5 Integrals

In this chapter we will be looking at the third and final major topic that will be covered in a typical first Calculus course, integrals. As with derivatives this chapter will be devoted almost exclusively to finding and computing integrals. Applications will be given in the following chapter. There are really two types of integrals that we'll be looking at in this chapter : Indefinite Integrals and Definite Integrals. The first half of this chapter is devoted to indefinite integrals and the last half is devoted to definite integrals.

As we investigate indefinite integrals we will see that as long as we understand basic differentiation we shouldn't have a lot of problems with basic indefinite integrals. The reason for this is that indefinite integration is basically "undoing" differentiation. In fact, indefinite integrals are sometimes called anti-derivatives to make this idea clear. Having said that however we will be using the phrase indefinite integral instead of anti-derivative as that is the more common phrase used.

We will also spend a fair amount of time learning the substitution rule for integrals. We will see that it is really just "undoing" the chain rule and so, again, if you understand the chain rule it will help when using the substitution rule. In addition, as we'll see as we go through the rest of the calculus course the substitution rule will come up time and again and so it is very important to make sure that we have that down so we don't have issues with it in later topics.

As we move over to investigating definite integrals we will quickly realize just how important it is to be able to do indefinite integrals. As we will see we will not be able to compute definite integrals unless we can fist compute indefinite integrals.

We will also take a look at an important interpretation of definite integrals. Namely, a definite integral can be interpreted as the net area between the graph of the function and the x-axis.

5.1 Indefinite Integrals

In the past two chapters we've been given a function, $f(x)$, and asking what the derivative of this function was. Starting with this section we are now going to turn things around. We now want to ask what function we differentiated to get the function $f(x)$.

Let's take a quick look at an example to get us started.

Example 1

What function did we differentiate to get the following function.

$$f(x) = x^4 + 3x - 9$$

Solution

Let's actually start by getting the derivative of this function to help us see how we're going to have to approach this problem. The derivative of this function is,

$$f'(x) = 4x^3 + 3$$

The point of this was to remind us of how differentiation works. When differentiating powers of x we multiply the term by the original exponent and then drop the exponent by one.

Now, let's go back and work the problem. In fact, let's just start with the first term. We got x^4 by differentiating a function and since we drop the exponent by one it looks like we must have differentiated x^5. However, if we had differentiated x^5 we would have $5x^4$ and we don't have a 5 in front our first term, so the 5 needs to cancel out after we've differentiated. It looks then like we would have to differentiate $\frac{1}{5}x^5$ in order to get x^4.

Likewise, for the second term, in order to get $3x$ after differentiating we would have to differentiate $\frac{3}{2}x^2$. Again, the fraction is there to cancel out the 2 we pick up in the differentiation.

The third term is just a constant and we know that if we differentiate x we get 1. So, it looks like we had to differentiate $-9x$ to get the last term.

Putting all of this together gives the following function,

$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x$$

Our answer is easy enough to check. Simply differentiate $F(x)$.

$$F'(x) = x^4 + 3x - 9 = f(x)$$

So, it looks like we got the correct function. Or did we? We know that the derivative of a constant is zero and so any of the following will also give $f(x)$ upon differentiating.

$$\begin{aligned} F(x) &= \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + 10 \\ F(x) &= \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x - 1954 \\ F(x) &= \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + \frac{3469}{123} \\ &etc. \end{aligned}$$

In fact, any function of the form,

$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c, \qquad c \text{ is a constant}$$

will give $f(x)$ upon differentiating.

There were two points to this last example. The first point was to get you thinking about how to do these problems. It is important initially to remember that we are really just asking what we differentiated to get the given function.

The other point is to recognize that there are actually an infinite number of functions that we could use and they will all differ by a constant.

Now that we've worked an example let's get some of the definitions and terminology out of the way.

Definitions

Given a function, $f(x)$, an **anti-derivative** of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x)$$

If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted,

$$\int f(x)\,dx = F(x) + c, \qquad c \text{ is an arbitrary constant}$$

In this definition the $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, x is called the **integration variable** and the "c" is called the **constant of integration**.

Note that often we will just say integral instead of indefinite integral (or definite integral for that matter when we get to those). It will be clear from the context of the problem that we are talking about an indefinite integral (or definite integral).

The process of finding the indefinite integral is called **integration** or **integrating** $f(x)$. If we need to be specific about the integration variable we will say that we are **integrating** $f(x)$ **with respect to** x.

Let's rework the first problem in light of the new terminology.

Example 2

Evaluate the following indefinite integral.

$$\int x^4 + 3x - 9\,dx$$

Solution

Since this is really asking for the most general anti-derivative we just need to reuse the final answer from the first example.

The indefinite integral is,

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

A couple of warnings are now in order. One of the more common mistakes that students make with integrals (both indefinite and definite) is to drop the dx at the end of the integral. This is required! Think of the integral sign and the dx as a set of parentheses. You already know and are probably quite comfortable with the idea that every time you open a parenthesis you must close it. With integrals, think of the integral sign as an "open parenthesis" and the dx as a "close parenthesis".

If you drop the *dx* it won't be clear where the integrand ends. Consider the following variations of the above example.

$$\begin{aligned}
\int x^4 + 3x - 9\,dx &= \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c \\
\int x^4 + 3x\,dx - 9 &= \frac{1}{5}x^5 + \frac{3}{2}x^2 + c - 9 \\
\int x^4\,dx + 3x - 9 &= \frac{1}{5}x^5 + c + 3x - 9
\end{aligned}$$

You only integrate what is between the integral sign and the dx. Each of the above integrals end in a different place and so we get different answers because we integrate a different number of terms each time. In the second integral the "-9" is outside the integral and so is left alone and not integrated. Likewise, in the third integral the "$3x-9$" is outside the integral and so is left alone.

Knowing which terms to integrate is not the only reason for writing the dx down. In the Substitution Rule section we will actually be working with the dx in the problem and if we aren't in the habit of writing it down it will be easy to forget about it and then we will get the wrong answer at that stage.

The moral of this is to make sure and put in the dx! At this stage it may seem like a silly thing to do, but it just needs to be there, if for no other reason than knowing where the integral stops.

On a side note, the dx notation should seem a little familiar to you. We saw things like this a couple of sections ago. We called the dx a differential in that section and yes that is exactly what it is. The dx that ends the integral is nothing more than a differential.

The next topic that we should discuss here is the integration variable used in the integral. Actually, there isn't really a lot to discuss here other than to note that the integration variable doesn't really matter. For instance,

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$
$$\int t^4 + 3t - 9\,dt = \frac{1}{5}t^5 + \frac{3}{2}t^2 - 9t + c$$
$$\int w^4 + 3w - 9\,dw = \frac{1}{5}w^5 + \frac{3}{2}w^2 - 9w + c$$

Changing the integration variable in the integral simply changes the variable in the answer. It is important to notice however that when we change the integration variable in the integral we also changed the differential (dx, dt, or dw) to match the new variable. This is more important than we might realize at this point.

Another use of the differential at the end of integral is to tell us what variable we are integrating with respect to. At this stage that may seem unimportant since most of the integrals that we're going to be working with here will only involve a single variable. However, if you are on a degree track that will take you into multi-variable calculus this will be very important at that stage since there will be more than one variable in the problem. You need to get into the habit of writing the correct differential at the end of the integral so when it becomes important in those classes you will already be in the habit of writing it down.

To see why this is important take a look at the following two integrals.

$$\int 2x\,dx \qquad\qquad \int 2t\,dx$$

The first integral is simple enough.

$$\int 2x\,dx = x^2 + c$$

The second integral is also fairly simple, but we need to be careful. The dx tells us that we are integrating x's. That means that we only integrate x's that are in the integrand and all other variables in the integrand are considered to be constants. The second integral is then,

$$\int 2t\,dx = 2tx + c$$

So, it may seem silly to always put in the dx, but it is a vital bit of notation that can cause us to get the incorrect answer if we neglect to put it in.

Now, there are some important properties of integrals that we should take a look at.

Properties of the Indefinite Integral

1. $\displaystyle\int k\,f(x)\,dx = k\int f(x)\,dx$ where k is any number.

   So, we can factor multiplicative constants out of indefinite integrals. See the Proof of Various Integral Formulas section of the Extras appendix to see the proof of this property.

2. $\displaystyle\int -f(x)\,dx = -\int f(x)\,dx$.

   This is really the first property with $k = -1$ and so no proof of this property will be given.

3. $\displaystyle\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$.

   In other words, the integral of a sum or difference of functions is the sum or difference of the individual integrals. This rule can be extended to as many functions as we need. See the Proof of Various Integral Formulas section of the Extras appendix to see the proof of this property.

Notice that when we worked the first example above we used the first and third property in the discussion. We integrated each term individually, put any constants back in and then put everything back together with the appropriate sign.

Not listed in the properties above were integrals of products and quotients. The reason for this is simple. Just like with derivatives each of the following will NOT work.

$$\int f(x)\,g(x)\,dx \ne \int f(x)\,dx \int g(x)\,dx \qquad\qquad \int \frac{f(x)}{g(x)}\,dx \ne \frac{\int f(x)\,dx}{\int g(x)\,dx}$$

With derivatives we had a product rule and a quotient rule to deal with these cases. However, with integrals there are no such rules. When faced with a product and quotient in an integral we will have a variety of ways of dealing with it depending on just what the integrand is.

There is one final topic to be discussed briefly in this section. On occasion we will be given $f'(x)$ and will ask what $f(x)$ was. We can now answer this question easily with an indefinite integral.

$$f(x) = \int f'(x)\,dx$$

Example 3

If $f'(x) = x^4 + 3x - 9$ what was $f(x)$?

Solution

By this point in this section this is a simple question to answer.

$$f(x) = \int f'(x)\,dx = \int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

In this section we kept evaluating the same indefinite integral in all of our examples. The point of this section was not to do indefinite integrals, but instead to get us familiar with the notation and some of the basic ideas and properties of indefinite integrals. The next couple of sections are devoted to actually evaluating indefinite integrals.

5.2 Computing Indefinite Integrals

In the previous section we started looking at indefinite integrals and in that section we concentrated almost exclusively on notation, concepts and properties of the indefinite integral. In this section we need to start thinking about how we actually compute indefinite integrals. We'll start off with some of the basic indefinite integrals.

The first integral that we'll look at is the integral of a power of x.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

The general rule when integrating a power of x we add one onto the exponent and then divide by the new exponent. It is clear (hopefully) that we will need to avoid $n = -1$ in this formula. If we allow $n = -1$ in this formula we will end up with division by zero. We will take care of this case in a bit.

Next is one of the easier integrals but always seems to cause problems for people.

$$\int k \, dx = kx + c, \qquad c \text{ and } k \text{ are constants}$$

If you remember that all we're asking is what did we differentiate to get the integrand this is pretty simple, but it does seem to cause problems on occasion.

Let's now take a look at the trig functions.

$$\int \sin(x) \, dx = -\cos(x) + c \qquad \int \cos(x) \, dx = \sin(x) + c$$

$$\int \sec^2(x) \, dx = \tan(x) + c \qquad \int \sec(x)\tan(x) \, dx = \sec(x) + c$$

$$\int \csc^2(x) \, dx = -\cot(x) + c \qquad \int \csc(x)\cot(x) \, dx = -\csc(x) + c$$

Notice that we only integrated two of the six trig functions here. The remaining four integrals are really integrals that give the remaining four trig functions. Also, be careful with signs here. It is easy to get the signs for derivatives and integrals mixed up. Again, remember that we're asking what function we differentiated to get the integrand.

We will be able to integrate the remaining four trig functions in a couple of sections, but they all require the Substitution Rule

Now, let's take care of exponential and logarithm functions.

$$\int \mathbf{e}^x \, dx = \mathbf{e}^x + c \qquad \int a^x \, dx = \frac{a^x}{\ln(a)} + c \qquad \int \frac{1}{x} \, dx = \int x^{-1} \, dx = \ln|x| + c$$

Integrating logarithms requires a topic that is usually taught in Calculus II and so we won't be integrating a logarithm in this class. Also note the third integrand can be written in a couple of ways and don't forget the absolute value bars in the x in the answer to the third integral.

Finally, let's take care of the inverse trig and hyperbolic functions.

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + c \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$$

$$\int \sinh(x)\,dx = \cosh(x) + c \qquad \int \cosh(x)\,dx = \sinh(x) + c$$

$$\int \operatorname{sech}^2(x)\,dx = \tanh(x) + c \qquad \int \operatorname{sech}(x)\tanh(x)\,dx = -\operatorname{sech}(x) + c$$

$$\int \operatorname{csch}^2(x)\,dx = -\coth(x) + c \qquad \int \operatorname{csch}(x)\coth(x)\,dx = -\operatorname{csch}(x) + c$$

As with logarithms integrating inverse trig functions requires a topic usually taught in Calculus II and so we won't be integrating them in this class. There is also a different answer for the second integral above. Recalling that since all we are asking here is what function did we differentiate to get the integrand the second integral could also be,

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1}(x) + c$$

Traditionally we use the first form of this integral.

Okay, now that we've got most of the basic integrals out of the way let's do some indefinite integrals. In all of these problems remember that we can always check our answer by differentiating and making sure that we get the integrand.

Example 1

Evaluate each of the following indefinite integrals.

(a) $\displaystyle\int 5t^3 - 10t^{-6} + 4\,dt$

(b) $\displaystyle\int x^8 + x^{-8}\,dx$

(c) $\displaystyle\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

(d) $\displaystyle\int dy$

(e) $\displaystyle\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw$

(f) $\displaystyle\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx$

Solution

Okay, in all of these remember the basic rules of indefinite integrals. First, to integrate sums and differences all we really do is integrate the individual terms and then put the terms back together with the appropriate signs. Next, we can ignore any coefficients until we are done with integrating that particular term and then put the coefficient back in. Also, do not forget the "+c" at the end it is important and must be there.

So, let's evaluate some integrals.

(a) $\displaystyle\int 5t^3 - 10t^{-6} + 4\,dt$

There's not really a whole lot to do here other than use the first two formulas from the beginning of this section. Remember that when integrating powers (that aren't -1 of course) we just add one onto the exponent and then divide by the new exponent.

$$\begin{aligned}\int 5t^3 - 10t^{-6} + 4\,dt &= 5\left(\frac{1}{4}\right)t^4 - 10\left(\frac{1}{-5}\right)t^{-5} + 4t + c\\ &= \frac{5}{4}t^4 + 2t^{-5} + 4t + c\end{aligned}$$

Be careful when integrating negative exponents. Remember to add one onto the exponent. One of the more common mistakes that students make when integrating negative exponents is to "add one" and end up with an exponent of "-7" instead of the correct exponent of "-5".

(b) $\displaystyle\int x^8 + x^{-8}\,dx$

This is here just to make sure we get the point about integrating negative exponents.

$$\int x^8 + x^{-8}\,dx = \frac{1}{9}x^9 - \frac{1}{7}x^{-7} + c$$

(c) $\displaystyle\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

In this case there isn't a formula for explicitly dealing with radicals or rational expressions. However, just like with derivatives we can write all these terms so they are in the numerator and they all have an exponent. This should always be your first step when faced with this kind of integral just as it was when differentiating.

$$\begin{aligned}\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx &= \int 3x^{\frac{3}{4}} + 7x^{-5} + \frac{1}{6}x^{-\frac{1}{2}}\,dx\\ &= 3\frac{1}{{}^7\!/_4}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{6}\left(\frac{1}{{}^1\!/_2}\right)x^{\frac{1}{2}} + c\\ &= \frac{12}{7}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{3}x^{\frac{1}{2}} + c\end{aligned}$$

When dealing with fractional exponents we usually don't "divide by the new exponent". Doing this is equivalent to multiplying by the reciprocal of the new exponent and so that is what we will usually do.

(d) $\displaystyle\int dy$

Don't make this one harder than it is...

$$\int dy = \int 1\, dy = y + c$$

In this case we are really just integrating a one!

(e) $\displaystyle\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right) dw$

We've got a product here and as we noted in the previous section there is no rule for dealing with products. However, in this case we don't need a rule. All that we need to do is multiply things out (taking care of the radicals at the same time of course) and then we will be able to integrate.

$$\begin{aligned}\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right) dw &= \int 4w - w^3 + 4w^{\frac{1}{3}} - w^{\frac{7}{3}} dw \\ &= 2w^2 - \frac{1}{4}w^4 + 3w^{\frac{4}{3}} - \frac{3}{10}w^{\frac{10}{3}} + c\end{aligned}$$

(f) $\displaystyle\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3} dx$

As with the previous part it's not really a problem that we don't have a rule for quotients for this integral. In this case all we need to do is break up the quotient and then integrate the individual terms.

$$\begin{aligned}\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3} dx &= \int \frac{4x^{10}}{x^3} - \frac{2x^4}{x^3} + \frac{15x^2}{x^3} dx \\ &= \int 4x^7 - 2x + \frac{15}{x} dx \\ &= \frac{1}{2}x^8 - x^2 + 15 \ln|x| + c\end{aligned}$$

Be careful to not think of the third term as x to a power for the purposes of integration. Using that rule on the third term will NOT work. The third term is simply a logarithm. Also, don't get excited about the 15. The 15 is just a constant and so it can be factored out of the integral. In other words, here is what we did to integrate the third term.

$$\int \frac{15}{x} dx = 15 \int \frac{1}{x} dx = 15 \ln|x| + c$$

Always remember that you can't integrate products and quotients in the same way that we integrate sums and differences. At this point the only way to integrate products and quotients is to multiply the product out or break up the quotient. Eventually we'll see some other products and quotients that can be dealt with in other ways. However, there will never be a single rule that will work for all products and there will never be a single rule that will work for all quotients. Every product and quotient is different and will need to be worked on a case by case basis.

The first set of examples focused almost exclusively on powers of x (or whatever variable we used in each example). It's time to do some examples that involve other functions.

Example 2

Evaluate each of the following integrals.

(a) $\displaystyle\int 3\mathbf{e}^{x} + 5\cos(x) - 10\sec^2(x)\,dx$

(b) $\displaystyle\int 2\sec(w)\tan(w) + \frac{1}{6w}\,dw$

(c) $\displaystyle\int \frac{23}{y^2+1} + 6\csc(y)\cot(y) + \frac{9}{y}\,dy$

(d) $\displaystyle\int \frac{3}{\sqrt{1-x^2}} + 6\sin(x) + 10\sinh(x)\,dx$

(e) $\displaystyle\int \frac{7-6\sin^2(\theta)}{\sin^2(\theta)}\,d\theta$

Solution

Most of the problems in this example will simply use the formulas from the beginning of this section. More complicated problems involving most of these functions will need to wait until we reach the Substitution Rule.

