words, we can only plug positive numbers into a logarithm! We can't plug in zero or a negative number.
2. The range of the logarithm function is $(-\infty, \infty)$.
3. $\log_b(b) = 1$
4. $\log_b(1) = 0$
5. $\log_b(b^x) = x$
6. $b^{\log_b(x)} = x$

The last two properties will be especially useful in the next section. Notice as well that these last two properties tell us that,

$$f(x) = b^x \qquad \text{and} \qquad g(x) = \log_b(x)$$

are inverses of each other.

Here are some more properties that are useful in the manipulation of logarithms.

More Properties

5. $\log_b(xy) = \log_b(x) + \log_b(y)$
6. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
7. $\log_b(x^r) = r\log_b(x)$

Note that there is no equivalent property to the first two for sums and differences. In other words,

$$\log_b(x+y) \neq \log_b(x) + \log_b(y)$$
$$\log_b(x-y) \neq \log_b(x) - \log_b(y)$$

Example 3

Write each of the following in terms of simpler logarithms.

(a) $\ln(x^3y^4z^5)$

(b) $\log_3\left(\frac{9x^4}{\sqrt{y}}\right)$

(c) $\log\left(\frac{x^2+y^2}{(x-y)^3}\right)$

Solution

What the instructions really mean here is to use as many of the properties of logarithms as we can to simplify things down as much as we can.

(a) $\ln(x^3y^4z^5)$

Property 7 above can be extended to products of more than two functions. Once we've used Property 7 we can then use Property 9.

$$\begin{aligned}\ln(x^3y^4z^5) &= \ln(x^3) + \ln(y^4) + \ln(z^5)\\ &= 3\ln(x) + 4\ln(y) + 5\ln(z)\end{aligned}$$

(b) $\log_3\left(\frac{9x^4}{\sqrt{y}}\right)$

When using property 8 above make sure that the logarithm that you subtract is the one that contains the denominator as its argument. Also, note that that we'll be converting the root to fractional exponents in the first step.

$$\begin{aligned}\log_3\left(\frac{9x^4}{\sqrt{y}}\right) &= \log_3(9x^4) - \log_3(y^{\frac{1}{2}}) \\ &= \log_3(9) + \log_3(x^4) - \log_3(y^{\frac{1}{2}}) \\ &= 2 + 4\log_3(x) - \frac{1}{2}\log_3(y)\end{aligned}$$

(c) $\log\left(\dfrac{x^2+y^2}{(x-y)^3}\right)$

The point to this problem is mostly the correct use of property 9 above.

$$\begin{aligned}\log\left(\frac{x^2+y^2}{(x-y)^3}\right) &= \log\left(x^2+y^2\right) - \log\left(x-y\right)^3 \\ &= \log\left(x^2+y^2\right) - 3\log\left(x-y\right)\end{aligned}$$

You can use Property 9 on the second term because the WHOLE term was raised to the 3, but in the first logarithm, only the individual terms were squared and not the term as a whole so the 2's must stay where they are!

The last topic that we need to look at in this section is the **change of base** formula for logarithms. The change of base formula is,

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

This is the most general change of base formula and will convert from base b to base a. However, the usual reason for using the change of base formula is to compute the value of a logarithm that is in a base that you can't easily deal with. Using the change of base formula means that you can write the logarithm in terms of a logarithm that you can deal with. The two most common change of base formulas are

$$\log_b(x) = \frac{\ln(x)}{\ln(b)} \qquad \text{and} \qquad \log_{\mathrm{b}}(x) = \frac{\log(x)}{\log(b)}$$

In fact, often you will see one or the other listed as THE change of base formula!

In the first part of this section we computed the value of a few logarithms, but we could do these without the change of base formula because all the arguments could be written in terms of the base to a power. For instance,

$$\log_7(49) = 2 \qquad \text{because} \qquad 7^2 = 49$$

However, this only works because 49 can be written as a power of 7! We would need the change of base formula to compute $\log_7 50$.

$$\log_7(50) = \frac{\ln(50)}{\ln(7)} = \frac{3.91202300543}{1.94591014906} = 2.0103821378$$

OR

$$\log_7(50) = \frac{\log(50)}{\log(7)} = \frac{1.69897000434}{0.845098040014} = 2.0103821378$$

So, it doesn't matter which we use, we will get the same answer regardless of the logarithm that we use in the change of base formula.

Note as well that we could use the change of base formula on $\log_7 49$ if we wanted to as well.

$$\log_7(49) = \frac{\ln(49)}{\ln(7)} = \frac{3.89182029811}{1.94591014906} = 2$$

This is a lot of work however, and is probably not the best way to deal with this.

So, in this section we saw how logarithms work and took a look at some of the properties of logarithms. We will run into logarithms on occasion so make sure that you can deal with them when we do run into them.

1.9 Exponential and Logarithm Equations

In this section we'll take a look at solving equations with exponential functions or logarithms in them.

We'll start with equations that involve exponential functions. The main property that we'll need for these equations is,

$$\log_b(b^x) = x$$

Example 1

Solve $7 + 15\mathbf{e}^{1-3z} = 10$.

Solution

The first step is to get the exponential all by itself on one side of the equation with a coefficient of one.

$$\begin{aligned} 7 + 15\mathbf{e}^{1-3z} &= 10 \\ 15\mathbf{e}^{1-3z} &= 3 \\ \mathbf{e}^{1-3z} &= \frac{1}{5} \end{aligned}$$

Now, we need to get the z out of the exponent so we can solve for it. To do this we will use the property above. Since we have an **e** in the equation we'll use the natural logarithm. First, we take the logarithm of both sides and then use the property to simplify the equation.

$$\begin{aligned} \ln\left(\mathbf{e}^{1-3z}\right) &= \ln\left(\frac{1}{5}\right) \\ 1 - 3z &= \ln\left(\frac{1}{5}\right) \end{aligned}$$

All we need to do now is solve this equation for z.

$$\begin{aligned} 1 - 3z &= \ln\left(\frac{1}{5}\right) \\ -3z &= -1 + \ln\left(\frac{1}{5}\right) \\ z &= -\frac{1}{3}\left(-1 + \ln\left(\frac{1}{5}\right)\right) = 0.8698126372 \end{aligned}$$

Example 2

Solve $10^{t^2-t} = 100$.

Solution

Now, in this case it looks like the best logarithm to use is the common logarithm since left hand side has a base of 10. There's no initial simplification to do, so just take the log of both sides and simplify.

$$\begin{aligned} \log 10^{t^2-t} &= \log 100 = \log 10^2 = 2 \\ t^2 - t &= 2 \end{aligned}$$

At this point, we've just got a quadratic that can be solved

$$\begin{aligned} t^2 - t - 2 &= 0 \\ (t-2)(t+1) &= 0 \end{aligned}$$

So, it looks like the solutions in this case are $t = 2$ and $t = -1$.

Now that we've seen a couple of equations where the variable only appears in the exponent we need to see an example with variables both in the exponent and out of it.

Example 3

Solve $x - x\mathbf{e}^{5x+2} = 0$.

Solution

The first step is to factor an x out of both terms.

DO NOT DIVIDE AN x FROM BOTH TERMS!!!!

Note that it is very tempting to "simplify" the equation by dividing an x out of both terms. However, if you do that you'll miss a solution as we'll see.

$$\begin{aligned} x - x\mathbf{e}^{5x+2} &= 0 \\ x\left(1 - \mathbf{e}^{5x+2}\right) &= 0 \end{aligned}$$

So, it's now a little easier to deal with. From this we can see that we get one of two possibilities.

$$x = 0 \qquad \text{OR} \qquad 1 - \mathbf{e}^{5x+2} = 0$$

The first possibility has nothing more to do, except notice that if we had divided both sides by an x we would have missed this one so be careful. In the second possibility we've got a little more to do. This is an equation similar to the first two that we did in this section.

$$\begin{aligned} \mathbf{e}^{5x+2} &= 1 \\ 5x+2 &= \ln 1 \\ 5x+2 &= 0 \\ x &= -\frac{2}{5} \end{aligned}$$

Don't forget that $\ln 1 = 0$.

So, the two solutions are $x = 0$ and $x = -\frac{2}{5}$.

The next equation is a more complicated (looking at least...) example similar to the previous one.

Example 4

Solve $5\left(x^2-4\right) = \left(x^2-4\right)\mathbf{e}^{7-x}$.

Solution

As with the previous problem do NOT divide an x^2-4 out of both sides. Doing this will lose solutions even though it "simplifies" the equation. Note however, that if you can divide a term out then you can also factor it out if the equation is written properly.