(a) $\displaystyle\int 3\mathbf{e}^{x} + 5\cos(x) - 10\sec^2(x)\,dx$

There isn't anything to this one other than using the formulas.

$$\int 3\mathbf{e}^{x} + 5\cos(x) - 10\sec^2(x)\,dx = 3\mathbf{e}^{x} + 5\sin(x) - 10\tan(x) + c$$

(b) $\displaystyle\int 2\sec(w)\tan(w) + \frac{1}{6w}\,dw$

Let's be a little careful with this one. First break it up into two integrals and note the rewritten integrand on the second integral.

$$\begin{aligned}\int 2\sec(w)\tan(w) + \frac{1}{6w}\,dw &= \int 2\sec(w)\tan(w)\,dw + \int \frac{1}{6}\frac{1}{w}\,dw\\ &= \int 2\sec(w)\tan(w)\,dw + \frac{1}{6}\int \frac{1}{w}\,dw\end{aligned}$$

Rewriting the second integrand will help a little with the integration at this early stage. We can think of the 6 in the denominator as a $\frac{1}{6}$ out in front of the term and then since this is a constant it can be factored out of the integral. The answer is then,

$$\int 2\sec(w)\tan(w) + \frac{1}{6w}\,dw = 2\sec(w) + \frac{1}{6}\ln|w| + c$$

Note that we didn't factor the 2 out of the first integral as we factored the $\frac{1}{6}$ out of the second. In fact, we will generally not factor the $\frac{1}{6}$ out either in later problems. It was only done here to make sure that you could follow what we were doing.

(c) $\displaystyle\int \frac{23}{y^2+1} + 6\csc(y)\cot(y) + \frac{9}{y}\,dy$

In this one we'll just use the formulas from above and don't get excited about the coefficients. They are just multiplicative constants and so can be ignored while we integrate each term and then once we're done integrating a given term we simply put the coefficients back in.

$$\int \frac{23}{y^2+1} + 6\csc(y)\cot(y) + \frac{9}{y}\,dy = 23\tan^{-1}(y) - 6\csc(y) + 9\ln|y| + c$$

(d) $\displaystyle\int \frac{3}{\sqrt{1-x^2}} + 6\sin(x) + 10\sinh(x)\,dx$

Again, there really isn't a whole lot to do with this one other than to use the appropriate formula from above while taking care of coefficients.

$$\int \frac{3}{\sqrt{1-x^2}} + 6\sin(x) + 10\sinh(x)\,dx = 3\sin^{-1}(x) - 6\cos(x) + 10\cosh(x) + c$$

(e) $\displaystyle\int \frac{7-6\sin^2(\theta)}{\sin^2(\theta)}\,d\theta$

This one can be a little tricky if you aren't ready for it. As discussed previously, at this point the only way we have of dealing with quotients is to break it up.

$$\begin{aligned}\int \frac{7-6\sin^2(\theta)}{\sin^2(\theta)}\,d\theta &= \int \frac{7}{\sin^2(\theta)} - 6\,d\theta \\ &= \int 7\csc^2(\theta) - 6\,d\theta\end{aligned}$$

Notice that upon breaking the integral up we further simplified the integrand by recalling the definition of cosecant. With this simplification we can do the integral.

$$\int \frac{7-6\sin^2(\theta)}{\sin^2(\theta)}\,d\theta = -7\cot(\theta) - 6\theta + c$$

As shown in the last part of this example we can do some fairly complicated looking quotients at this point if we remember to do simplifications when we see them. In fact, this is something that you should always keep in mind. In almost any problem that we're doing here don't forget to simplify where possible. In almost every case this can only help the problem and will rarely complicate the problem.

In the next problem we're going to take a look at a product and this time we're not going to be able to just multiply the product out. However, if we recall the comment about simplifying a little this problem becomes fairly simple.

Example 3

Integrate $\displaystyle\int \sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\,dt$.

Solution

There are several ways to do this integral and most of them require the next section. However, there is a way to do this integral using only the material from this section. All that is required is to remember the trig formula that we can use to simplify the integrand up a little. Recall the following double angle formula.

$$\sin(2t) = 2\sin(t)\cos(t)$$

A small rewrite of this formula gives,

$$\sin(t)\cos(t) = \frac{1}{2}\sin(2t)$$

If we now replace all the t's with $\frac{t}{2}$ we get,

$$\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) = \frac{1}{2}\sin(t)$$

Using this formula, the integral becomes something we can do.

$$\begin{aligned}\int \sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\,dt &= \int \frac{1}{2}\sin(t)\,dt \\ &= -\frac{1}{2}\cos(t) + c\end{aligned}$$

As noted earlier there is another method for doing this integral. In fact, there are two alternate methods. To see all three check out the section on Constant of Integrationin the Extras appendix but be aware that the other two do require the material covered in the next section.

The formula/simplification in the previous problem is a nice "trick" to remember. It can be used on occasion to greatly simplify some problems.

There is one more set of examples that we should do before moving out of this section.

Example 4

Given the following information determine the function $f(x)$.

(a) $f'(x) = 4x^3 - 9 + 2\sin(x) + 7\mathbf{e}^x, \quad f(0) = 15$

(b) $f''(x) = 15\sqrt{x} + 5x^3 + 6, \quad f(1) = -\frac{5}{4}, \quad f(4) = 404$

In both of these we will need to remember that

$$f(x) = \int f'(x)\,dx$$

Also note that because we are giving values of the function at specific points we are also going to be determining what the constant of integration will be in these problems.

Solution

(a) $f'(x) = 4x^3 - 9 + 2\sin(x) + 7\mathbf{e}^x, \quad f(0) = 15$

The first step here is integrating to determine the most general possible $f(x)$.

$$\begin{aligned} f(x) &= \int 4x^3 - 9 + 2\sin(x) + 7\mathbf{e}^x \, dx \\ &= x^4 - 9x - 2\cos(x) + 7\mathbf{e}^x + c \end{aligned}$$

Now we have a value of the function so let's plug in $x = 0$ and determine the value of the constant of integration c.

$$\begin{aligned} 15 = f(0) &= 0^4 - 9(0) - 2\cos(0) + 7\mathbf{e}^0 + c \\ &= -2 + 7 + c \\ &= 5 + c \end{aligned}$$

So, from this it looks like $c = 10$. This means that the function is,

$$f(x) = x^4 - 9x - 2\cos(x) + 7\mathbf{e}^x + 10$$

(b) $f''(x) = 15\sqrt{x} + 5x^3 + 6, \quad f(1) = -\frac{5}{4}, \quad f(4) = 404$

This one is a little different form the first one. In order to get the function we will need the first derivative and we have the second derivative. We can however, use an integral to get the first derivative from the second derivative, just as we used an integral to get the function from the first derivative.

So, let's first get the most general possible first derivative by integrating the second derivative.

$$\begin{aligned} f'(x) &= \int f''(x) \, dx \\ &= \int 15x^{\frac{1}{2}} + 5x^3 + 6 \, dx \\ &= 15\left(\frac{2}{3}\right)x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \\ &= 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \end{aligned}$$

Don't forget the constant of integration!

We can now find the most general possible function by integrating the first derivative which we found above.

$$\begin{aligned} f(x) &= \int 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \, dx \\ &= 4x^{\frac{5}{2}} + \frac{1}{4}x^5 + 3x^2 + cx + d \end{aligned}$$

Do not get excited about integrating the c. It's just a constant and we know how to integrate constants. Also, there will be no reason to think the constants of integration from the integration in each step will be the same and so we'll need to call each constant of integration something different, d in this case.

Now, plug in the two values of the function that we've got.

$$-\frac{5}{4} = f(1) = 4 + \frac{1}{4} + 3 + c + d = \frac{29}{4} + c + d$$
$$404 = f(4) = 4(32) + \frac{1}{4}(1024) + 3(16) + c(4) + d = 432 + 4c + d$$

This gives us a system of two equations in two unknowns that we can solve.

$$\begin{aligned} -\frac{5}{4} &= \frac{29}{4} + c + d \\ 404 &= 432 + 4c + d \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c &= -\frac{13}{2} \\ d &= -2 \end{aligned}$$

The function is then,

$$f(x) = 4x^{\frac{5}{2}} + \frac{1}{4}x^5 + 3x^2 - \frac{13}{2}x - 2$$

Don't remember how to solve systems? Check out the Solving Systems portion of the Algebra/Trig Review.

In this section we've started the process of integration. We've seen how to do quite a few basic integrals and we also saw a quick application of integrals in the last example.

There are many new formulas in this section that we'll now have to know. However, if you think about it, they aren't really new formulas. They are really nothing more than derivative formulas that we should already know written in terms of integrals. If you remember that you should find it easier to remember the formulas in this section.

Always remember that integration is asking nothing more than what function did we differentiate to get the integrand. If you can remember that many of the basic integrals that we saw in this section and many of the integrals in the coming sections aren't too bad.

5.3 Substitution Rule for Indefinite Integrals

After the last section we now know how to do the following integrals.

$$\int \sqrt[4]{x}\,dx \qquad \int \frac{1}{t^3}\,dt \qquad \int \cos(w)\,dw \qquad \int \mathbf{e}^y\,dy$$

All of the integrals we've done to this point have required that we just had an x, or a t, or a w, *etc*. and not more complicated terms such as,

$$\int 18x^2\,\sqrt[4]{6x^3+5}\,dx \qquad \int \frac{2t^3+1}{\left(t^4+2t\right)^3}\,dt$$
$$\int \left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right)\,dw \qquad \int \left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy$$

All of these look considerably more difficult than the first set. However, they aren't too bad once you see how to do them. Let's start with the first one.

$$\int 18x^2\,\sqrt[4]{6x^3+5}\,dx$$

In this case let's notice that if we let

$$u = 6x^3+5$$

and we compute the differential (you remember how to compute these right?) for this we get,

$$du = 18x^2 dx$$

Now, let's go back to our integral and notice that we can eliminate every x that exists in the integral and write the integral completely in terms of u using both the definition of u and its differential.

$$\begin{aligned}\int 18x^2\,\sqrt[4]{6x^3+5}\,dx &= \int \left(6x^3+5\right)^{\frac{1}{4}}\,\left(18x^2 dx\right)\\ &= \int u^{\frac{1}{4}}\,du\end{aligned}$$

In the process of doing this we've taken an integral that looked very difficult and with a quick substitution we were able to rewrite the integral into a very simple integral that we can do.

Evaluating the integral gives,

$$\int 18x^2\,\sqrt[4]{6x^3+5}\,dx = \int u^{\frac{1}{4}}\,du = \frac{4}{5}u^{\frac{5}{4}} + c = \frac{4}{5}\left(6x^3+5\right)^{\frac{5}{4}} + c$$

As always, we can check our answer with a quick derivative if we'd like to and don't forget to "back substitute" and get the integral back into terms of the original variable.

What we've done in the work above is called the **Substitution Rule**. Here is the substitution rule in general.

Substitution Rule

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du, \qquad \text{where, } u = g(x)$$

A natural question at this stage is how to identify the correct substitution. Unfortunately, the answer is it depends on the integral. However, there is a general rule of thumb that will work for many of the integrals that we're going to be running across.

When faced with an integral we'll ask ourselves what we know how to integrate. With the integral above we can quickly recognize that we know how to integrate

$$\int \sqrt[4]{x}\, dx$$

However, we didn't have just the root we also had stuff in front of the root and (more importantly in this case) stuff under the root. Since we can only integrate roots if there is just an x under the root a good first guess for the substitution is then to make u be the stuff under the root.

Another way to think of this is to ask yourself if you were to differentiate the integrand (we're not of course, but just for a second pretend that we were) is there a chain rule and what is the inside function for the chain rule. If there is a chain rule (for a derivative) then there is a pretty good chance that the inside function will be the substitution that will allow us to do the integral.

We will have to be careful however. There are times when using this general rule can get us in trouble or overly complicate the problem. We'll eventually see problems where there are more than one "inside function" and/or integrals that will look very similar and yet use completely different substitutions. The reality is that the only way to really learn how to do substitutions is to just work lots of problems and eventually you'll start to get a feel for how these work and you'll find it easier and easier to identify the proper substitutions.

Now, with that out of the way we should ask the following question. How, do we know if we got the correct substitution? Well, upon computing the differential and actually performing the substitution every x in the integral (including the x in the dx) must disappear in the substitution process and the only letters left should be u's (including a du) and we should be left with an integral that we can do.

If there are x's left over or we have an integral that cannot be evaluated then there is a pretty good chance that we chose the wrong substitution. Unfortunately, however there is at least one case (we'll be seeing an example of this in the next section) where the correct substitution will actually leave some x's and we'll need to do a little more work to get it to work.

Again, it cannot be stressed enough at this point that the only way to really learn how to do substitutions is to just work lots of problems. There are lots of different kinds of problems and after working many problems you'll start to get a real feel for these problems and after you work enough of them you'll reach the point where you'll be able to do simple substitutions in your head without having to actually write anything down.

As a final note we should point out that often (in fact in almost every case) the differential will not appear exactly in the integrand as it did in the example above and sometimes we'll need to do some manipulation of the integrand and/or the differential to get all the x's to disappear in the substitution.

Let's work some examples so we can get a better idea on how the substitution rule works.

Example 1

Evaluate each of the following integrals.

(a) $\displaystyle\int \left(1 - \frac{1}{w}\right) \cos\left(w - \ln(w)\right) \, dw$

(b) $\displaystyle\int 3\,(8y - 1)\, \mathbf{e}^{4y^2 - y} \, dy$

(c) $\displaystyle\int x^2 \left(3 - 10x^3\right)^4 dx$

(d) $\displaystyle\int \frac{x}{\sqrt{1 - 4x^2}} \, dx$

Solution

(a) $\displaystyle\int \left(1 - \frac{1}{w}\right) \cos\left(w - \ln(w)\right) \, dw$

In this case it looks like we have a cosine with an inside function and so let's use that as the substitution.

$$u = w - \ln(w) \qquad\qquad du = \left(1 - \frac{1}{w}\right) dw$$

So, as with the first example we worked the stuff in front of the cosine appears exactly in the differential. The integral is then,

$$\begin{aligned} \int \left(1 - \frac{1}{w}\right) \cos\left(w - \ln(w)\right) \, dw &= \int \cos(u) \, du \\ &= \sin(u) + c \\ &= \sin\left(w - \ln(w)\right) + c \end{aligned}$$

Don't forget to go back to the original variable in the problem.

(b) $\displaystyle\int 3\left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy$

Again, it looks like we have an exponential function with an inside function (*i.e.* the exponent) and it looks like the substitution should be,

$$u = 4y^2 - y \qquad\qquad du = \left(8y-1\right)dy$$

Now, with the exception of the 3 the stuff in front of the exponential appears exactly in the differential. Recall however that we can factor the 3 out of the integral and so it won't cause any problems. The integral is then,

$$\begin{aligned}\int 3\left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy &= 3\int \mathbf{e}^{u}\,du\\ &= 3\mathbf{e}^{u}+c\\ &= 3\mathbf{e}^{4y^2-y}+c\end{aligned}$$

(c) $\displaystyle\int x^2\left(3-10x^3\right)^4 dx$

In this case it looks like the following should be the substitution.

$$u = 3-10x^3 \qquad\qquad du = -30x^2\,dx$$

Okay, now we have a small problem. We've got an x^2 out in front of the parenthesis but we don't have a "-30". This is not really the problem it might appear to be at first. We will simply rewrite the differential as follows.

$$x^2\,dx = -\frac{1}{30}\,du$$

With this we can now substitute the $x^2\,dx$ away. In the process we will pick up a constant, but that isn't a problem since it can always be factored out of the integral.

We can now do the integral.

$$\begin{aligned}\int x^2\left(3-10x^3\right)^4 dx &= \int \left(3-10x^3\right)^4 x^2\,dx\\ &= \int u^4\left(-\frac{1}{30}\right)du\\ &= -\frac{1}{30}\left(\frac{1}{5}\right)u^5+c\\ &= -\frac{1}{150}\left(3-10x^3\right)^5+c\end{aligned}$$

Note that in most problems when we pick up a constant as we did in this example we will generally factor it out of the integral in the same step that we substitute it in.

(d) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

In this example don't forget to bring the root up to the numerator and change it into fractional exponent form. Upon doing this we can see that the substitution is,

$$u = 1-4x^2 \qquad du = -8x\,dx \qquad \Rightarrow \qquad xdx = -\frac{1}{8}du$$

The integral is then,

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\,dx &= \int x\left(1-4x^2\right)^{-\frac{1}{2}}\,dx \\ &= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du \\ &= -\frac{1}{4}u^{\frac{1}{2}} + c \\ &= -\frac{1}{4}\left(1-4x^2\right)^{\frac{1}{2}} + c\end{aligned}$$

In the previous set of examples the substitution was generally pretty clear. There was exactly one term that had an "inside function" and so there wasn't really much in the way of options for the substitution. Let's take a look at some more complicated problems to make sure we don't come to expect all substitutions are like those in the previous set of examples.

Example 2

Evaluate each of the following integrals.

(a) $\displaystyle\int \sin\left(1-x\right)\left(2-\cos\left(1-x\right)\right)^4\,dx$

(b) $\displaystyle\int \cos\left(3z\right)\sin^{10}\left(3z\right)\,dz$

(c) $\displaystyle\int \sec^2\left(4t\right)\left(3-\tan\left(4t\right)\right)^3\,dt$

Solution

(a) $\displaystyle\int \sin\left(1-x\right)\left(2-\cos\left(1-x\right)\right)^4\,dx$

In this problem there are two "inside functions". There is the $1-x$ that is inside the two trig functions and there is also the term that is raised to the 4^{th} power.

There are two ways to proceed with this problem. The first idea that many students have is substitute the $1 - x$ away. There is nothing wrong with doing this but it doesn't eliminate the problem of the term to the 4^{th} power. That's still there and if we used this idea we would then need to do a second substitution to deal with that.

The second (and much easier) way of doing this problem is to just deal with the stuff raised to the 4^{th} power and see what we get. The substitution in this case would be,

$$u = 2 - \cos(1 - x) \qquad du = -\sin(1 - x)\,dx \qquad \Rightarrow \qquad \sin(1 - x)\,dx = -du$$

Two things to note here. First, don't forget to correctly deal with the "$-$". A common mistake at this point is to lose it. Secondly, notice that the $1-x$ turns out to not really be a problem after all. Because the $1 - x$ was "buried" in the substitution that we actually used it was also taken care of at the same time. The integral is then,

$$\begin{aligned}\int \sin(1 - x)\left(2 - \cos(1 - x)\right)^4 dx &= -\int u^4\,du \\ &= -\frac{1}{5}u^5 + c \\ &= -\frac{1}{5}(2 - \cos(1 - x))^5 + c\end{aligned}$$

As seen in this example sometimes there will seem to be two substitutions that will need to be done however, if one of them is buried inside of another substitution then we'll only really need to do one. Recognizing this can save a lot of time in working some of these problems.

(b) $\displaystyle\int \cos(3z)\sin^{10}(3z)\;dz$

This one is a little tricky at first. We can see the correct substitution by recalling that,

$$\sin^{10}(3z) = \left(\sin(3z)\right)^{10}$$

Using this it looks like the correct substitution is,

$$u = \sin(3z) \qquad du = 3\cos(3z)\,dz \qquad \Rightarrow \qquad \cos(3z)\,dz = \frac{1}{3}du$$

Notice that we again had two apparent substitutions in this integral but again the $3z$ is buried in the substitution we're using and so we didn't need to worry about it.

Here is the integral.

$$\begin{aligned}\int \cos(3z)\sin^{10}(3z)\;dz &= \frac{1}{3}\int u^{10}\,du \\ &= \frac{1}{3}\left(\frac{1}{11}\right)u^{11} + c \\ &= \frac{1}{33}\sin^{11}(3z) + c\end{aligned}$$

Note that the one third in front of the integral came about from the substitution on the differential and we just factored it out to the front of the integral. This is what we will usually do with these constants.