So, the first step here is to move everything to one side of the equation and then to factor out the x^2-4.

$$\begin{aligned} 5\left(x^2-4\right) - \left(x^2-4\right)\mathbf{e}^{7-x} &= 0 \\ \left(x^2-4\right)\left(5-\mathbf{e}^{7-x}\right) &= 0 \end{aligned}$$

At this point all we need to do is set each factor equal to zero and solve each.

$$\begin{aligned} x^2-4 &= 0 & \qquad 5-\mathbf{e}^{7-x} &= 0 \\ x &= \pm 2 & \mathbf{e}^{7-x} &= 5 \\ & & 7-x &= \ln(5) \\ & & x &= 7-\ln(5) = 5.390562088 \end{aligned}$$

The three solutions are then $x = \pm 2$ and $x = 5.3906$.

As a final example let's take a look at an equation that contains two different exponentials.

Example 5

Solve $4\mathbf{e}^{1+3x} - 9\mathbf{e}^{5-2x} = 0$.

Solution

The first step here is to get one exponential on each side and then we'll divide both sides by one of them (which doesn't matter for the most part) so we'll have a quotient of two exponentials. The quotient can then be simplified and we'll finally get both coefficients on the other side. Doing all of this gives,

$$\begin{aligned} 4\mathbf{e}^{1+3x} &= 9\mathbf{e}^{5-2x} \\ \frac{\mathbf{e}^{1+3x}}{\mathbf{e}^{5-2x}} &= \frac{9}{4} \\ \mathbf{e}^{1+3x-(5-2x)} &= \frac{9}{4} \\ \mathbf{e}^{5x-4} &= \frac{9}{4} \end{aligned}$$

Note that while we said that it doesn't really matter which exponential we divide out by doing it the way we did here we'll avoid a negative coefficient on the x. Not a major issue, but those minus signs on coefficients are really easy to lose on occasion.

This is now in a form that we can deal with so here's the rest of the solution.

$$\begin{aligned} \mathbf{e}^{5x-4} &= \frac{9}{4} \\ 5x-4 &= \ln\left(\frac{9}{4}\right) \\ 5x &= 4+\ln\left(\frac{9}{4}\right) \\ x &= \frac{1}{5}\left(4+\ln\left(\frac{9}{4}\right)\right) = 0.9621860432 \end{aligned}$$

This equation has a single solution of $x = 0.9622$.

Now let's take a look at some equations that involve logarithms. The main property that we'll be using to solve these kinds of equations is,

$$b^{\log_b x} = x$$

Example 6

Solve $3 + 2\ln\left(\frac{x}{7}+3\right) = -4$.

Solution

This first step in this problem is to get the logarithm by itself on one side of the equation with a coefficient of 1.

$$\begin{aligned} 2\ln\left(\frac{x}{7}+3\right) &= -7 \\ \ln\left(\frac{x}{7}+3\right) &= -\frac{7}{2} \end{aligned}$$

Now, we need to get the x out of the logarithm and the best way to do that is to "exponentiate" both sides using **e**. In other words,

$$\mathbf{e}^{\ln\left(\frac{x}{7}+3\right)} = \mathbf{e}^{-\frac{7}{2}}$$

So, using the property above with **e**, since there is a natural logarithm in the equation, we get,

$$\frac{x}{7} + 3 = \mathbf{e}^{-\frac{7}{2}}$$

Now all that we need to do is solve this for x.

$$\begin{aligned} \frac{x}{7} + 3 &= \mathbf{e}^{-\frac{7}{2}} \\ \frac{x}{7} &= -3 + \mathbf{e}^{-\frac{7}{2}} \\ x &= 7\left(-3 + \mathbf{e}^{-\frac{7}{2}}\right) = -20.78861832 \end{aligned}$$

At this point we might be tempted to say that we're done and move on. However, we do need to be careful. Recall from the previous section that we can't plug a negative number into a logarithm. This, by itself, doesn't mean that our answer won't work since its negative. What we need to do is plug it into the logarithm and make sure that $\frac{x}{7}+3$ will not be negative. I'll leave it to you to verify that this is in fact positive upon plugging our solution into the logarithm and so $x = -20.78861832$ is a solution to the equation.