(c) $\displaystyle\int \sec^2(4t)\left(3-\tan(4t)\right)^3 dt$

In this case we've got a $4t$, a secant squared as well as a term cubed. However, it looks like if we use the following substitution the first two issues are going to be taken care of for us.

$$u = 3 - \tan(4t) \qquad du = -4\sec^2(4t)\,dt \qquad \Rightarrow \qquad \sec^2(4t)\,dt = -\frac{1}{4}du$$

The integral is now,

$$\begin{aligned}\int \sec^2(4t)\left(3-\tan(4t)\right)^3 dt &= -\frac{1}{4}\int u^3\,du \\ &= -\frac{1}{16}u^4 + c \\ &= -\frac{1}{16}\left(3-\tan(4t)\right)^4 + c\end{aligned}$$

The most important thing to remember in substitution problems is that after the substitution all the original variables need to disappear from the integral. After the substitution the only variables that should be present in the integral should be the new variable from the substitution (usually u). Note as well that this includes the variables in the differential!

This next set of examples, while not particularly difficult, can cause trouble if we aren't paying attention to what we're doing.

Example 3

Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{3}{5y+4}\,dy$

(b) $\displaystyle\int \frac{3y}{5y^2+4}\,dy$

(c) $\displaystyle\int \frac{3y}{(5y^2+4)^2}\,dy$

(d) $\displaystyle\int \frac{3}{5y^2+4}\,dy$

Solution

(a) $\displaystyle\int \frac{3}{5y+4}\,dy$

We haven't seen a problem quite like this one yet. Let's notice that if we take the denominator and differentiate it we get just a constant and the only thing that we have in the numerator is also a constant. This is a pretty good indication that we can use the denominator for our substitution so,

$$u = 5y + 4 \qquad du = 5\,dy \qquad \Rightarrow \qquad dy = \frac{1}{5}du$$

The integral is now,

$$\begin{aligned}\int \frac{3}{5y+4}\,dy &= \frac{3}{5}\int \frac{1}{u}\,du\\ &= \frac{3}{5}\ln|u| + c\\ &= \frac{3}{5}\ln|5y+4| + c\end{aligned}$$

Remember that we can just factor the 3 in the numerator out of the integral and that makes the integral a little clearer in this case.

(b) $\displaystyle\int \frac{3y}{5y^2+4}\,dy$

The integral is very similar to the previous one with a couple of minor differences but notice that again if we differentiate the denominator we get something that is different from the numerator by only a multiplicative constant. Therefore, we'll again take the denominator as our substitution.

$$u = 5y^2 + 4 \qquad du = 10y\,dy \qquad \Rightarrow \qquad y\,dy = \frac{1}{10}du$$

The integral is,

$$\begin{aligned}\int \frac{3y}{5y^2+4}\,dy &= \frac{3}{10}\int \frac{1}{u}\,du\\ &= \frac{3}{10}\ln|u| + c\\ &= \frac{3}{10}\ln\left|5y^2+4\right| + c\end{aligned}$$

(c) $\displaystyle\int \frac{3y}{\left(5y^2+4\right)^2}\,dy$

Now, this one is almost identical to the previous part except we added a power onto the denominator. Notice however that if we ignore the power and differentiate what's left we get the same thing as the previous example so we'll use the same substitution here.

$$u = 5y^2 + 4 \qquad du = 10y\,dy \qquad \Rightarrow \qquad y\,dy = \frac{1}{10}du$$

The integral in this case is,

$$\begin{aligned}\int \frac{3y}{\left(5y^2+4\right)^2}\,dy &= \frac{3}{10}\int u^{-2}\,du \\ &= -\frac{3}{10}u^{-1} + c \\ &= -\frac{3}{10}\left(5y^2+4\right)^{-1} + c = -\frac{3}{10\left(5y^2+4\right)} + c\end{aligned}$$

Be careful in this case to not turn this into a logarithm. After working problems like the first two in this set a common error is to turn every rational expression into a logarithm. If there is a power on the whole denominator then there is a good chance that it isn't a logarithm.

The idea that we used in the last three parts to determine the substitution is not a bad idea to remember. If we've got a rational expression try differentiating the denominator (ignoring any powers that are on the whole denominator) and if the result is the numerator or only differs from the numerator by a multiplicative constant then we can usually use that as our substitution.

(d) $\displaystyle\int \frac{3}{5y^2+4}\,dy$

Now, this part is completely different from the first three and yet seems similar to them as well. In this case if we differentiate the denominator we get a y that is not in the numerator and so we can't use the denominator as our substitution.

In fact, because we have y^2 in the denominator and no y in the numerator is an indication of how to work this problem. This integral is going to be an inverse tangent when we are done. The key to seeing this is to recall the following formula,

$$\int \frac{1}{1+u^2}\,du = \tan^{-1}u + c$$

We clearly don't have exactly this but we do have something that is similar. The denominator has a squared term plus a constant and the numerator is just a constant. So, with a little work and the proper substitution we should be able to get our integral into a form that will allow us to use this formula.

There is one part of this formula that is really important and that is the "$1+$" in the denominator. The "$1+$" must be there and we've got a "$4+$" but it is easy enough to take care of that. We'll just factor a 4 out of the denominator and at the same time we'll factor the 3 in the numerator out of the integral as well. Doing this gives,

$$\begin{aligned}\int \frac{3}{5y^2+4}\,dy &= \int \frac{3}{4\left(\frac{5y^2}{4}+1\right)}\,dy\\ &= \frac{3}{4}\int \frac{1}{\frac{5y^2}{4}+1}\,dy\\ &= \frac{3}{4}\int \frac{1}{\left(\frac{\sqrt{5}\,y}{2}\right)^2+1}\,dy\end{aligned}$$

Notice that in the last step we rewrote things a little in the denominator. This will help us to see what the substitution needs to be. In order to get this integral into the formula above we need to end up with a u^2 in the denominator. Our substitution will then need to be something that upon squaring gives us $\frac{5y^2}{4}$. With the rewrite we can see what that we'll need to use the following substitution.

$$u = \frac{\sqrt{5}y}{2} \qquad du = \frac{\sqrt{5}}{2}\,dy \qquad \Rightarrow \qquad dy = \frac{2}{\sqrt{5}}du$$

Don't get excited about the root in the substitution, these will show up on occasion. Upon plugging our substitution in we get,

$$\int \frac{3}{5y^2+4}\,dy = \frac{3}{4}\left(\frac{2}{\sqrt{5}}\right)\int \frac{1}{u^2+1}\,du$$

After doing the substitution, and factoring any constants out, we get exactly the integral that gives an inverse tangent and so we know that we did the correct substitution for this integral. The integral is then,

$$\begin{aligned}\int \frac{3}{5y^2+4}\,dy &= \frac{3}{2\sqrt{5}}\int \frac{1}{u^2+1}\,du\\ &= \frac{3}{2\sqrt{5}}\tan^{-1}(u)+c\\ &= \frac{3}{2\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}\,y}{2}\right)+c\end{aligned}$$

In this last set of integrals we had four integrals that were similar to each other in many ways and yet all either yielded different answer using the same substitution or used a completely different substitution than one that was similar to it.

This is a fairly common occurrence and so you will need to be able to deal with these kinds of issues.

There are many integrals that on the surface look very similar and yet will use a completely different substitution or will yield a completely different answer when using the same substitution.

Let's take a look at another set of examples to give us more practice in recognizing these kinds of issues. Note however that we won't be putting as much detail into these as we did with the previous examples.

Example 4

Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{2t^3+1}{\left(t^4+2t\right)^3}\,dt$

(b) $\displaystyle\int \frac{2t^3+1}{t^4+2t}\,dt$

(c) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

(d) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$

Solution

(a) $\displaystyle\int \frac{2t^3+1}{\left(t^4+2t\right)^3}\,dt$

Clearly the derivative of the denominator, ignoring the exponent, differs from the numerator only by a multiplicative constant and so the substitution is,

$$u = t^4+2t \qquad du = \left(4t^3+2\right)dt = 2\left(2t^3+1\right)dt \qquad \Rightarrow \qquad \left(2t^3+1\right)dt = \frac{1}{2}du$$

After a little manipulation of the differential we get the following integral.

$$\begin{aligned}\int \frac{2t^3+1}{\left(t^4+2t\right)^3}\,dt &= \frac{1}{2}\int \frac{1}{u^3}\,du \\ &= \frac{1}{2}\int u^{-3}\,du \\ &= \frac{1}{2}\left(-\frac{1}{2}\right)u^{-2}+c \\ &= -\frac{1}{4}\left(t^4+2t\right)^{-2}+c\end{aligned}$$

(b) $\displaystyle\int \frac{2t^3+1}{t^4+2t}\,dt$

The only difference between this problem and the previous one is the denominator. In the previous problem the whole denominator is cubed and in this problem the denominator has no power on it. The same substitution will work in this problem but because we no longer have the power the problem will be different.

So, using the substitution from the previous example the integral is,

$$\begin{aligned}\int \frac{2t^3+1}{t^4+2t}\,dt &= \frac{1}{2}\int \frac{1}{u}\,du\\ &= \frac{1}{2}\ln|u|+c\\ &= \frac{1}{2}\ln\left|t^4+2t\right|+c\end{aligned}$$

So, in this case we get a logarithm from the integral.

(c) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

Here, if we ignore the root we can again see that the derivative of the stuff under the radical differs from the numerator by only a multiplicative constant and so we'll use that as the substitution.

$$u = 1-4x^2 \qquad du = -8x\,dx \qquad \Rightarrow \qquad x\,dx = -\frac{1}{8}du$$

The integral is then,

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\,dx &= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du\\ &= -\frac{1}{8}(2)\,u^{\frac{1}{2}}+c\\ &= -\frac{1}{4}\sqrt{1-4x^2}+c\end{aligned}$$

(d) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$

In this case we are missing the x in the numerator and so the substitution from the last part will do us no good here. This integral is another inverse trig function integral that is similar to the last part of the previous set of problems. In this case we need to following formula.

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u)+c$$

The integral in this problem is nearly this. The only difference is the presence of the coefficient of 4 on the x^2. With the correct substitution this can be dealt with however.

To see what this substitution should be let's rewrite the integral a little. We need to figure out what we squared to get $4x^2$ and that will be our substitution.

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-(2x)^2}}\,dx$$

With this rewrite it looks like we can use the following substitution.

$$u = 2x \qquad du = 2dx \qquad \Rightarrow \qquad dx = \frac{1}{2}du$$

The integral is then,

$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{2}\sin^{-1}(u) + c \\ &= \frac{1}{2}\sin^{-1}(2x) + c\end{aligned}$$

Since this document is also being presented on the web we're going to put the rest of the substitution rule examples in the next section. With all the examples in one section the section was becoming too large for web presentation.

5.4 More Substitution Rule

In order to allow these pages to be displayed on the web we've broken the substitution rule examples into two sections. The previous section contains the introduction to the substitution rule and some fairly basic examples. The examples in this section tend towards the slightly more difficult side. Also, we'll not be putting quite as much explanation into the solutions here as we did in the previous section.

In the first couple of sets of problems in this section the difficulty is not with the actual integration itself, but with the set up for the integration. Most of the integrals are fairly simple and most of the substitutions are fairly simple. The problems arise in getting the integral set up properly for the substitution(s) to be done. Once you see how these are done it's easy to see what you have to do, but the first time through these can cause problems if you aren't on the lookout for potential problems.

Example 1

Evaluate each of the following integrals.

(a) $\displaystyle\int \mathbf{e}^{2t} + \sec(2t)\tan(2t)\,dt$

(b) $\displaystyle\int \sin(t)\left(4\cos^3(t) + 6\cos^2(t) - 8\right)\,dt$

(c) $\displaystyle\int x\cos\left(x^2+1\right) + \frac{x}{x^2+1}\,dx$

Solution

(a) $\displaystyle\int \mathbf{e}^{2t} + \sec(2t)\tan(2t)\,dt$

This first integral has two terms in it and both will require the same substitution. This means that we won't have to do anything special to the integral. One of the more common "mistakes" here is to break the integral up and do a separate substitution on each part. This isn't really mistake but will definitely increase the amount of work we'll need to do. So, since both terms in the integral use the same substitution we'll just do everything as a single integral using the following substitution.

$$u = 2t \qquad du = 2dt \qquad \Rightarrow \qquad dt = \frac{1}{2}du$$

The integral is then,

$$\begin{aligned}\int \mathbf{e}^{2t} + \sec(2t)\tan(2t)\,dt &= \frac{1}{2}\int \mathbf{e}^{u} + \sec(u)\tan(u)\,du\\ &= \frac{1}{2}\left(\mathbf{e}^{u} + \sec(u)\right) + c\\ &= \frac{1}{2}\left(\mathbf{e}^{2t} + \sec(2t)\right) + c\end{aligned}$$

Often a substitution can be used multiple times in an integral so don't get excited about that if it happens. Also note that since there was a $\frac{1}{2}$ in front of the whole integral there must also be a $\frac{1}{2}$ in front of the answer from the integral.

(b) $\displaystyle\int \sin(t)\left(4\cos^3(t) + 6\cos^2(t) - 8\right)\,dt$

This integral is similar to the previous one, but it might not look like it at first glance. Here is the substitution for this problem,

$$u = \cos(t) \qquad du = -\sin(t)\,dt \qquad \Rightarrow \qquad \sin(t)\,dt = -du$$

We'll plug the substitution into the problem twice (since there are two cosines) and will only work because there is a sine multiplying everything. Without that sine in front we would not be able to use this substitution.

The integral in this case is,

$$\begin{aligned}\int \sin(t)\left(4\cos^3(t) + 6\cos^2(t) - 8\right)\,dt &= -\int 4u^3 + 6u^2 - 8\,du\\ &= -\left(u^4 + 2u^3 - 8u\right) + c\\ &= -\left(\cos^4(t) + 2\cos^3(t) - 8\cos(t)\right) + c\end{aligned}$$

Again, be careful with the minus sign in front of the whole integral.

(c) $\displaystyle\int x\cos\left(x^2+1\right) + \frac{x}{x^2+1}\,dx$

It should be fairly clear that each term in this integral will use the same substitution, but let's rewrite things a little to make things really clear.

$$\int x\cos\left(x^2+1\right) + \frac{x}{x^2+1}\,dx = \int x\left(\cos\left(x^2+1\right) + \frac{1}{x^2+1}\right)\,dx$$

Since each term had an x in it and we'll need that for the differential we factored that out of both terms to get it into the front. This integral is now very similar to the previous one. Here's the substitution.

$$u = x^2 + 1 \qquad du = 2x\,dx \qquad \Rightarrow \qquad x\,dx = \frac{1}{2}du$$

The integral is,

$$\begin{aligned}\int x\cos\left(x^2+1\right)+\frac{x}{x^2+1}\,dx &= \frac{1}{2}\int \cos\left(u\right)+\frac{1}{u}\,du\\ &= \frac{1}{2}\left(\sin\left(u\right)+\ln\left|u\right|\right)+c\\ &= \frac{1}{2}\left(\sin\left(x^2+1\right)+\ln\left|x^2+1\right|\right)+c\end{aligned}$$

So, as we've seen in the previous set of examples sometimes we can use the same substitution more than once in an integral and doing so will simplify the work.

Example 2

Evaluate each of the following integrals.

(a) $\displaystyle\int x^2+\mathbf{e}^{1-x}\,dx$

(b) $\displaystyle\int x\cos\left(x^2+1\right)+\frac{1}{x^2+1}\,dx$

Solution

(a) $\displaystyle\int x^2+\mathbf{e}^{1-x}\,dx$

In this integral the first term does not need any substitution while the second term does need a substitution. So, to deal with that we'll need to split the integral up as follows,

$$\int x^2+\mathbf{e}^{1-x}\,dx=\int x^2\,dx+\int \mathbf{e}^{1-x}\,dx$$

The substitution for the second integral is then,

$$u=1-x \qquad du=-dx \qquad \Rightarrow \qquad dx=-du$$

The integral is,

$$\begin{aligned}\int x^2+\mathbf{e}^{1-x}\,dx &= \int x^2\,dx-\int \mathbf{e}^u\,du\\ &= \frac{1}{3}x^3-\mathbf{e}^u+c\\ &= \frac{1}{3}x^3-\mathbf{e}^{1-x}+c\end{aligned}$$

Be careful with this kind of integral. One of the more common mistakes here is do the following "shortcut".

$$\int x^2 + \mathbf{e}^{1-x}\,dx = -\int x^2 + \mathbf{e}^u\,du$$

In other words, some students will try do the substitution just the second term without breaking up the integral. There are two issues with this. First, there is a "$-$" in front of the whole integral that shouldn't be there. It should only be on the second term because that is the term getting the substitution. Secondly, and probably more importantly, there are x's in the integral and we have a du for the differential. We can't mix variables like this. When we do integrals all the variables in the integrand must match the variable in the differential.

(b) $\displaystyle\int x\cos\left(x^2+1\right) + \frac{1}{x^2+1}\,dx$

This integral looks very similar to Example 1c above, but it is different. In this integral we no longer have the x in the numerator of the second term and that means that the substitution we'll use for the first term will no longer work for the second term. In fact, the second term doesn't need a substitution at all since it is just an inverse tangent.

The substitution for the first term is then,

$$u = x^2 + 1 \qquad du = 2x\,dx \qquad \Rightarrow \qquad x\,dx = \frac{1}{2}du$$

Now let's do the integral. Remember to first break it up into two terms before using the substitution.

$$\begin{aligned}\int x\cos\left(x^2+1\right) + \frac{1}{x^2+1}\,dx &= \int x\cos\left(x^2+1\right)\,dx + \int \frac{1}{x^2+1}\,dx\\ &= \frac{1}{2}\int \cos(u)\,du + \int\frac{1}{x^2+1}\,dx\\ &= \frac{1}{2}\sin(u) + \tan^{-1}(x) + c\\ &= \frac{1}{2}\sin\left(x^2+1\right) + \tan^{-1}(x) + c\end{aligned}$$

In this set of examples we saw that sometimes one (or potentially more than one) term in the integrand will not require a substitution. In these cases we'll need to break up the integral into two integrals, one involving the terms that don't need a substitution and another with the term(s) that do need a substitution.

Example 3

Evaluate each of the following integrals.

(a) $\displaystyle\int \mathbf{e}^{-z} + \sec^2\left(\frac{z}{10}\right)\,dz$

(b) $\displaystyle\int \sin w\sqrt{1-2\cos w} + \frac{1}{7w+2}\,dw$

(c) $\displaystyle\int \frac{10x+3}{x^2+16}\,dx$

Solution

(a) $\displaystyle\int \mathbf{e}^{-z} + \sec^2\left(\frac{z}{10}\right)\,dz$

In this integral, unlike any integrals that we've yet done, there are two terms and each will require a different substitution. So, to do this integral we'll first need to split up the integral as follows,

$$\int \mathbf{e}^{-z} + \sec^2\left(\frac{z}{10}\right)\,dz = \int \mathbf{e}^{-z}dz + \int \sec^2\left(\frac{z}{10}\right)\,dz$$

Here are the substitutions for each integral.

$$\begin{array}{llll} u = -z & du = -dz & \Rightarrow & dz = -du \\ v = \dfrac{z}{10} & dv = \dfrac{1}{10}dz & \Rightarrow & dz = 10dv \end{array}$$

Notice that we used different letters for each substitution to avoid confusion when we go to plug back in for u and v.

Here is the integral.

$$\begin{aligned}\int \mathbf{e}^{-z} + \sec^2\left(\frac{z}{10}\right)\,dz &= -\int \mathbf{e}^{u}du + 10\int \sec^2(v)\,dv \\ &= -\mathbf{e}^{u} + 10\tan(v) + c \\ &= -\mathbf{e}^{-z} + 10\tan\left(\frac{z}{10}\right) + c\end{aligned}$$

(b) $\displaystyle\int \sin w\sqrt{1-2\cos w}+\frac{1}{7w+2}\,dw$

As with the last problem this integral will require two separate substitutions. Let's first break up the integral.

$$\int \sin w\sqrt{1-2\cos w}+\frac{1}{7w+2}\,dw=\int \sin w(1-2\cos w)^{\frac{1}{2}}\,dw+\int \frac{1}{7w+2}\,dw$$

Here are the substitutions for this integral.

$$u=1-2\cos(w) \qquad du=2\sin(w)\,dw \qquad \Rightarrow \qquad \sin(w)\,dw=\frac{1}{2}du$$
$$v=7w+2 \qquad dv=7dw \qquad \Rightarrow \qquad dw=\frac{1}{7}dv$$

The integral is then,

$$\begin{aligned}\int \sin w\sqrt{1-2\cos w}+\frac{1}{7w+2}\,dw&=\frac{1}{2}\int u^{\frac{1}{2}}\,du+\frac{1}{7}\int\frac{1}{v}\,dv\\&=\frac{1}{2}\left(\frac{2}{3}\right)u^{\frac{3}{2}}+\frac{1}{7}\ln|v|+c\\&=\frac{1}{3}(1-2\cos w)^{\frac{3}{2}}+\frac{1}{7}\ln|7w+2|+c\end{aligned}$$

(c) $\displaystyle\int\frac{10x+3}{x^2+16}\,dx$

The last problem in this set can be tricky. If there was just an x in the numerator we could do a quick substitution to get a natural logarithm. Likewise, if there wasn't an x in the numerator we would get an inverse tangent after a quick substitution.

To get this integral into a form that we can work with we will first need to break it up as follows.

$$\begin{aligned}\int\frac{10x+3}{x^2+16}\,dx&=\int\frac{10x}{x^2+16}\,dx+\int\frac{3}{x^2+16}\,dx\\&=\int\frac{10x}{x^2+16}\,dx+\frac{1}{16}\int\frac{3}{\frac{x^2}{16}+1}\,dx\end{aligned}$$

We now have two integrals each requiring a different substitution. The substitutions for each of the integrals above are,

$$u=x^2+16 \qquad du=2xdx \qquad \Rightarrow \qquad xdx=\frac{1}{2}du$$
$$v=\frac{x}{4} \qquad dv=\frac{1}{4}dx \qquad \Rightarrow \qquad dx=4dv$$

The integral is then,

$$\begin{aligned}\int \frac{10x+3}{x^2+16}\,dx &= 5\int \frac{1}{u}\,du + \frac{3}{4}\int \frac{1}{v^2+1}\,dv \\ &= 5\ln|u| + \frac{3}{4}\tan^{-1}(v) + c \\ &= 5\ln\left|x^2+16\right| + \frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right) + c\end{aligned}$$

We've now seen a set of integrals in which we need to do more than one substitution. In these cases we will need to break up the integral into separate integrals and do separate substitutions for each.

We now need to move onto a different set of examples that can be a little tricky. Once you've seen how to do these they aren't too bad but doing them for the first time can be difficult if you aren't ready for them.

Example 4

Evaluate each of the following integrals.

(a) $\displaystyle\int \tan(x)\,dx$

(b) $\displaystyle\int \sec(y)\,dy$

(c) $\displaystyle\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

(d) $\displaystyle\int \mathbf{e}^{t+\mathbf{e}^t}\,dt$

(e) $\displaystyle\int 2x^3\sqrt{x^2+1}\,dx$

Solution

(a) $\displaystyle\int \tan(x)\,dx$

The first question about this problem is probably why is it here? Substitution rule problems generally require more than a single function. The key to this problem is to realize that there really are two functions here. All we need to do is remember the definition of tangent and we can write the integral as,

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

Written in this way we can see that the following substitution will work for us,

$$u = \cos(x) \qquad du = -\sin(x)\,dx \qquad \Rightarrow \qquad \sin(x)\,dx = -du$$

The integral is then,

$$\begin{aligned}\int \tan(x)\,dx &= -\int \frac{1}{u}\,du \\ &= -\ln|u| + c \\ &= -\ln|\cos(x)| + c\end{aligned}$$

Now, while this is a perfectly serviceable answer that minus sign in front is liable to cause problems if we aren't careful. So, let's rewrite things a little. Recalling a property of logarithms we can move the minus sign in front to an exponent on the cosine and then do a little simplification.

$$\begin{aligned}\int \tan(x)\,dx &= -\ln|\cos(x)| + c \\ &= \ln|\cos(x)|^{-1} + c \\ &= \ln\frac{1}{|\cos(x)|} + c \\ &= \ln|\sec(x)| + c\end{aligned}$$

This is the formula that is typically given for the integral of tangent.

Note that we could integrate cotangent in a similar manner.

(b) $\int \sec(y)\,dy$

This problem also at first appears to not belong in the substitution rule problems. This is even more of a problem upon noticing that we can't just use the definition of the secant function to write this in a form that will allow the use of the substitution rule.

This problem is going to require a technique that isn't used terribly often at this level but is a useful technique to be aware of. Sometimes we can make an integral doable by multiplying the top and bottom by a common term. This will not always work and even when it does it is not always clear what we should multiply by but when it works it is very useful.

Here is how we'll use this idea for this problem.

$$\int \sec(y)\,dy = \int \frac{\sec(y)}{1}\frac{\big(\sec(y)+\tan(y)\big)}{\big(\sec(y)+\tan(y)\big)}\,dy$$

First, we will think of the secant as a fraction and then multiply the top and bottom of the fraction by the same term. It is probably not clear why one would want to do this here but doing this will actually allow us to use the substitution rule. To see how this will work let's simplify the integrand somewhat.

$$\int \sec(y)\,dy = \int \frac{\sec^2(y)+\tan(y)\sec(y)}{\sec(y)+\tan(y)}\,dy$$

We can now use the following substitution.

$$u = \sec(y)+\tan(y) \qquad du = \big(\sec(y)\tan(y)+\sec^2(y)\big)\,dy$$

The integral is then,

$$\begin{aligned}\int \sec(y)\,dy &= \int \frac{1}{u}\,du\\ &= \ln|u| + c\\ &= \ln\big|\sec(y)+\tan(y)\big| + c\end{aligned}$$

Sometimes multiplying the top and bottom of a fraction by a carefully chosen term will allow us to work a problem. It does however take some thought sometimes to determine just what the term should be.

We can use a similar process for integrating cosecant.

(c) $\displaystyle\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

This next problem has a subtlety to it that can get us in trouble if we aren't paying attention. Because of the root in the cosine it makes some sense to use the following substitution.

$$u = x^{\frac{1}{2}} \qquad du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$$

This is where we need to be careful. Upon rewriting the differential we get,

$$2du = \frac{1}{\sqrt{x}}\,dx$$

The root that is in the denominator will not become a u as we might have been tempted to do. Instead it will get taken care of in the differential.

The integral is,

$$\begin{aligned}\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx &= 2\int \cos(u)\,du\\ &= 2\sin(u)+c\\ &= 2\sin\left(\sqrt{x}\right)+c\end{aligned}$$

(d) $\displaystyle\int \mathbf{e}^{t+\mathbf{e}^t}\,dt$

With this problem we need to very carefully pick our substitution. As the problem is written we might be tempted to use the following substitution,

$$u = t+\mathbf{e}^t \qquad du = \left(1+\mathbf{e}^t\right)dt$$

However, this won't work as you can probably see. The differential doesn't show up anywhere in the integrand and we just wouldn't be able to eliminate all the t's with this substitution.

In order to work this problem we will need to rewrite the integrand as follows,

$$\int \mathbf{e}^{t+\mathbf{e}^t}\,dt = \int \mathbf{e}^t\mathbf{e}^{\mathbf{e}^t}\,dt$$

We will now use the substitution,

$$u=\mathbf{e}^t \qquad du=\mathbf{e}^t dt$$

The integral is,

$$\begin{aligned}\int \mathbf{e}^{t+\mathbf{e}^t}\,dt &= \int \mathbf{e}^u\,du\\ &= \mathbf{e}^u+c\\ &= \mathbf{e}^{\mathbf{e}^t}+c\end{aligned}$$

Some substitutions can be really tricky to see and it's not unusual that you'll need to do some simplification and/or rewriting to get a substitution to work.

(e) $\displaystyle\int 2x^3\sqrt{x^2+1}\,dx$

This last problem in this set is different from all the other substitution problems that we've worked to this point. Given the fact that we've got more than an x under the root it makes sense that the substitution pretty much has to be,

$$u=x^2+1 \qquad du=2x\,dx$$

At first glance it looks like this might not work for the substitution because we have an x^3 in front of the root. However, if we first rewrite $2x^3 = x^2(2x)$ we could then move the $2x$ to the end of the integral so at least the du will show up explicitly in the integral. Doing this gives the following,

$$\begin{aligned} \int 2x^3\sqrt{x^2+1}\,dx &= \int x^2\sqrt{x^2+1}\,(2x)\,dx \\ &= \int x^2 u^{\frac{1}{2}}\,du \end{aligned}$$

This is a real problem. Our integrals can't have two variables in them. Normally this would mean that we chose our substitution incorrectly. However, in this case we can rewrite the substitution as follows,

$$x^2 = u - 1$$

and now, we can eliminate the remaining x's from our integral. Doing this gives,

$$\begin{aligned} \int 2x^3\sqrt{x^2+1}\,dx &= \int (u-1)\,u^{\frac{1}{2}}\,du \\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{2}{5}\left(x^2+1\right)^{\frac{5}{2}} - \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + c \end{aligned}$$

Sometimes, we will need to use a substitution more than once.

This kind of problem doesn't arise all that often and when it does there will sometimes be alternate methods of doing the integral. However, it will often work out that the easiest method of doing the integral is to do what we just did here.

This final set of examples isn't too bad once you see the substitutions and that is the point with this set of problems. These all involve substitutions that we've not seen prior to this and so we need to see some of these kinds of problems.

Example 5

Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{1}{x \ln(x)}\, dx$

(b) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{2t}}\, dt$

(c) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\, dt$

(d) $\displaystyle\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\, dx$

Solution

(a) $\displaystyle\int \frac{1}{x \ln(x)}\, dx$

In this case we know that we can't integrate a logarithm by itself and so it makes some sense (hopefully) that the logarithm will need to be in the substitution. Here is the substitution for this problem.

$$u = \ln(x) \qquad du = \frac{1}{x}dx$$

So, the x in the denominator of the integrand will get substituted away with the differential. Here is the integral for this problem.

$$\begin{aligned}\int \frac{1}{x \ln(x)}\, dx &= \int \frac{1}{u}\, du \\ &= \ln|u| + c \\ &= \ln|\ln(x)| + c\end{aligned}$$

(b) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{2t}}\, dt$

Again, the substitution here may seem a little tricky. In this case the substitution is,

$$u = 1+\mathbf{e}^{2t} \qquad du = 2\mathbf{e}^{2t}dt \qquad \Rightarrow \qquad \mathbf{e}^{2t}dt = \frac{1}{2}du$$

The integral is then,

$$\begin{aligned}\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{2t}}\, dt &= \frac{1}{2}\int \frac{1}{u}\, du \\ &= \frac{1}{2}\ln\left|1+\mathbf{e}^{2t}\right| + c\end{aligned}$$

(c) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\,dt$

In this case we can't use the same type of substitution that we used in the previous problem. In order to use the substitution in the previous example the exponential in the numerator and the denominator need to be the same and in this case they aren't.

To see the correct substitution for this problem note that,

$$\mathbf{e}^{4t} = \left(\mathbf{e}^{2t}\right)^2$$

Using this, the integral can be written as follows,

$$\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\,dt = \int \frac{\mathbf{e}^{2t}}{1+\left(\mathbf{e}^{2t}\right)^2}\,dt$$

We can now use the following substitution.

$$u = \mathbf{e}^{2t} \qquad du = 2\mathbf{e}^{2t}dt \qquad \Rightarrow \qquad \mathbf{e}^{2t}dt = \frac{1}{2}du$$

The integral is then,

$$\begin{aligned}\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\,dt &= \frac{1}{2}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{2}\tan^{-1}(u)+c\\ &= \frac{1}{2}\tan^{-1}\left(\mathbf{e}^{2t}\right)+c\end{aligned}$$

(d) $\displaystyle\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$

This integral is similar to the first problem in this set. Since we don't know how to integrate inverse sine functions it seems likely that this will be our substitution. If we use this as our substitution we get,

$$u = \sin^{-1}(x) \qquad du = \frac{1}{\sqrt{1-x^2}}dx$$

So, the root in the integral will get taken care of in the substitution process and this will eliminate all the x's from the integral. Therefore, this was the correct substitution.

The integral is,

$$\begin{aligned}\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx &= \int u\,du\\ &= \frac{1}{2}u^2+c\\ &= \frac{1}{2}\left(\sin^{-1}(x)\right)^2+c\end{aligned}$$

Over the last couple of sections we've seen a lot of substitution rule examples. There are a couple of general rules that we will need to remember when doing these problems. First, when doing a substitution remember that when the substitution is done all the x's in the integral (or whatever variable is being used for that particular integral) should all be substituted away. This includes the x in the dx. After the substitution only u's should be left in the integral. Also, sometimes the correct substitution is a little tricky to find and more often than not there will need to be some manipulation of the differential or integrand in order to actually do the substitution.

Also, many integrals will require us to break them up so we can do multiple substitutions so be on the lookout for those kinds of integrals/substitutions.

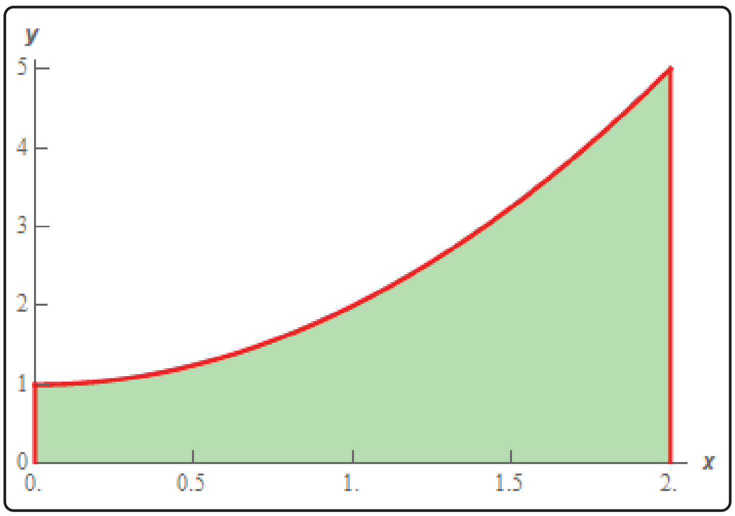
5.5 Area Problem

As noted in the first section of this section there are two kinds of integrals and to this point we've looked at indefinite integrals. It is now time to start thinking about the second kind of integral : Definite Integrals. However, before we do that we're going to take a look at the Area Problem. The area problem is to definite integrals what the tangent and rate of change problems are to derivatives.

The area problem will give us one of the interpretations of a definite integral and it will lead us to the definition of the definite integral.

To start off we are going to assume that we've got a function $f(x)$ that is positive on some interval $[a,b]$. What we want to do is determine the area of the region between the function and the x-axis.

It's probably easiest to see how we do this with an example. So, let's determine the area between $f(x) = x^2 + 1$ on $[0,2]$. In other words, we want to determine the area of the shaded region below.

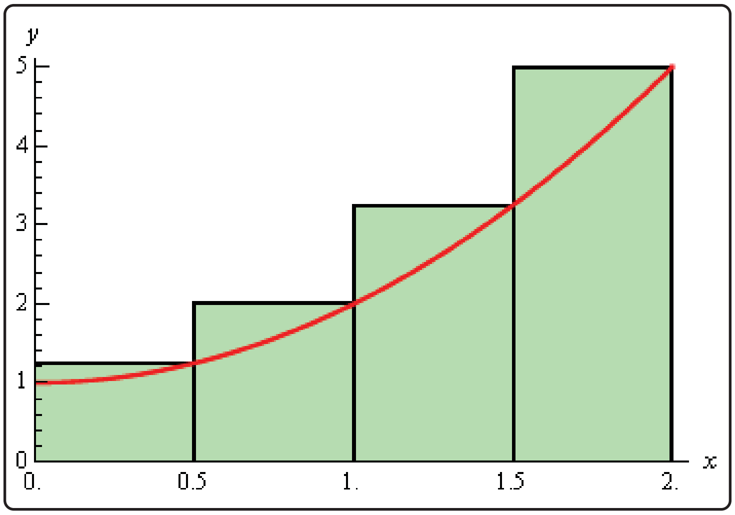


Now, at this point, we can't do this exactly. However, we can estimate the area. We will estimate the area by dividing up the interval into n subintervals each of width,

$$\Delta x = \frac{b-a}{n}$$

Then in each interval we can form a rectangle whose height is given by the function value at a specific point in the interval. We can then find the area of each of these rectangles, add them up and this will be an estimate of the area.