Let's now take a look at a more complicated equation. Often there will be more than one logarithm in the equation. When this happens we will need to use one or more of the following properties to combine all the logarithms into a single logarithm. Once this has been done we can proceed as we did in the previous example.

$$\log_b(xy) = \log_b(x) + \log_b(y) \qquad \log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y) \qquad \log_b\left(x^r\right) = r\log_b(x)$$

Example 7

Solve $2\ln\left(\sqrt{x}\right) - \ln(1-x) = 2$.

Solution

First get the two logarithms combined into a single logarithm.

$$\begin{aligned} 2\ln\left(\sqrt{x}\right) - \ln(1-x) &= 2 \\ \ln\left(\left(\sqrt{x}\right)^2\right) - \ln(1-x) &= 2 \\ \ln(x) - \ln(1-x) &= 2 \\ \ln\left(\frac{x}{1-x}\right) &= 2 \end{aligned}$$

Now, exponentiate both sides and solve for x.

$$\begin{aligned} \frac{x}{1-x} &= \mathbf{e}^2 \\ x &= \mathbf{e}^2(1-x) \\ x &= \mathbf{e}^2 - \mathbf{e}^2 x \\ x\left(1+\mathbf{e}^2\right) &= \mathbf{e}^2 \\ x &= \frac{\mathbf{e}^2}{1+\mathbf{e}^2} = 0.8807970780 \end{aligned}$$

The solution work here was a little messy but this is work that you will need to be able to do on occasion so make sure you can do it!

Finally, we just need to make sure that the solution, $x = 0.8807970780$, doesn't produce negative numbers in both of the original logarithms. It doesn't, so this is in fact our solution to this problem.

Let's take a look at another example.

Example 8

Solve $\log x + \log(x-3) = 1$.

Solution

As with the last example, first combine the logarithms into a single logarithm.

$$\begin{aligned} \log x + \log(x-3) &= 1 \\ \log(x(x-3)) &= 1 \end{aligned}$$

Now exponentiate, using 10 this time instead of **e** because we've got common logs in the equation, both sides.

$$\begin{aligned} 10^{\log(x^2-3x)} &= 10^1 \\ x^2 - 3x &= 10 \\ x^2 - 3x - 10 &= 0 \\ (x-5)(x+2) &= 0 \end{aligned}$$

So, potential solutions are $x = 5$ and $x = -2$. Note, however that if we plug $x = -2$ into either of the two original logarithms we would get negative numbers so this can't be a solution. We can however, use $x = 5$.

Therefore, the solution to this equation is $x = 5$.

When solving equations with logarithms it is important to check your potential solutions to make sure that they don't generate logarithms of negative numbers or zero. It is also important to make sure that you do the checks in the **original** equation. If you check them in the second logarithm above (after we've combined the two logs) both solutions will appear to work! This is because in combining the two logarithms we've actually changed the problem. In fact, it is this change that introduces the extra solution that we couldn't use!

Also, be careful in solving equations containing logarithms to not get locked into the idea that you will get two potential solutions and only one of these will work. It is possible to have problems where both are solutions and where neither are solutions.

There is one more problem that we should work.

Example 9

Solve $\ln(x-2)+\ln(x+1)=2$.

Solution

The first step of this problem is the same as we've been doing up to this point. So, let's combine the logarithms.

$$\begin{aligned}\ln((x-2)(x+1)) &= 2\\ \ln\left(x^2-x-2\right) &= 2\end{aligned}$$

Now we can exponentiate both sides with respect to **e** to eliminate the logarithm. Doing this along with a little simplification gives,

$$\begin{aligned}x^2-x-2 &= \mathbf{e}^2\\ x^2-x-2-\mathbf{e}^2 &= 0\end{aligned}$$

We've reached the point of this problem. We need to solve this quadratic and without the $\mathbf{e}^2$ everyone would be able to do that. However, with the $\mathbf{e}^2$ people tend to decide that they can't do it.

This is just a quadratic equation and everyone in this class should be able to solve that. The only difference between this quadratic equation and those you are probably used to seeing is that there are numbers in it that are not integers, or at worst, fractions. In this case the constant in the quadratic is just $-2-\mathbf{e}^2$ and so all we need to do is use the quadratic formula to get the solutions.