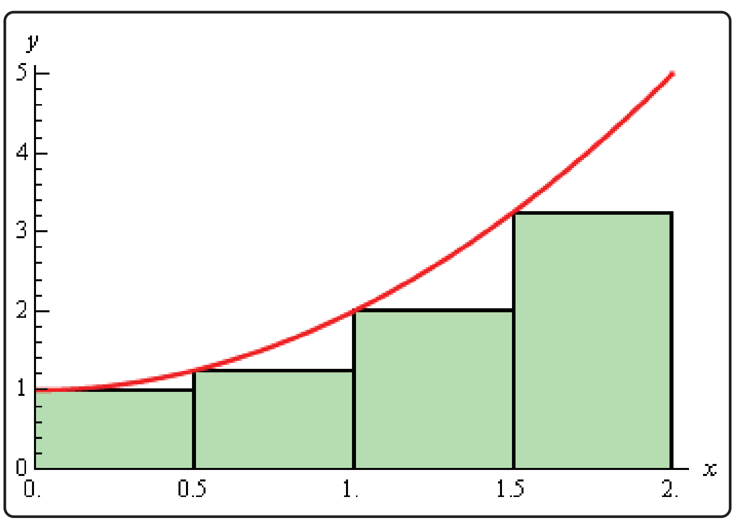
It's probably easier to see this with a sketch of the situation. So, let's divide up the interval into 4 subintervals and use the function value at the right endpoint of each interval to define the height of the rectangle. This gives,



Note that by choosing the height as we did each of the rectangles will over estimate the area since each rectangle takes in more area than the graph each time. Now let's estimate the area. First, the width of each of the rectangles is $\frac{1}{2}$. The height of each rectangle is determined by the function value at the right endpoint and so the height of each rectangle is nothing more that the function value at the right endpoint. Here is the estimated area.

$$\begin{aligned} A_r &= \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right) + \frac{1}{2} f(2) \\ &= \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) + \frac{1}{2}(5) \\ &= 5.75 \end{aligned}$$

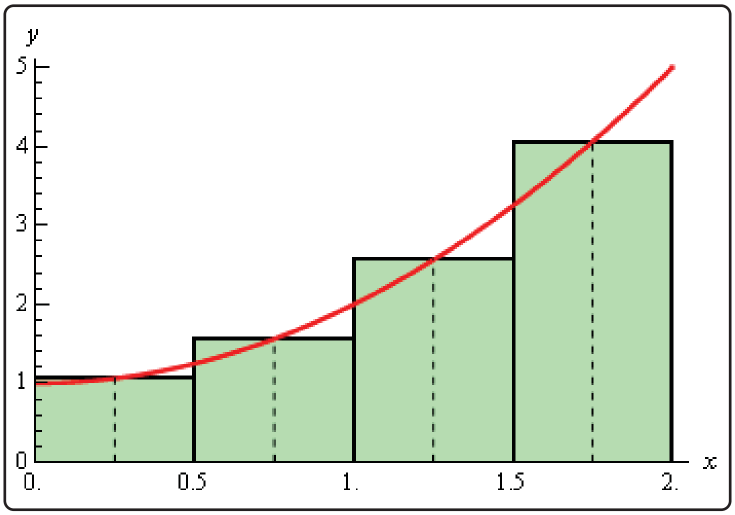
Of course, taking the rectangle heights to be the function value at the right endpoint is not our only option. We could have taken the rectangle heights to be the function value at the left endpoint. Using the left endpoints as the heights of the rectangles will give the following graph and estimated area.



$$\begin{aligned}A_l &= \frac{1}{2}f(0) + \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right)\\ &= \frac{1}{2}(1) + \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right)\\ &= 3.75\end{aligned}$$

In this case we can see that the estimation will be an underestimation since each rectangle misses some of the area each time.

There is one more common point for getting the heights of the rectangles that is often more accurate. Instead of using the right or left endpoints of each sub interval we could take the midpoint of each subinterval as the height of each rectangle. Here is the graph for this case.



So, it looks like each rectangle will over and under estimate the area. This means that the approximation this time should be much better than the previous two choices of points. Here is the estimation for this case.

$$\begin{aligned}A_m &= \frac{1}{2}f\left(\frac{1}{4}\right) + \frac{1}{2}f\left(\frac{3}{4}\right) + \frac{1}{2}f\left(\frac{5}{4}\right) + \frac{1}{2}f\left(\frac{7}{4}\right)\\ &= \frac{1}{2}\left(\frac{17}{16}\right) + \frac{1}{2}\left(\frac{25}{16}\right) + \frac{1}{2}\left(\frac{41}{16}\right) + \frac{1}{2}\left(\frac{65}{16}\right)\\ &= 4.625\end{aligned}$$

We've now got three estimates. For comparison's sake the exact area is

$$A = \frac{14}{3} = 4.66\bar{6}$$

So, both the right and left endpoint estimation did not do all that great of a job at the estimation. The midpoint estimation however did quite well.

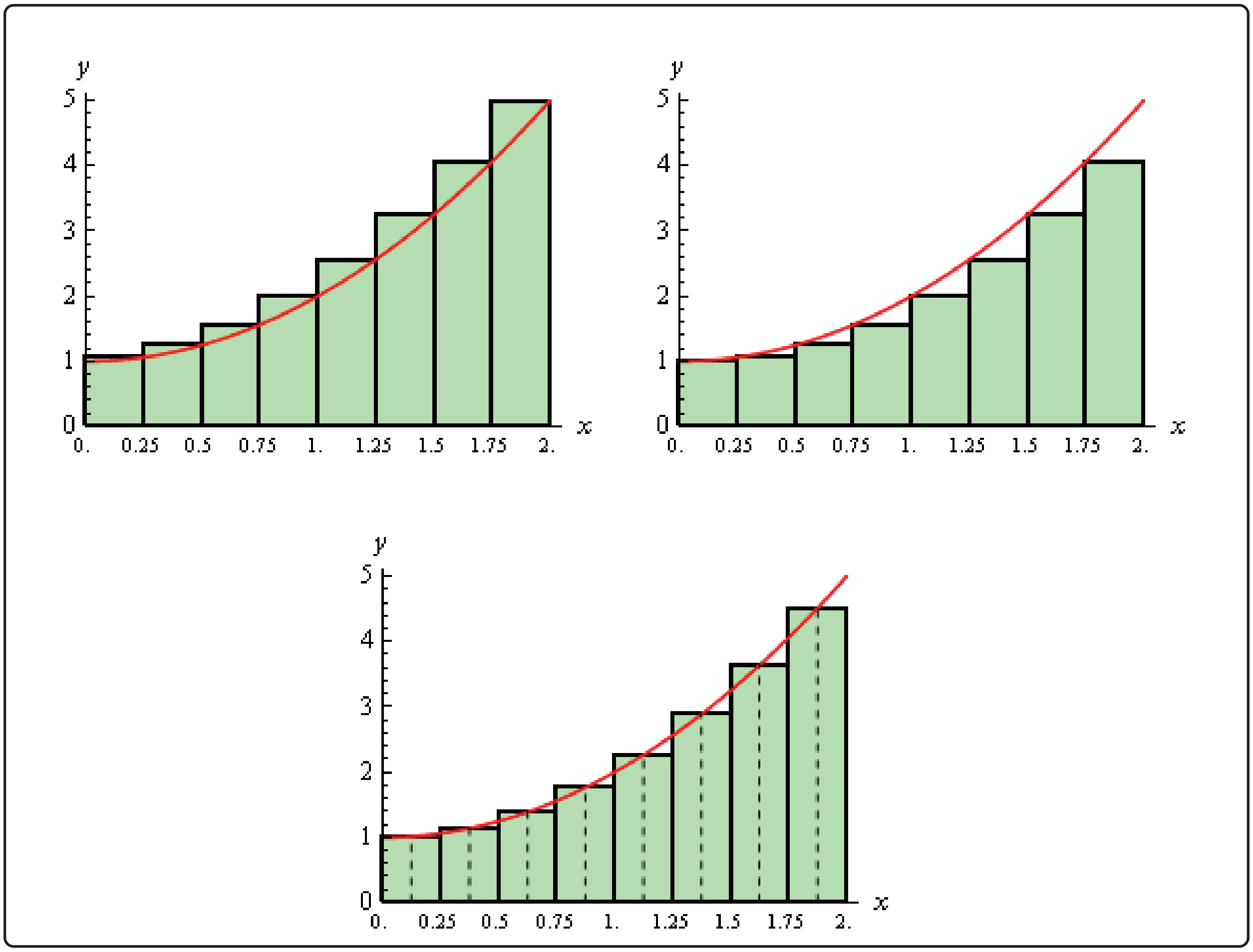
Be careful to not draw any conclusion about how choosing each of the points will affect our estimation. In this case, because we are working with an increasing function choosing the right endpoints will overestimate and choosing left endpoint will underestimate.

If we were to work with a decreasing function we would get the opposite results. For decreasing functions the right endpoints will underestimate and the left endpoints will overestimate.

Also, if we had a function that both increased and decreased in the interval we would, in all likelihood, not even be able to determine if we would get an overestimation or underestimation.

Now, let's suppose that we want a better estimation, because none of the estimations above really did all that great of a job at estimating the area. We could try to find a different point to use for the height of each rectangle but that would be cumbersome and there wouldn't be any guarantee that the estimation would in fact be better. Also, we would like a method for getting better approximations that would work for any function we would chose to work with and if we just pick new points that may not work for other functions.

The easiest way to get a better approximation is to take more rectangles (*i.e.* increase n). Let's double the number of rectangles that we used and see what happens. Here are the graphs showing the eight rectangles and the estimations for each of the three choices for rectangle heights that we used above.



Here are the area estimations for each of these cases.

$$A_r = 5.1875 \qquad A_l = 4.1875 \qquad A_m = 4.65625$$

So, increasing the number of rectangles did improve the accuracy of the estimation as we'd guessed that it would.

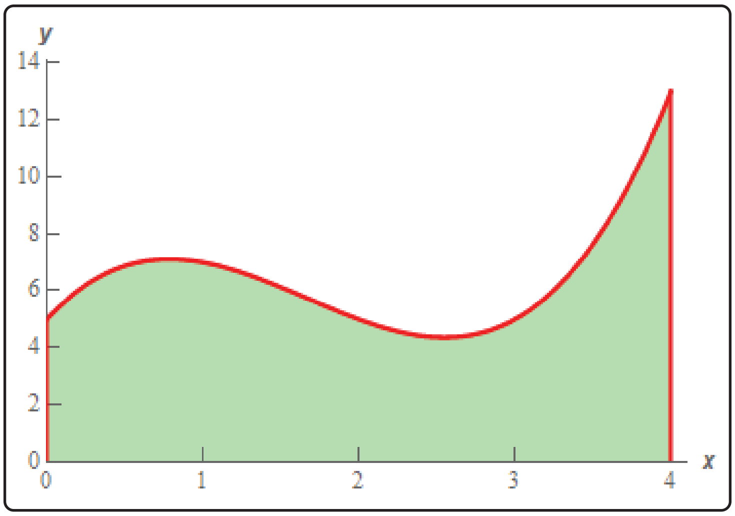
Let's work a slightly more complicated example.

Example 1

Estimate the area between $f(x) = x^3 - 5x^2 + 6x + 5$ and the x-axis on $[0, 4]$ using $n = 5$ subintervals and all three cases above for the heights of each rectangle.

Solution

First, let's get the graph to make sure that the function is positive.



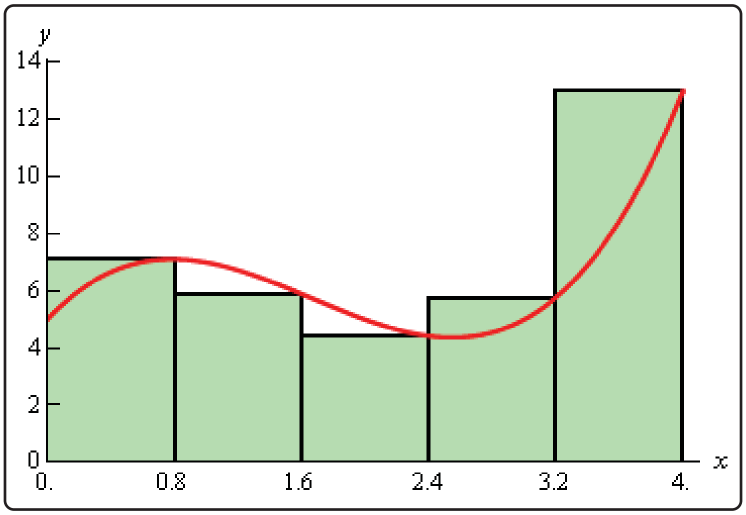
So, the graph is positive and the width of each subinterval will be,

$$\Delta x = \frac{4}{5} = 0.8$$

This means that the endpoints of the subintervals are,

$$0, \ 0.8, \ 1.6, \ 2.4, \ 3.2, \ 4$$

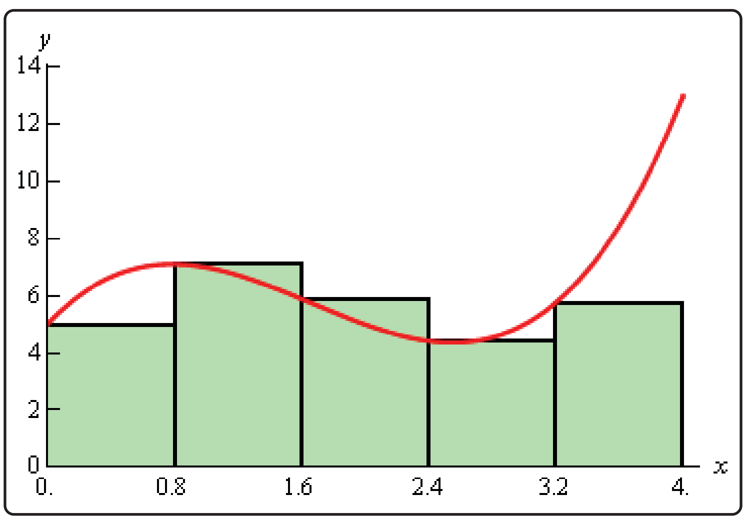
Let's first look at using the right endpoints for the function height. Here is the graph for this case.



Notice, that unlike the first area we looked at, the choosing the right endpoints here will both over and underestimate the area depending on where we are on the curve. This will often be the case with a more general curve that the one we initially looked at. The area estimation using the right endpoints of each interval for the rectangle height is,

$$\begin{aligned} A_r &= 0.8f(0.8) + 0.8f(1.6) + 0.8f(2.4) + 0.8f(3.2) + 0.8f(4) \\ &= 28.96 \end{aligned}$$

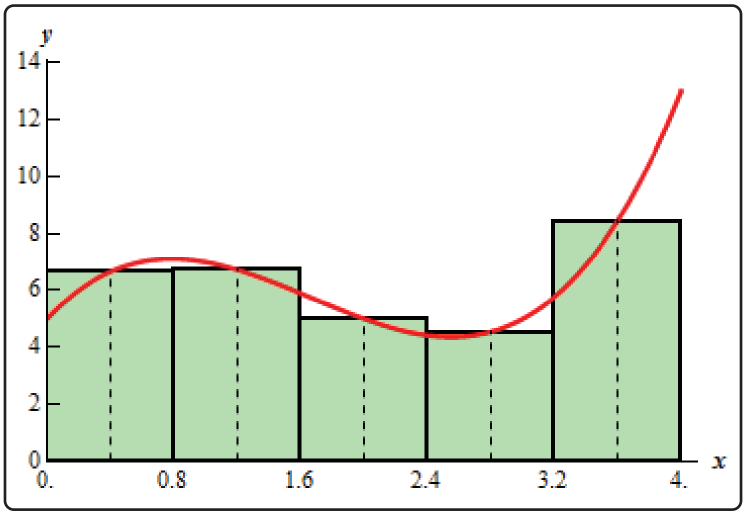
Now let's take a look at left endpoints for the function height. Here is the graph.



The area estimation using the left endpoints of each interval for the rectangle height is,

$$\begin{aligned} A_l &= 0.8f(0) + 0.8f(0.8) + 0.8f(1.6) + 0.8f(2.4) + 0.8f(3.2) \\ &= 22.56 \end{aligned}$$

Finally, let's take a look at the midpoints for the heights of each rectangle. Here is the graph,



The area estimation using the midpoint is then,

$$\begin{aligned} A_m &= 0.8f(0.4) + 0.8f(1.2) + 0.8f(2) + 0.8f(2.8) + 0.8f(3.6) \\ &= 25.12 \end{aligned}$$

For comparison purposes the exact area is,

$$A = \frac{76}{3} = 25.33\overline{3}$$

So, again the midpoint did a better job than the other two. While this will be the case more often than not, it won't always be the case and so don't expect this to always happen.

Now, let's move on to the general case. Let's start out with $f(x) \geq 0$ on $[a,b]$ and we'll divide the interval into n subintervals each of length,

$$\Delta x = \frac{b-a}{n}$$

Note that the subintervals don't have to be equal length, but it will make our work significantly easier. The endpoints of each subinterval are,

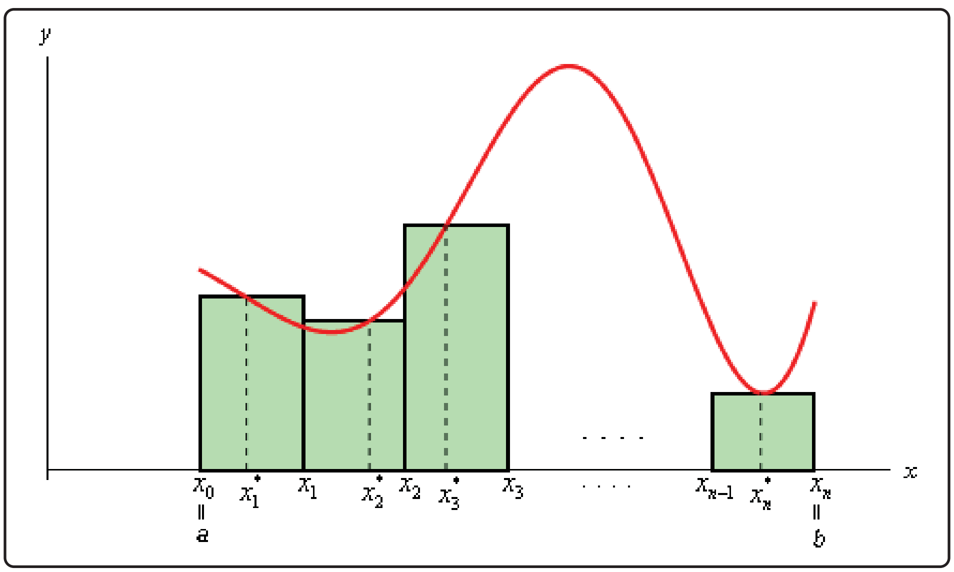
$$\begin{aligned} x_0 &= a \\ x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ &\vdots \\ x_i &= a + i\Delta x \\ &\vdots \\ x_{n-1} &= a + (n-1)\,\Delta x \\ x_n &= a + n\Delta x = b \end{aligned}$$

Next in each interval,

$$[x_0, x_1]\,,\ [x_1, x_2]\,, \ldots, [x_{i-1}, x_i]\,, \ldots, [x_{n-1}, x_n]$$

we choose a point $x_1^*, x_2^*, \ldots, x_i^*, \ldots x_n^*$. These points will define the height of the rectangle in each subinterval. Note as well that these points do not have to occur at the same point in each subinterval. However, they are usually the left end point of the interval, right end point of the interval or the midpoint of the interval.

Here is a sketch of this situation.



The area under the curve on the given interval is then approximately,

$$A \approx f\left(x_1^*\right)\Delta x + f\left(x_2^*\right)\Delta x + \cdots + f\left(x_i^*\right)\Delta x + \cdots + f\left(x_n^*\right)\Delta x$$

We will use **summation notation** or **sigma notation** at this point to simplify up our notation a little. If you need a refresher on summation notation check out the section devoted to this in the Extras appendix.

Using summation notation the area estimation is,

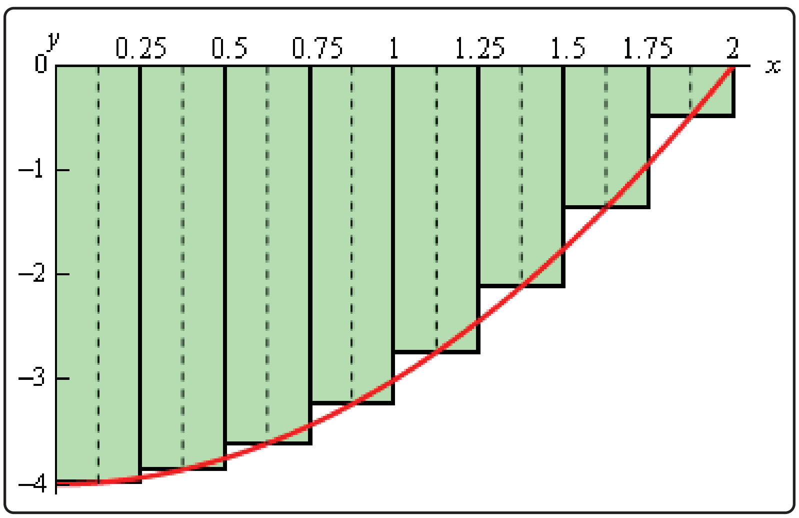
$$A \approx \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

The summation in the above equation is called a **Riemann Sum**.

To get a better estimation we will take n larger and larger. In fact, if we let n go out to infinity we will get the exact area. In other words,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

Before leaving this section let's address one more issue. To this point we've required the function to be positive in our work. Many functions are not positive however. Consider the case of $f\left(x\right) = x^2 - 4$ on $[0, 2]$. If we use $n = 8$ and the midpoints for the rectangle height we get the following graph,

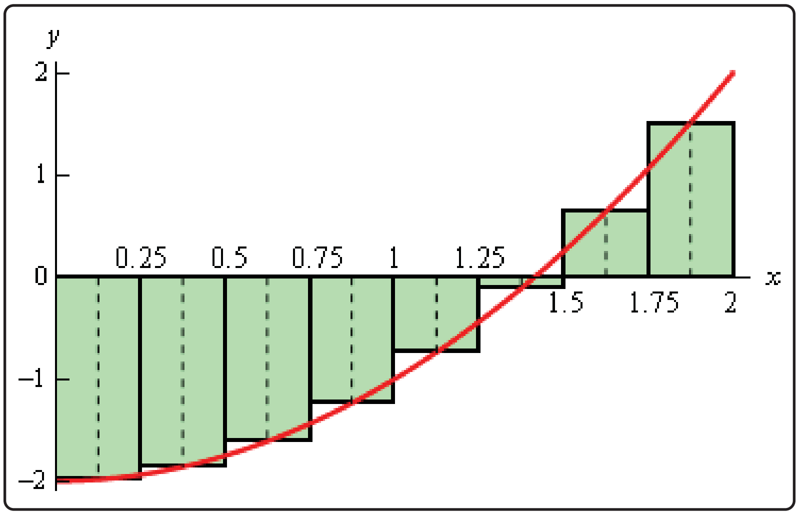


In this case let's notice that the function lies completely below the x-axis and hence is always negative. If we ignore the fact that the function is always negative and use the same ideas above to estimate the area between the graph and the x-axis we get,

$$\begin{aligned} A_m &= \frac{1}{4} f\left(\frac{1}{8}\right) + \frac{1}{4} f\left(\frac{3}{8}\right) + \frac{1}{4} f\left(\frac{5}{8}\right) + \frac{1}{4} f\left(\frac{7}{8}\right) + \frac{1}{4} f\left(\frac{9}{8}\right) + \\ &\qquad \frac{1}{4} f\left(\frac{11}{8}\right) + \frac{1}{4} f\left(\frac{13}{8}\right) + \frac{1}{4} f\left(\frac{15}{8}\right) \\ &= -5.34375 \end{aligned}$$

Our answer is negative as we might have expected given that all the function evaluations are negative.

So, using the technique in this section it looks like if the function is above the x-axis we will get a positive area and if the function is below the x-axis we will get a negative area. Now, what about a function that is both positive and negative in the interval? For example, $f(x) = x^2 - 2$ on $[0, 2]$. Using $n = 8$ and midpoints the graph is,



Some of the rectangles are below the x-axis and so will give negative areas while some are above the x-axis and will give positive areas. Since more rectangles are below the x-axis than above it looks like we should probably get a negative area estimation for this case. In fact that is correct. Here the area estimation for this case.

$$\begin{aligned} A_m &= \frac{1}{4}f\left(\frac{1}{8}\right) + \frac{1}{4}f\left(\frac{3}{8}\right) + \frac{1}{4}f\left(\frac{5}{8}\right) + \frac{1}{4}f\left(\frac{7}{8}\right) + \frac{1}{4}f\left(\frac{9}{8}\right) + \\ &\qquad \frac{1}{4}f\left(\frac{11}{8}\right) + \frac{1}{4}f\left(\frac{13}{8}\right) + \frac{1}{4}f\left(\frac{15}{8}\right) \\ &= -1.34375 \end{aligned}$$

In cases where the function is both above and below the x-axis the technique given in the section will give the *net* area between the function and the x-axis with areas below the x-axis negative and areas above the x-axis positive. So, if the net area is negative then there is more area under the x-axis than above while a positive net area will mean that more of the area is above the x-axis.

5.6 Definition of the Definite Integral

In this section we will formally define the definite integral and give many of the properties of definite integrals. Let's start off with the definition of a definite integral.

Definite Integral

Given a function $f(x)$ that is continuous on the interval $[a, b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of** $f(x)$ **from** a **to** b is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

The definite integral is defined to be exactly the limit and summation that we looked at in the last section to find the net area between a function and the x-axis. Also note that the notation for the definite integral is very similar to the notation for an indefinite integral. The reason for this will be apparent eventually.

There is also a little bit of terminology that we should get out of the way here. The number "a" that is at the bottom of the integral sign is called the **lower limit** of the integral and the number "b" at the top of the integral sign is called the **upper limit** of the integral. Also, despite the fact that a and b were given as an interval the lower limit does not necessarily need to be smaller than the upper limit. Collectively we'll often call a and b the **interval of integration**.

Let's work a quick example. This example will use many of the properties and facts from the brief review of summation notation in the Extras appendix.

Example 1

Using the definition of the definite integral compute the following.

$$\int_0^2 x^2 + 1\, dx$$

Solution

First, we can't actually use the definition unless we determine which points in each interval that well use for x_i^*. In order to make our life easier we'll use the right endpoints of each interval.

From the previous section we know that for a general n the width of each subinterval is,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

The subintervals are then,

$$\left[0, \frac{2}{n}\right], \quad \left[\frac{2}{n}, \frac{4}{n}\right], \quad \left[\frac{4}{n}, \frac{6}{n}\right], \ \ldots \ , \ \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right], \ \ldots \ , \ \left[\frac{2(n-1)}{n}, 2\right]$$

As we can see the right endpoint of the i^{th} subinterval is

$$x_i^* = \frac{2i}{n}$$

The summation in the definition of the definite integral is then,

$$\begin{aligned}\sum_{i=1}^{n} f(x_i^*)\,\Delta x &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^2 + 1\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left(\frac{8i^2}{n^3} + \frac{2}{n}\right)\end{aligned}$$

Now, we are going to have to take a limit of this. That means that we are going to need to "evaluate" this summation. In other words, we are going to have to use the formulas given in the summation notation review to eliminate the actual summation and get a formula for this for a general n.

To do this we will need to recognize that n is a constant as far as the summation notation is concerned. As we cycle through the integers from 1 to n in the summation only i changes and so anything that isn't an i will be a constant and can be factored out of the summation. In particular any n that is in the summation can be factored out if we need to.

Here is the summation "evaluation".

$$\begin{aligned}\sum_{i=1}^{n} f(x_i^*)\,\Delta x &= \sum_{i=1}^{n}\frac{8i^2}{n^3} + \sum_{i=1}^{n}\frac{2}{n} \\ &= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n}\sum_{i=1}^{n} 2 \\ &= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{n}(2n) \\ &= \frac{4(n+1)(2n+1)}{3n^2} + 2 \\ &= \frac{14n^2 + 12n + 4}{3n^2}\end{aligned}$$

We can now compute the definite integral.

$$\begin{aligned}\int_0^2 x^2+1\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x\\ &= \lim_{n\to\infty}\frac{14n^2+12n+4}{3n^2}\\ &= \frac{14}{3}\end{aligned}$$

We've seen several methods for dealing with the limit in this problem so we'll leave it to you to verify the results.

Wow, that was a lot of work for a fairly simple function. There is a much simpler way of evaluating these and we will get to it eventually. The main purpose to this section is to get the main properties and facts about the definite integral out of the way. We'll discuss how we compute these in practice starting with the next section.

So, let's start taking a look at some of the properties of the definite integral.

Properties

1. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

   We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2. $\displaystyle\int_a^a f(x)\,dx = 0$.

   If the upper and lower limits are the same then there is no work to do, the integral is zero.

3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any number.

   So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

4. $\displaystyle\int_a^b f(x)\pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

   We can break up definite integrals across a sum or difference.

5. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is any number.

   This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, $[a,c]$ and $[c,b]$. Note however that c doesn't need to be between a and b.

6. $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

   The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

See the Proof of Various Integral Properties section of the Extras appendix for the proof of properties 1 – 4. Property 5 is not easy to prove and so is not shown there. Property 6 is not really a property in the full sense of the word. It is only here to acknowledge that as long as the function and limits are the same it doesn't matter what letter we use for the variable. The answer will be the same.

Let's do a couple of examples dealing with these properties.

Example 2

Use the results from the first example to evaluate each of the following.

(a) $\displaystyle\int_2^0 x^2 + 1\,dx$

(b) $\displaystyle\int_0^2 10x^2 + 10\,dx$

(c) $\displaystyle\int_0^2 t^2 + 1\,dt$

Solution

All of the solutions to these problems will rely on the fact we proved in the first example. Namely that,

$$\int_0^2 x^2 + 1\,dx = \frac{14}{3}$$

(a) $\displaystyle\int_{2}^{0} x^2 + 1\,dx$

In this case the only difference between the two is that the limits have interchanged. So, using the first property gives,

$$\begin{aligned}\int_{2}^{0} x^2 + 1\,dx &= -\int_{0}^{2} x^2 + 1\,dx \\ &= -\frac{14}{3}\end{aligned}$$

(b) $\displaystyle\int_{0}^{2} 10x^2 + 10\,dx$

For this part notice that we can factor a 10 out of both terms and then out of the integral using the third property.

$$\begin{aligned}\int_{0}^{2} 10x^2 + 10\,dx &= \int_{0}^{2} 10\left(x^2 + 1\right)\,dx \\ &= 10\int_{0}^{2} x^2 + 1\,dx \\ &= 10\left(\frac{14}{3}\right) \\ &= \frac{140}{3}\end{aligned}$$

(c) $\displaystyle\int_{0}^{2} t^2 + 1\,dt$

In this case the only difference is the letter used and so this is just going to use property 6.

$$\int_{0}^{2} t^2 + 1\,dt = \int_{0}^{2} x^2 + 1\,dx = \frac{14}{3}$$

Here are a couple of examples using the other properties.

Example 3

Evaluate the following definite integral.

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2+1}\,dx$$

Solution

There really isn't anything to do with this integral once we notice that the limits are the same. Using the second property this is,

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2+1}\,dx = 0$$

Example 4

Given that $\int_{6}^{-10} f(x)\,dx = 23$ and $\int_{-10}^{6} g(x)\,dx = -9$ determine the value of

$$\int_{-10}^{6} 2f(x) - 10g(x)\,dx$$

Solution

We will first need to use the fourth property to break up the integral and the third property to factor out the constants.

$$\begin{aligned}\int_{-10}^{6} 2f(x) - 10g(x)\,dx &= \int_{-10}^{6} 2f(x)\,dx - \int_{-10}^{6} 10g(x)\,dx \\ &= 2\int_{-10}^{6} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx\end{aligned}$$

Now notice that the limits on the first integral are interchanged with the limits on the given integral so switch them using the first property above (and adding a minus sign of course). Once this is done we can plug in the known values of the integrals.

$$\begin{aligned}\int_{-10}^{6} 2f(x) - 10g(x)\,dx &= -2\int_{6}^{-10} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx \\ &= -2(23) - 10(-9) \\ &= 44\end{aligned}$$

Example 5

Given that $\int_{12}^{-10} f(x)\,dx = 6$, $\int_{100}^{-10} f(x)\,dx = -2$, and $\int_{100}^{-5} f(x)\,dx = 4$ determine the value of $\int_{-5}^{12} f(x)\,dx$.

Solution

This example is mostly an example of property 5 although there are a couple of uses of property 1 in the solution as well.

We need to figure out how to correctly break up the integral using property 5 to allow us to use the given pieces of information. First, we'll note that there is an integral that has a "-5" in one of the limits. It's not the lower limit, but we can use property 1 to correct that eventually. The other limit is 100 so this is the number c that we'll use in property 5.

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{12} f(x)\,dx$$

We'll be able to get the value of the first integral, but the second still isn't in the list of know integrals. However, we do have second integral that has a limit of 100 in it. The other limit for this second integral is -10 and this will be c in this application of property 5.

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{-10} f(x)\,dx + \int_{-10}^{12} f(x)\,dx$$

At this point all that we need to do is use the property 1 on the first and third integral to get the limits to match up with the known integrals. After that we can plug in for the known integrals.

$$\begin{aligned}\int_{-5}^{12} f(x)\,dx &= -\int_{100}^{-5} f(x)\,dx + \int_{100}^{-10} f(x)\,dx - \int_{12}^{-10} f(x)\,dx\\ &= -4 - 2 - 6\\ &= -12\end{aligned}$$

There are also some nice properties that we can use in comparing the general size of definite integrals. Here they are.

More Properties

7. $\displaystyle\int_a^b c\,dx = c\,(b-a)$, c is any number.

8. If $f(x) \geq 0$ for $a \leq x \leq b$ then $\displaystyle\int_a^b f(x)\,dx \geq 0$.

9. If $f(x) \geq g(x)$ for $a \leq x \leq b$ then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

10. If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then $m\,(b-a) \leq \int_a^b f(x)\,dx \leq M\,(b-a)$.

11. $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

See the Proof of Various Integral Properties section of the Extras appendix for the proof of these properties.

Interpretations of Definite Integral

There are a couple of quick interpretations of the definite integral that we can give here.

First, as we alluded to in the previous section one possible interpretation of the definite integral is to give the net area between the graph of $f(x)$ and the x-axis on the interval $[a,b]$. So, the net area between the graph of $f(x) = x^2 + 1$ and the x-axis on $[0,2]$ is,

$$\int_0^2 x^2 + 1\,dx = \frac{14}{3}$$

If you look back in the last section this was the exact area that was given for the initial set of problems that we looked at in this area.

Another interpretation is sometimes called the Net Change Theorem. This interpretation says that if $f(x)$ is some quantity (so $f'(x)$ is the rate of change of $f(x)$) then,

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

is the net change in $f(x)$ on the interval $[a,b]$. In other words, compute the definite integral of a rate of change and you'll get the net change in the quantity. We can see that the value of the definite integral, $f(b) - f(a)$, does in fact give us the net change in $f(x)$ and so there really isn't anything to prove with this statement. This is really just an acknowledgment of what the definite integral of a rate of change tells us.

So as a quick example, if $V(t)$ is the volume of water in a tank then,

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the net change in the volume as we go from time t_1 to time t_2.

Likewise, if $s(t)$ is the function giving the position of some object at time t we know that the velocity of the object at any time t is : $v(t) = s'(t)$. Therefore, the displacement of the object from time t_1 to time t_2 is,

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

Note that in this case if $v(t)$ is both positive and negative (*i.e.* the object moves to both the right and left) in the time frame this will NOT give the total distance traveled. It will only give the displacement, *i.e.* the difference between where the object started and where it ended up. To get the total distance traveled by an object we'd have to compute,

$$\int_{t_1}^{t_2} |v(t)|\,dt$$

It is important to note here that the Net Change Theorem only really makes sense if we're integrating a derivative of a function.

Fundamental Theorem of Calculus, Part I

As noted by the title above this is only the first part to the Fundamental Theorem of Calculus. We will give the second part in the next section as it is the key to easily computing definite integrals and that is the subject of the next section.

The first part of the Fundamental Theorem of Calculus tells us how to differentiate certain types of definite integrals and it also tells us about the very close relationship between integrals and derivatives.

Fundamental Theorem of Calculus, Part I

If $f(x)$ is continuous on $[a,b]$ then,

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and it is differentiable on (a,b) and,

$$g'(x) = f(x)$$

An alternate notation for the derivative portion of this is,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

To see the proof of this see the Proof of Various Integral Properties section of the Extras appendix.

Let's check out a couple of quick examples using this.

Example 6

Differentiate each of the following.

(a) $g\left(x\right) = \displaystyle\int_{-4}^{x} \mathbf{e}^{2t}\cos^2\left(1 - 5t\right)\,dt$

(b) $\displaystyle\int_{x^2}^{1} \frac{t^4 + 1}{t^2 + 1}\,dt$

Solution

(a) $g\left(x\right) = \displaystyle\int_{-4}^{x} \mathbf{e}^{2t}\cos^2\left(1 - 5t\right)\,dt$

This one is nothing more than a quick application of the Fundamental Theorem of Calculus.

$$g'\left(x\right) = \mathbf{e}^{2x}\cos^2\left(1 - 5x\right)$$

(b) $\displaystyle\int_{x^2}^{1} \frac{t^4 + 1}{t^2 + 1}\,dt$

This one needs a little work before we can use the Fundamental Theorem of Calculus. The first thing to notice is that the Fundamental Theorem of Calculus requires the lower limit to be a constant and the upper limit to be the variable. So, using a property of definite integrals we can interchange the limits of the integral we just need to remember to add in a minus sign after we do that. Doing this gives,

$$\frac{d}{dx}\int_{x^2}^{1} \frac{t^4 + 1}{t^2 + 1}\,dt = \frac{d}{dx}\left(-\int_{1}^{x^2} \frac{t^4 + 1}{t^2 + 1}\,dt\right) = -\frac{d}{dx}\int_{1}^{x^2} \frac{t^4 + 1}{t^2 + 1}\,dt$$

The next thing to notice is that the Fundamental Theorem of Calculus also requires an x in the upper limit of integration and we've got x^2. To do this derivative we're going to need the following version of the chain rule.

$$\frac{d}{dx}\Big(g\left(u\right)\Big) = \frac{d}{du}\Big(g\left(u\right)\Big)\,\frac{du}{dx} \qquad \text{where } u = f\left(x\right)$$

So, if we let $u = x^2$ we use the chain rule to get,

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}\,dt &= -\frac{d}{dx}\int_{1}^{x^2}\frac{t^4+1}{t^2+1}\,dt\\ &= -\frac{d}{du}\int_{1}^{u}\frac{t^4+1}{t^2+1}\,dt\;\frac{du}{dx} \qquad \text{where } u = x^2\\ &= -\frac{u^4+1}{u^2+1}\left(2x\right)\\ &= -2x\frac{u^4+1}{u^2+1}\end{aligned}$$

The final step is to get everything back in terms of x.

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}\,dt &= -2x\frac{\left(x^2\right)^4+1}{\left(x^2\right)^2+1}\\ &= -2x\frac{x^8+1}{x^4+1}\end{aligned}$$

Using the chain rule as we did in the last part of this example we can derive some general formulas for some more complicated problems.

First,

$$\frac{d}{dx}\int_{a}^{u(x)} f\left(t\right)\,dt = u'\left(x\right)f\left(u\left(x\right)\right)$$

This is simply the chain rule for these kinds of problems.