The solutions to this quadratic equation are,

$$x=\frac{1\pm\sqrt{1-4(1)\left(-2-\mathbf{e}^2\right)}}{2}=\frac{1\pm\sqrt{9+4\mathbf{e}^2}}{2}=-2.6047,\quad 3.6047$$

Do not get excited about the "messy" solutions to this quadratic. We will get these kinds of solutions on occasion.

The last step to this problem is to check the two solutions to the quadratic equation in the original equation. Doing that we can see that the first solution, -2.6047, will give negative numbers in the logarithms and so can't be a solution. On the other hand, the second solution, 3.6047, does not give negative numbers in the logarithms and so is okay.

The solution to the original equation is $x=3.6047$.

3.6 Derivatives of Exponential and Logarithm Functions

The next set of functions that we want to take a look at are exponential and logarithm functions. The most common exponential and logarithm functions in a calculus course are the natural exponential function, $\mathbf{e}^x$, and the natural logarithm function, $\ln(x)$. We will take a more general approach however and look at the general exponential and logarithm function.

Exponential Functions

We'll start off by looking at the exponential function,

$$f(x) = a^x$$

We want to differentiate this. The power rule that we looked at a couple of sections ago won't work as that required the exponent to be a fixed number and the base to be a variable. That is exactly the opposite from what we've got with this function. So, we're going to have to start with the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x\left(a^h - 1\right)}{h} \end{aligned}$$

Now, the a^x is not affected by the limit since it doesn't have any h's in it and so is a constant as far as the limit is concerned. We can therefore factor this out of the limit. This gives,

$$f'(x) = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

Now let's notice that the limit we've got above is exactly the definition of the derivative of $f(x) = a^x$ at $x = 0$, *i.e.* $f'(0)$. Therefore, the derivative becomes,

$$f'(x) = f'(0)\, a^x$$

So, we are kind of stuck. We need to know the derivative in order to get the derivative!

There is one value of a that we can deal with at this point. Back in the Exponential Functions section of the Review chapter we stated that $\mathbf{e} = 2.718281828459 05\ldots$ What we didn't do however is actually define where $\mathbf{e}$ comes from. There are in fact a variety of ways to define $\mathbf{e}$. Here are three of them.

Some Definitions of e

1. $\mathbf{e} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

2. $\mathbf{e}$ is the unique positive number for which $\lim_{h \to 0} \frac{\mathbf{e}^h - 1}{h} = 1$

3. $\mathbf{e} = \sum_{n=0}^{\infty} \frac{1}{n!}$

The second one is the important one for us because that limit is exactly the limit that we're working with above. So, this definition leads to the following fact,

Fact 1

For the natural exponential function, $f(x) = \mathbf{e}^x$ we have $f'(0) = \lim_{h \to 0} \frac{\mathbf{e}^h - 1}{h} = 1$.

So, provided we are using the natural exponential function we get the following.

$$f(x) = \mathbf{e}^x \qquad \Rightarrow \qquad f'(x) = \mathbf{e}^x$$

At this point we're missing some knowledge that will allow us to easily get the derivative for a general function. Eventually we will be able to show that for a general exponential function we have,

$$f(x) = a^x \qquad \Rightarrow \qquad f'(x) = a^x \ln(a)$$

Logarithm Functions

Let's now briefly get the derivatives for logarithms. In this case we will need to start with the following fact about functions that are inverses of each other.

Fact 2

If $f(x)$ and $g(x)$ are inverses of each other then,

$$g'(x) = \frac{1}{f'(g(x))}$$

So, how is this fact useful to us? Well recall that the natural exponential function and the natural logarithm function are inverses of each other and we know what the derivative of the natural exponential function is!

So, if we have $f(x) = \mathbf{e}^x$ and $g(x) = \ln x$ then,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\mathbf{e}^{g(x)}} = \frac{1}{\mathbf{e}^{\ln x}} = \frac{1}{x}$$

The last step just uses the fact that the two functions are inverses of each other.

Putting this all together gives,

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} \qquad x > 0$$

Note that we need to require that $x > 0$ since this is required for the logarithm and so must also be required for its derivative. It can also be shown that,

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \qquad x \neq 0$$

Using this all we need to avoid is $x = 0$.