Next, we can get a formula for integrals in which the upper limit is a constant and the lower limit is a function of x. All we need to do here is interchange the limits on the integral (adding in a minus sign of course) and then use the formula above to get,

$$\frac{d}{dx}\int_{v(x)}^{b} f\left(t\right)\,dt = -\frac{d}{dx}\int_{b}^{v(x)} f\left(t\right)\,dt = -v'\left(x\right)f\left(v\left(x\right)\right)$$

Finally, we can also get a version for both limits being functions of x. In this case we'll need to use Property 5 above to break up the integral as follows,

$$\int_{v(x)}^{u(x)} f\left(t\right)\,dt = \int_{v(x)}^{a} f\left(t\right)\,dt + \int_{a}^{u(x)} f\left(t\right)\,dt$$

We can use pretty much any value of a when we break up the integral. The only thing that we need to do is to make sure that $f\left(a\right)$ exists. So, assuming that $f\left(a\right)$ exists after we break up the integral we can then differentiate and use the two formulas above to get,

$$\begin{aligned}\frac{d}{dx}\int_{v(x)}^{u(x)} f\left(t\right)\,dt &= \frac{d}{dx}\left(\int_{v(x)}^{a} f\left(t\right)\,dt + \int_{a}^{u(x)} f\left(t\right)\,dt\right)\\ &= -v'\left(x\right)f\left(v\left(x\right)\right) + u'\left(x\right)f\left(u\left(x\right)\right)\end{aligned}$$

Let's work a quick example.

Example 7

Differentiate the following integral.

$$\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right)\, dt$$

Solution

This will use the final formula that we derived above.

$$\begin{aligned}\frac{d}{dx}\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right)\, dt &= -\frac{1}{2}x^{-\frac{1}{2}}\left(\sqrt{x}\right)^2 \sin\left(1+\left(\sqrt{x}\right)^2\right) + (3)\left(3x\right)^2 \sin\left(1+\left(3x\right)^2\right)\\ &= -\frac{1}{2}\sqrt{x}\sin\left(1+x\right) + 27x^2 \sin\left(1+9x^2\right)\end{aligned}$$

5.7 Computing Definite Integrals

In this section we are going to concentrate on how we actually evaluate definite integrals in practice. To do this we will need the Fundamental Theorem of Calculus, Part II.

Fundamental Theorem of Calculus, Part II

Suppose $f(x)$ is a continuous function on $[a,b]$ and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$$

To see the proof of this see the Proof of Various Integral Properties section of the Extras appendix.

Recall that when we talk about an anti-derivative for a function we are really talking about the indefinite integral for the function. So, to evaluate a definite integral the first thing that we're going to do is evaluate the indefinite integral for the function. This should explain the similarity in the notations for the indefinite and definite integrals.

Also notice that we require the function to be continuous in the interval of integration. This was also a requirement in the definition of the definite integral. We didn't make a big deal about this in the last section. In this section however, we will need to keep this condition in mind as we do our evaluations.

Next let's address the fact that we can use any anti-derivative of $f(x)$ in the evaluation. Let's take a final look at the following integral.

$$\int_0^2 x^2 + 1\,dx$$

Both of the following are anti-derivatives of the integrand.

$$F(x) = \frac{1}{3}x^3 + x \qquad \text{and} \qquad F(x) = \frac{1}{3}x^3 + x - \frac{18}{31}$$

Using the Fundamental Theorem of Calculus to evaluate this integral with the first anti-derivatives gives,

$$\begin{aligned}\int_0^2 x^2 + 1\,dx &= \left.\left(\frac{1}{3}x^3 + x\right)\right|_0^2 \\ &= \frac{1}{3}(2)^3 + 2 - \left(\frac{1}{3}(0)^3 + 0\right) \\ &= \frac{14}{3}\end{aligned}$$

Much easier than using the definition wasn't it? Let's now use the second anti-derivative to evaluate this definite integral.

$$\begin{aligned}\int_0^2 x^2+1\,dx &= \left(\frac{1}{3}x^3+x-\frac{18}{31}\right)\Big|_0^2\\ &= \frac{1}{3}(2)^3+2-\frac{18}{31}-\left(\frac{1}{3}(0)^3+0-\frac{18}{31}\right)\\ &= \frac{14}{3}-\frac{18}{31}+\frac{18}{31}\\ &= \frac{14}{3}\end{aligned}$$

The constant that we tacked onto the second anti-derivative canceled in the evaluation step. So, when choosing the anti-derivative to use in the evaluation process make your life easier and don't bother with the constant as it will only end up canceling in the long run.

Also, note that we're going to have to be very careful with minus signs and parenthesis with these problems. It's very easy to get in a hurry and mess them up.

Let's start our examples with the following set designed to make a couple of quick points that are very important.

Example 1

Evaluate each of the following.

(a) $\displaystyle\int y^2+y^{-2}\,dy$

(b) $\displaystyle\int_1^2 y^2+y^{-2}\,dy$

(c) $\displaystyle\int_{-1}^2 y^2+y^{-2}\,dy$

Solution

(a) $\displaystyle\int y^2+y^{-2}\,dy$

This is the only indefinite integral in this section and by now we should be getting pretty good with these so we won't spend a lot of time on this part. This is here only to make sure that we understand the difference between an indefinite and a definite integral. The integral is,

$$\int y^2+y^{-2}\,dy=\frac{1}{3}y^3-y^{-1}+c$$

(b) $\displaystyle\int_1^2 y^2 + y^{-2}\,dy$

Recall from our first example above that all we really need here is any anti-derivative of the integrand. We just computed the most general anti-derivative in the first part so we can use that if we want to. However, recall that as we noted above any constants we tack on will just cancel in the long run and so we'll use the answer from (a) without the "$+c$".

Here's the integral,

$$\begin{aligned}\int_1^2 y^2 + y^{-2}\,dy &= \left(\frac{1}{3}y^3 - \frac{1}{y}\right)\Bigg|_1^2 \\ &= \frac{1}{3}(2)^3 - \frac{1}{2} - \left(\frac{1}{3}(1)^3 - \frac{1}{1}\right) \\ &= \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 \\ &= \frac{17}{6}\end{aligned}$$

Remember that the evaluation is always done in the order of evaluation at the upper limit minus evaluation at the lower limit. Also, be very careful with minus signs and parenthesis. It's very easy to forget them or mishandle them and get the wrong answer.

Notice as well that, in order to help with the evaluation, we rewrote the indefinite integral a little. In particular we got rid of the negative exponent on the second term. It's generally easier to evaluate the term with positive exponents.

(c) $\displaystyle\int_{-1}^2 y^2 + y^{-2}\,dy$

This integral is here to make a point. Recall that in order for us to do an integral the integrand must be continuous in the range of the limits. In this case the second term will have division by zero at $y = 0$ and since $y = 0$ is in the interval of integration, *i.e.* it is between the lower and upper limit, this integrand is not continuous in the interval of integration and so we can't do this integral.

Note that this problem will not prevent us from doing the integral in (b) since $y = 0$ is not in the interval of integration.

So, what have we learned from this example?

First, in order to do a definite integral the first thing that we need to do is the indefinite integral. So, we aren't going to get out of doing indefinite integrals, they will be in every integral that we'll be doing in the rest of this course so make sure that you're getting good at computing them.

Second, we need to be on the lookout for functions that aren't continuous at any point between the limits of integration. Also, it's important to note that this will only be a problem if the point(s) of discontinuity occur between the limits of integration or at the limits themselves. If the point of discontinuity occurs outside of the limits of integration the integral can still be evaluated.

In the following sets of examples we won't make too much of an issue with continuity problems, or lack of continuity problems, unless it affects the evaluation of the integral. Do not let this convince you that you don't need to worry about this idea. It arises often enough that it can cause real problems if you aren't on the lookout for it.

Finally, note the difference between indefinite and definite integrals. Indefinite integrals are functions while definite integrals are numbers.

Let's work some more examples.

Example 2

Evaluate each of the following.

(a) $\displaystyle\int_{-3}^{1} 6x^2 - 5x + 2\,dx$

(b) $\displaystyle\int_{4}^{0} \sqrt{t}\,(t-2)\;dt$

(c) $\displaystyle\int_{1}^{2} \frac{2w^5 - w + 3}{w^2}\,dw$

(d) $\displaystyle\int_{25}^{-10} dR$

Solution

(a) $\displaystyle\int_{-3}^{1} 6x^2 - 5x + 2\,dx$

There isn't a lot to this one other than simply doing the work.

$$\begin{aligned}\int_{-3}^{1} 6x^2 - 5x + 2\,dx &= \left(2x^3 - \frac{5}{2}x^2 + 2x\right)\Big|_{-3}^{1}\\ &= \left(2 - \frac{5}{2} + 2\right) - \left(-54 - \frac{45}{2} - 6\right)\\ &= 84\end{aligned}$$

(b) $\displaystyle\int_4^0 \sqrt{t}\,(t-2)\ dt$

Recall that we can't integrate products as a product of integrals and so we first need to multiply the integrand out before integrating, just as we did in the indefinite integral case.

$$\begin{aligned}\int_4^0 \sqrt{t}\,(t-2)\ dt &= \int_4^0 t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\ dt \\ &= \left(\frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}}\right)\Bigg|_4^0 \\ &= 0 - \left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{4}{3}(4)^{\frac{3}{2}}\right) \\ &= -\frac{32}{15}\end{aligned}$$

In the evaluation process recall that,

$$(4)^{\frac{5}{2}} = \left((4)^{\frac{1}{2}}\right)^5 = (2)^5 = 32$$

$$(4)^{\frac{3}{2}} = \left((4)^{\frac{1}{2}}\right)^3 = (2)^3 = 8$$

Also, don't get excited about the fact that the lower limit of integration is larger than the upper limit of integration. That will happen on occasion and there is absolutely nothing wrong with this.

(c) $\displaystyle\int_1^2 \frac{2w^5 - w + 3}{w^2}\,dw$

First, notice that we will have a division by zero issue at $w = 0$, but since this isn't in the interval of integration we won't have to worry about it.

Next again recall that we can't integrate quotients as a quotient of integrals and so the first step that we'll need to do is break up the quotient so we can integrate the function.

$$\begin{aligned}\int_1^2 \frac{2w^5 - w + 3}{w^2}\,dw &= \int_1^2 2w^3 - \frac{1}{w} + 3w^{-2}dw \\ &= \left(\frac{1}{2}w^4 - \ln|w| - \frac{3}{w}\right)\Bigg|_1^2 \\ &= \left(8 - \ln(2) - \frac{3}{2}\right) - \left(\frac{1}{2} - \ln 1 - 3\right) \\ &= 9 - \ln(2)\end{aligned}$$

Don't get excited about answers that don't come down to a simple integer or fraction. Often times they won't. Also, don't forget that $\ln(1) = 0$.

(d) $\displaystyle\int_{25}^{-10} dR$

This one is actually pretty easy. Recall that we're just integrating 1.

$$\begin{aligned}\int_{25}^{-10} dR &= R\Big|_{25}^{-10}\\ &= -10 - 25\\ &= -35\end{aligned}$$

The last set of examples dealt exclusively with integrating powers of x. Let's work a couple of examples that involve other functions.

Example 3

Evaluate each of the following.

(a) $\displaystyle\int_{0}^{1} 4x - 6\sqrt[3]{x^2}\,dx$

(b) $\displaystyle\int_{0}^{\frac{\pi}{3}} 2\sin(\theta) - 5\cos(\theta)\,d\theta$

(c) $\displaystyle\int_{\pi/6}^{\pi/4} 5 - 2\sec(z)\tan(z)\,dz$

(d) $\displaystyle\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\,dz$

(e) $\displaystyle\int_{-2}^{3} 5t^6 - 10t + \frac{1}{t}\,dt$

Solution

(a) $\displaystyle\int_0^1 4x - 6\sqrt[3]{x^2}\,dx$

This one is here mostly here to contrast with the next example.

$$\begin{aligned}\int_0^1 4x - 6\sqrt[3]{x^2}\,dx &= \int_0^1 4x - 6x^{\frac{2}{3}}\,dx \\ &= \left(2x^2 - \frac{18}{5}x^{\frac{5}{3}}\right)\Big|_0^1 \\ &= 2 - \frac{18}{5} - (0) \\ &= -\frac{8}{5}\end{aligned}$$

(b) $\displaystyle\int_0^{\frac{\pi}{3}} 2\sin(\theta) - 5\cos(\theta)\,d\theta$

Be careful with signs with this one. Recall from the indefinite integral sections that it's easy to mess up the signs when integrating sine and cosine.

$$\begin{aligned}\int_0^{\frac{\pi}{3}} 2\sin(\theta) - 5\cos(\theta)\,d\theta &= \left(-2\cos(\theta) - 5\sin(\theta)\right)\Big|_0^{\pi/3} \\ &= -2\cos\left(\frac{\pi}{3}\right) - 5\sin\left(\frac{\pi}{3}\right) - \left(-2\cos(0) - 5\sin(0)\right) \\ &= -1 - \frac{5\sqrt{3}}{2} + 2 \\ &= 1 - \frac{5\sqrt{3}}{2}\end{aligned}$$

Compare this answer to the previous answer, especially the evaluation at zero. It's very easy to get into the habit of just writing down zero when evaluating a function at zero. This is especially a problem when many of the functions that we integrate involve only x's raised to positive integers; these evaluate is zero of course. After evaluating many of these kinds of definite integrals it's easy to get into the habit of just writing down zero when you evaluate at zero. However, there are many functions out there that aren't zero when evaluated at zero so be careful.

(c) $\displaystyle\int_{\pi/6}^{\pi/4} 5 - 2\sec(z)\tan(z)\,dz$

Not much to do other than do the integral.

$$\begin{aligned}\int_{\pi/6}^{\pi/4} 5 - 2\sec(z)\tan(z)\,dz &= \left(5z - 2\sec(z)\right)\Big|_{\pi/6}^{\pi/4} \\ &= 5\left(\frac{\pi}{4}\right) - 2\sec\left(\frac{\pi}{4}\right) - \left(5\left(\frac{\pi}{6}\right) - 2\sec\left(\frac{\pi}{6}\right)\right) \\ &= \frac{5\pi}{12} - 2\sqrt{2} + \frac{4}{\sqrt{3}}\end{aligned}$$

For the evaluation, recall that

$$\sec(z) = \frac{1}{\cos(z)}$$

and so if we can evaluate cosine at these angles we can evaluate secant at these angles.

(d) $\displaystyle\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\,dz$

In order to do this one will need to rewrite both of the terms in the integral a little as follows,

$$\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\,dz = \int_{-20}^{-1} 3\mathbf{e}^{z} - \frac{1}{3}\frac{1}{z}\,dz$$

For the first term recall we used the following fact about exponents.

$$x^{-a} = \frac{1}{x^a} \qquad\qquad \frac{1}{x^{-a}} = x^a$$

In the second term, taking the 3 out of the denominator will just make integrating that term easier.

Now the integral.

$$\begin{aligned}\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\,dz &= \left(3\mathbf{e}^{z} - \frac{1}{3}\ln|z|\right)\Bigg|_{-20}^{-1}\\ &= 3\mathbf{e}^{-1} - \frac{1}{3}\ln|-1| - \left(3\mathbf{e}^{-20} - \frac{1}{3}\ln|-20|\right)\\ &= 3\mathbf{e}^{-1} - 3\mathbf{e}^{-20} + \frac{1}{3}\ln(20)\end{aligned}$$

Just leave the answer like this. It's messy, but it's also exact.

Note that the absolute value bars on the logarithm are required here. Without them we couldn't have done the evaluation.

(e) $\displaystyle\int_{-2}^{3} 5t^6 - 10t + \frac{1}{t}\,dt$

This integral can't be done. There is division by zero in the third term at $t = 0$ and $t = 0$ lies in the interval of integration. The fact that the first two terms can be integrated doesn't matter. If even one term in the integral can't be integrated then the whole integral can't be done.

So, we've computed a fair number of definite integrals at this point. Remember that the vast majority of the work in computing them is first finding the indefinite integral. Once we've found that the rest is just some number crunching.

There are a couple of particularly tricky definite integrals that we need to take a look at next. Actually they are only tricky until you see how to do them, so don't get too excited about them. The first one involves integrating a piecewise function.

Example 4

Given,

$$f(x) = \begin{cases} 6 & \text{if } x > 1 \\ 3x^2 & \text{if } x \le 1 \end{cases}$$

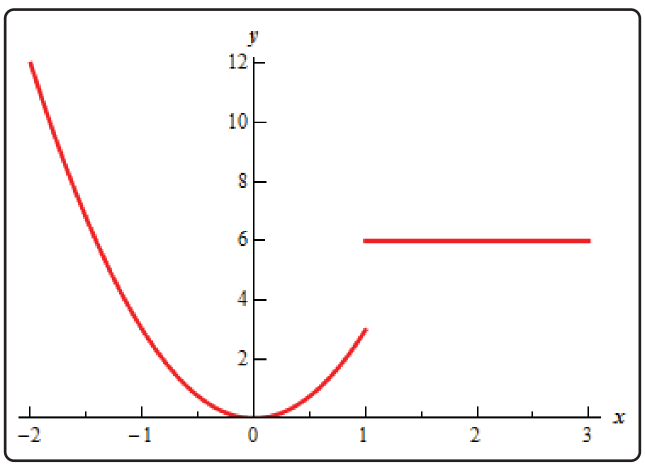
Evaluate each of the following integrals.

(a) $\displaystyle\int_{10}^{22} f(x)\, dx$

(b) $\displaystyle\int_{-2}^{3} f(x)\, dx$

Solution

Let's first start with a graph of this function.



The graph reveals a problem. This function is not continuous at $x = 1$ and we're going to have to watch out for that.

(a) $\displaystyle\int_{10}^{22} f(x)\, dx$

For this integral notice that $x = 1$ is not in the interval of integration and so that is something that we'll not need to worry about in this part.

Also note the limits for the integral lie entirely in the range for the first function. What this means for us is that when we do the integral all we need to do is plug in the first function into the integral.

Here is the integral.

$$\begin{aligned}\int_{10}^{22} f(x)\,dx &= \int_{10}^{22} 6\,dx \\ &= 6x\Big|_{10}^{22} \\ &= 132-60 \\ &= 72\end{aligned}$$

(b) $\displaystyle\int_{-2}^{3} f(x)\,dx$

In this part $x = 1$ is between the limits of integration. This means that the integrand is no longer continuous in the interval of integration and that is a show stopper as far we're concerned. As noted above we simply can't integrate functions that aren't continuous in the interval of integration.

Also, even if the function was continuous at $x = 1$ we would still have the problem that the function is actually two different equations depending where we are in the interval of integration.

Let's first address the problem of the function not being continuous at $x = 1$. As we'll see, in this case, if we can find a way around this problem the second problem will also get taken care of at the same time.