In this case, unlike the exponential function case, we can actually find the derivative of the general logarithm function. All that we need is the derivative of the natural logarithm, which we just found, and the change of base formula. Using the change of base formula we can write a general logarithm as,

$$\log_a x = \frac{\ln x}{\ln a}$$

Differentiation is then fairly simple.

$$\begin{aligned}\frac{d}{dx}\left(\log_a(x)\right) &= \frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right)\\ &= \frac{1}{\ln(a)}\frac{d}{dx}\left(\ln(x)\right)\\ &= \frac{1}{x\ln(a)}\end{aligned}$$

We took advantage of the fact that a was a constant and so $\ln a$ is also a constant and can be factored out of the derivative. Putting all this together gives,

$$\frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x\ln(a)}$$

Here is a summary of the derivatives in this section.

Derivative of Exponential and Logarithm Functions

$$\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x \qquad \frac{d}{dx}\left(a^x\right) = a^x\ln(a)$$

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} \qquad \frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x\ln(a)}$$

Okay, now that we have the derivations of the formulas out of the way let's compute a couple of derivatives.

Example 1

Differentiate each of the following functions.

(a) $R(w) = 4^w - 5\log_9(w)$

(b) $f(x) = 3\mathbf{e}^x + 10x^3 \ln(x)$

(c) $y = \dfrac{5\mathbf{e}^x}{3\mathbf{e}^x + 1}$

Solution

(a) $R(w) = 4^w - 5\log_9$

This will be the only example that doesn't involve the natural exponential and natural logarithm functions.

$$R'(w) = 4^w \ln(4) - \frac{5}{w\ln(9)}$$

(b) $f(x) = 3\mathbf{e}^x + 10x^3 \ln(x)$

Not much to this one. Just remember to use the product rule on the second term.

$$\begin{aligned} f'(x) &= 3\mathbf{e}^x + 30x^2\ln(x) + 10x^3\left(\frac{1}{x}\right) \\ &= 3\mathbf{e}^x + 30x^2\ln(x) + 10x^2 \end{aligned}$$

(c) $y = \dfrac{5\mathbf{e}^x}{3\mathbf{e}^x + 1}$

We'll need to use the quotient rule on this one.

$$\begin{aligned} y' &= \frac{5\mathbf{e}^x\left(3\mathbf{e}^x+1\right) - \left(5\mathbf{e}^x\right)\left(3\mathbf{e}^x\right)}{\left(3\mathbf{e}^x+1\right)^2} \\ &= \frac{15\mathbf{e}^{2x} + 5\mathbf{e}^x - 15\mathbf{e}^{2x}}{\left(3\mathbf{e}^x+1\right)^2} \\ &= \frac{5\mathbf{e}^x}{\left(3\mathbf{e}^x+1\right)^2} \end{aligned}$$

There's really not a lot to differentiating natural logarithms and natural exponential functions at this point as long as you remember the formulas. In later sections as we get more formulas under our belt they will become more complicated.

Next, we need to do our obligatory application/interpretation problem so we don't forget about them.

Example 2

Suppose that the position of an object is given by

$$s(t) = t\mathbf{e}^t$$

Does the object ever stop moving?

Solution

First, we will need the derivative. We need this to determine if the object ever stops moving since at that point (provided there is one) the velocity will be zero and recall that the derivative of the position function is the velocity of the object.

The derivative is,

$$s'(t) = \mathbf{e}^t + t\mathbf{e}^t = (1+t)\,\mathbf{e}^t$$

So, we need to determine if the derivative is ever zero. To do this we will need to solve,

$$(1+t)\,\mathbf{e}^t = 0$$

Now, we know that exponential functions are never zero and so this will only be zero at $t = -1$. So, if we are going to allow negative values of t then the object will stop moving once at $t = -1$. If we aren't going to allow negative values of t then the object will never stop moving.

Before moving on to the next section we need to go back over a couple of derivatives to make sure that we don't confuse the two. The two derivatives are,

$\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$	Power Rule
$\dfrac{d}{dx}\left(a^x\right) = a^x \ln(a)$	Derivative of an exponential function

It is important to note that with the Power rule the exponent MUST be a constant and the base MUST be a variable while we need exactly the opposite for the derivative of an exponential function. For an exponential function the exponent MUST be a variable and the base MUST be a constant.

It is easy to get locked into one of these formulas and just use it for both of these. We also haven't even talked about what to do if both the exponent and the base involve variables. We'll see this situation in a later section.