In the previous examples where we had functions that weren't continuous we had division by zero and no matter how hard we try we can't get rid of that problem. Division by zero is a real problem and we can't really avoid it. In this case the discontinuity does not stem from problems with the function not existing at $x = 1$. Instead the function is not continuous because it takes on different values on either sides of $x = 1$. We can "remove" this problem by recalling Property 5 from the previous section. This property tells us that we can write the integral as follows,

$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$$

On each of these intervals the function is continuous. In fact we can say more. In the first integral we will have x between -2 and 1 and this means that we can use the second equation for $f(x)$ and likewise for the second integral x will be between 1 and

3 and so we can use the first function for $f(x)$. The integral in this case is then,

$$\begin{aligned}\int_{-2}^{3} f(x)\,dx &= \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx \\ &= \int_{-2}^{1} 3x^2\,dx + \int_{1}^{3} 6\,dx \\ &= x^3\Big|_{-2}^{1} + 6x\Big|_{1}^{3} \\ &= 1-(-8)+(18-6) \\ &= 21\end{aligned}$$

So, to integrate a piecewise function, all we need to do is break up the integral at the break point(s) that happen to occur in the interval of integration and then integrate each piece.

Next, we need to look at is how to integrate an absolute value function.

Example 5

Evaluate the following integral.

$$\int_{0}^{3} |3t-5|\,dt$$

Solution

Recall that the point behind indefinite integration (which we'll need to do in this problem) is to determine what function we differentiated to get the integrand. To this point we've not seen any functions that will differentiate to get an absolute value nor will we ever see a function that will differentiate to get an absolute value.

The only way that we can do this problem is to get rid of the absolute value. To do this we need to recall the definition of absolute value.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Once we remember that we can define absolute value as a piecewise function we can use the work from Example 4 as a guide for doing this integral.

What we need to do is determine where the quantity on the inside of the absolute value bars is negative and where it is positive. It looks like if $t > \frac{5}{3}$ the quantity inside the absolute value is positive and if $t < \frac{5}{3}$ the quantity inside the absolute value is negative.

Next, note that $t = \frac{5}{3}$ is in the interval of integration and so, if we break up the integral at this point we get,

$$\int_0^3 |3t-5|\ dt = \int_0^{\frac{5}{3}} |3t-5|\ dt + \int_{\frac{5}{3}}^3 |3t-5|\ dt$$

Now, in the first integral we have $t < \frac{5}{3}$ and so $3t-5 < 0$ in this interval of integration. That means we can drop the absolute value bars if we put in a minus sign. Likewise, in the second integral we have $t > \frac{5}{3}$ which means that in this interval of integration we have $3t-5 > 0$ and so we can just drop the absolute value bars in this integral.

After getting rid of the absolute value bars in each integral we can do each integral. So, doing the integration gives,

$$\begin{aligned}
\int_0^3 |3t-5|\ dt &= \int_0^{\frac{5}{3}} -(3t-5)\ dt + \int_{\frac{5}{3}}^3 3t-5\,dt \\
&= \int_0^{\frac{5}{3}} -3t+5\,dt + \int_{\frac{5}{3}}^3 3t-5\,dt \\
&= \left(-\frac{3}{2}t^2+5t\right)\Bigg|_0^{\frac{5}{3}} + \left(\frac{3}{2}t^2-5t\right)\Bigg|_{\frac{5}{3}}^3 \\
&= -\frac{3}{2}\left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) - (0) + \left(\frac{3}{2}(3)^2 - 5(3) - \left(\frac{3}{2}\left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right)\right)\right) \\
&= \frac{25}{6} + \frac{8}{3} \\
&= \frac{41}{6}
\end{aligned}$$

Integrating absolute value functions isn't too bad. It's a little more work than the "standard" definite integral, but it's not really all that much more work. First, determine where the quantity inside the absolute value bars is negative and where it is positive. When we've determined that point all we need to do is break up the integral so that in each range of limits the quantity inside the absolute value bars is always positive or always negative. Once this is done we can drop the absolute value bars (adding negative signs when the quantity is negative) and then we can do the integral as we've always done.

Even and Odd Functions

This is the last topic that we need to discuss in this section.

First, recall that an even function is any function which satisfies,

$$f(-x) = f(x)$$

Typical examples of even functions are,

$$f(x) = x^2 \qquad\qquad f(x) = \cos(x)$$

An odd function is any function which satisfies,

$$f(-x) = -f(x)$$

The typical examples of odd functions are,

$$f(x) = x^3 \qquad\qquad f(x) = \sin(x)$$

There are a couple of nice facts about integrating even and odd functions over the interval $[-a, a]$. If $f(x)$ is an even function then,

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$$

Likewise, if $f(x)$ is an odd function then,

$$\int_{-a}^{a} f(x)\, dx = 0$$

Note that in order to use these facts the limit of integration must be the same number, but opposite signs!

Example 6

Integrate each of the following.

(a) $\displaystyle\int_{-2}^{2} 4x^4 - x^2 + 1\, dx$

(b) $\displaystyle\int_{-10}^{10} x^5 + \sin(x)\, dx$

Solution

Neither of these are terribly difficult integrals, but we can use the facts on them anyway.

(a) $\displaystyle\int_{-2}^{2} 4x^4 - x^2 + 1\, dx$

In this case the integrand is even and the interval is correct so,

$$\begin{aligned}\int_{-2}^{2} 4x^4 - x^2 + 1\, dx &= 2\int_{0}^{2} 4x^4 - x^2 + 1\, dx \\ &= 2\left(\frac{4}{5}x^5 - \frac{1}{3}x^3 + x\right)\Bigg|_{0}^{2} \\ &= \frac{748}{15}\end{aligned}$$

So, using the fact cut the evaluation in half (in essence since one of the new limits was zero).

(b) $\displaystyle\int_{-10}^{10} x^5 + \sin(x)\, dx$

The integrand in this case is odd and the interval is in the correct form and so we don't even need to integrate. Just use the fact.

$$\int_{-10}^{10} x^5 + \sin(x)\, dx = 0$$

Note that the limits of integration are important here. Take the last integral as an example. A small change to the limits will not give us zero.

$$\int_{-10}^{9} x^5 + \sin(x)\, dx = \cos(10) - \cos(9) - \frac{468559}{6} = -78093.09461$$

The moral here is to be careful and not misuse these facts.

5.8 Substitution Rule for Definite Integrals

We now need to go back and revisit the substitution rule as it applies to definite integrals. At some level there really isn't a lot to do in this section. Recall that the first step in doing a definite integral is to compute the indefinite integral and that hasn't changed. We will still compute the indefinite integral first. This means that we already know how to do these. We use the substitution rule to find the indefinite integral and then do the evaluation.

There are however, two ways to deal with the evaluation step. One of the ways of doing the evaluation is the probably the most obvious at this point, but also has a point in the process where we can get in trouble if we aren't paying attention.

Let's work an example illustrating both ways of doing the evaluation step.

Example 1

Evaluate the following definite integral.

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt$$

Solution

Let's start off looking at the first way of dealing with the evaluation step. We'll need to be careful with this method as there is a point in the process where if we aren't paying attention we'll get the wrong answer.

Solution 1 :

We'll first need to compute the indefinite integral using the substitution rule. Note however, that we will constantly remind ourselves that this is a definite integral by putting the limits on the integral at each step. Without the limits it's easy to forget that we had a definite integral when we've gotten the indefinite integral computed.

In this case the substitution is,

$$u = 1-4t^3 \qquad du = -12t^2dt \qquad \Rightarrow \qquad t^2dt = -\frac{1}{12}du$$

Plugging this into the integral gives,

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{6}\int_{-2}^{0} u^{\frac{1}{2}}\,du \\ &= -\frac{1}{9}u^{\frac{3}{2}}\bigg|_{-2}^{0}\end{aligned}$$

Notice that we didn't do the evaluation yet. This is where the potential problem arises with this solution method. The limits given here are from the original integral and hence are values of t. We have u's in our solution. We can't plug values of t in for u.

Therefore, we will have to go back to t's before we do the substitution. This is the standard step in the substitution process, but it is often forgotten when doing definite integrals. Note as well that in this case, if we don't go back to t's we will have a small problem in that one of the evaluations will end up giving us a complex number.

So, finishing this problem gives,

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{9}\left(1-4t^3\right)^{\frac{3}{2}}\Big|_{-2}^{0}\\ &= -\frac{1}{9}-\left(-\frac{1}{9}(33)^{\frac{3}{2}}\right)\\ &= \frac{1}{9}\left(33\sqrt{33}-1\right)\end{aligned}$$

So, that was the first solution method. Let's take a look at the second method.

Solution 2 :

Note that this solution method isn't really all that different from the first method. In this method we are going to remember that when doing a substitution we want to eliminate all the t's in the integral and write everything in terms of u.

When we say all here we really mean all. In other words, remember that the limits on the integral are also values of t and we're going to convert the limits into u values. Converting the limits is pretty simple since our substitution will tell us how to relate t and u so all we need to do is plug in the original t limits into the substitution and we'll get the new u limits.

Here is the substitution (it's the same as the first method) as well as the limit conversions.

$$\begin{array}{llll} u = 1-4t^3 & du = -12t^2dt & \Rightarrow & t^2dt = -\frac{1}{12}du \\ t=-2 & \Rightarrow & u = 1-4(-2)^3 = 33 & \\ t = 0 & \Rightarrow & u = 1-4(0)^3 = 1 & \end{array}$$

The integral is now,

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{6}\int_{33}^{1} u^{\frac{1}{2}}\,du\\ &= -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1}\end{aligned}$$

As with the first method let's pause here a moment to remind us what we're doing. In this case, we've converted the limits to u's and we've also got our integral in terms of u's and so here we can just plug the limits directly into our integral. Note that in this case we won't plug our substitution back in. Doing this here would cause problems as we would have t's in the integral and our limits would be u's. Here's the rest of this problem.

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1} \\ &= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right) = \frac{1}{9}\left(33\sqrt{33}-1\right)\end{aligned}$$

We got exactly the same answer and this time didn't have to worry about going back to t's in our answer.

So, we've seen two solution techniques for computing definite integrals that require the substitution rule. Both are valid solution methods and each have their uses. We will be using the second almost exclusively however since it makes the evaluation step a little easier.

Let's work some more examples.

Example 2

Evaluate each of the following.

(a) $\displaystyle\int_{-1}^{5} (1+w)\left(2w+w^2\right)^5\,dw$

(b) $\displaystyle\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x}\,dx$

(c) $\displaystyle\int_{0}^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\,dy$

(d) $\displaystyle\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz$

Solution

Since we've done quite a few substitution rule integrals to this time we aren't going to put a lot of effort into explaining the substitution part of things here.

(a) $\displaystyle\int_{-1}^{5} (1+w)\left(2w+w^2\right)^5 dw$

The substitution and converted limits are,

$$u = 2w + w^2 \qquad du = (2+2w)\,dw \qquad \Rightarrow \qquad (1+w)\,dw = \frac{1}{2}du$$
$$w = -1 \quad \Rightarrow \quad u = -1$$
$$w = 5 \quad \Rightarrow \quad u = 35$$

Sometimes a limit will remain the same after the substitution. Don't get excited when it happens and don't expect it to happen all the time.

Here is the integral,

$$\begin{aligned}\int_{-1}^{5} (1+w)\left(2w+w^2\right)^5 dw &= \frac{1}{2}\int_{-1}^{35} u^5\,du \\ &= \frac{1}{12}u^6\bigg|_{-1}^{35} = 153188802\end{aligned}$$

Don't get excited about large numbers for answers here. Sometimes they are. That's life.

(b) $\displaystyle\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x}\,dx$

Here is the substitution and converted limits for this problem,

$$u = 1 + 2x \qquad du = 2dx \qquad \Rightarrow \qquad dx = \frac{1}{2}du$$
$$x = -2 \quad \Rightarrow \quad u = -3$$
$$x = -6 \quad \Rightarrow \quad u = -11$$

The integral is then,

$$\begin{aligned}\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x}\,dx &= \frac{1}{2}\int_{-3}^{-11} 4u^{-3} - \frac{5}{u}\,du \\ &= \frac{1}{2}\left(-2u^{-2} - 5\ln|u|\right)\bigg|_{-3}^{-11} \\ &= \frac{1}{2}\left(-\frac{2}{121} - 5\ln(11)\right) - \frac{1}{2}\left(-\frac{2}{9} - 5\ln(3)\right) \\ &= \frac{112}{1089} - \frac{5}{2}\ln(11) + \frac{5}{2}\ln(3)\end{aligned}$$

(c) $\displaystyle\int_0^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\, dy$

This integral needs to be split into two integrals since the first term doesn't require a substitution and the second does.

$$\int_0^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\, dy = \int_0^{\frac{1}{2}} \mathbf{e}^y\, dy + \int_0^{\frac{1}{2}} 2\cos(\pi y)\, dy$$

Here is the substitution and converted limits for the second term.

$$\begin{aligned} &u = \pi y \qquad du = \pi dy \qquad \Rightarrow \qquad dy = \frac{1}{\pi} du \\ &y = 0 \qquad \Rightarrow \qquad u = 0 \\ &y = \frac{1}{2} \qquad \Rightarrow \qquad u = \frac{\pi}{2} \end{aligned}$$

Here is the integral.

$$\begin{aligned} \int_0^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\, dy &= \int_0^{\frac{1}{2}} \mathbf{e}^y\, dy + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(u)\, du \\ &= \mathbf{e}^y\Big|_0^{\frac{1}{2}} + \frac{2}{\pi}\sin(u)\Big|_0^{\frac{\pi}{2}} \\ &= \mathbf{e}^{\frac{1}{2}} - \mathbf{e}^0 + \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) - \frac{2}{\pi}\sin(0) \\ &= \mathbf{e}^{\frac{1}{2}} - 1 + \frac{2}{\pi} \end{aligned}$$

(d) $\displaystyle\int_{\frac{\pi}{3}}^0 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\, dz$

This integral will require two substitutions. So first split up the integral so we can do a substitution on each term.

$$\int_{\frac{\pi}{3}}^0 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\, dz = \int_{\frac{\pi}{3}}^0 3\sin\left(\frac{z}{2}\right) dz - \int_{\frac{\pi}{3}}^0 5\cos(\pi - z)\, dz$$

There are the two substitutions for these integrals.

$$\begin{aligned} &u = \frac{z}{2} \qquad du = \frac{1}{2}dz \qquad \Rightarrow \qquad dz = 2du \\ &z = \frac{\pi}{3} \qquad \Rightarrow \qquad u = \frac{\pi}{6} \\ &z = 0 \qquad \Rightarrow \qquad u = 0 \end{aligned}$$

$$\begin{aligned} &v = \pi - z \qquad dv = -dz \qquad \Rightarrow \qquad dz = -dv \\ &z = \frac{\pi}{3} \qquad \Rightarrow \qquad v = \frac{2\pi}{3} \\ &z = 0 \qquad \Rightarrow \qquad v = \pi \end{aligned}$$

Here is the integral for this problem.

$$\begin{aligned}\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos\left(\pi - z\right)\,dz &= 6\int_{\frac{\pi}{6}}^{0} \sin\left(u\right)\,du + 5\int_{\frac{2\pi}{3}}^{\pi} \cos\left(v\right)\,dv \\ &= -6\cos\left(u\right)\Big|_{\frac{\pi}{6}}^{0} + 5\sin\left(v\right)\Big|_{\frac{2\pi}{3}}^{\pi} \\ &= 3\sqrt{3} - 6 + \left(-\frac{5\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{2} - 6\end{aligned}$$

The next set of examples is designed to make sure that we don't forget about a very important point about definite integrals.

Example 3

Evaluate each of the following.

(a) $\displaystyle\int_{-5}^{5} \frac{4t}{2 - 8t^2}\,dt$

(b) $\displaystyle\int_{3}^{5} \frac{4t}{2 - 8t^2}\,dt$

Solution

(a) $\displaystyle\int_{-5}^{5} \frac{4t}{2 - 8t^2}\,dt$

Be careful with this integral. The denominator is zero at $t = \pm\frac{1}{2}$ and both of these are in the interval of integration. Therefore, this integrand is not continuous in the interval and so the integral can't be done.

Be careful with definite integrals and be on the lookout for division by zero problems. In the previous section they were easy to spot since all the division by zero problems that we had there were where the variable was itself zero. Once we move into substitution problems however they will not always be so easy to spot so make sure that you first take a quick look at the integrand and see if there are any continuity problems with the integrand and if they occur in the interval of integration.

(b) $\displaystyle\int_{3}^{5} \frac{4t}{2-8t^2}\,dt$

Now, in this case the integral can be done because the two points of discontinuity, $t = \pm\frac{1}{2}$, are both outside of the interval of integration. The substitution and converted limits in this case are,

$$u = 2 - 8t^2 \qquad du = -16t\,dt \qquad \Rightarrow \qquad t dt = -\frac{1}{16} du$$
$$t = 3 \quad \Rightarrow \quad u = -70$$
$$t = 5 \quad \Rightarrow \quad u = -198$$

The integral is then,

$$\begin{aligned}\int_{3}^{5} \frac{4t}{2-8t^2}\,dt &= -\frac{4}{16}\int_{-70}^{-198} \frac{1}{u}\,du \\ &= -\frac{1}{4}\ln|u|\Big|_{-70}^{-198} \\ &= -\frac{1}{4}\left(\ln(198) - \ln(70)\right)\end{aligned}$$

Let's work another set of examples. These are a little tougher (at least in appearance) than the previous sets.

Example 4

Evaluate each of the following.

(a) $\displaystyle\int_{0}^{\ln(1+\pi)} \mathbf{e}^{x}\cos\left(1-\mathbf{e}^{x}\right)dx$

(b) $\displaystyle\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{[\ln t]^4}{t}\,dt$

(c) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\,dP$

(d) $\displaystyle\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx$

(e) $\displaystyle\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw$

Solution

(a) $\displaystyle\int_0^{\ln(1+\pi)} \mathbf{e}^x \cos(1-\mathbf{e}^x)\,dx$

The limits are a little unusual in this case, but that will happen sometimes so don't get too excited about it. Here is the substitution.

$$\begin{aligned} &u = 1-\mathbf{e}^x && du = -\mathbf{e}^x dx \\ &x = 0 && \Rightarrow \quad u = 1-\mathbf{e}^0 = 1-1 = 0 \\ &x = \ln(1+\pi) && \Rightarrow \quad u = 1-\mathbf{e}^{\ln(1+\pi)} = 1-(1+\pi) = -\pi \end{aligned}$$

The integral is then,

$$\begin{aligned} \int_0^{\ln(1+\pi)} \mathbf{e}^x \cos(1-\mathbf{e}^x)\,dx &= -\int_0^{-\pi} \cos u\,du \\ &= -\sin(u)\Big|_0^{-\pi} \\ &= -(\sin(-\pi)-\sin 0) = 0 \end{aligned}$$

(b) $\displaystyle\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{[\ln t]^4}{t}\,dt$

Here is the substitution and converted limits for this problem.

$$\begin{aligned} &u = \ln t && du = \frac{1}{t}dt \\ &t = \mathbf{e}^2 && \Rightarrow \quad u = \ln \mathbf{e}^2 = 2 \\ &t = \mathbf{e}^6 && \Rightarrow \quad u = \ln \mathbf{e}^6 = 6 \end{aligned}$$

The integral is,

$$\begin{aligned} \int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{[\ln t]^4}{t}\,dt &= \int_2^6 u^4\,du \\ &= \frac{1}{5}u^5\Big|_2^6 \\ &= \frac{7744}{5} \end{aligned}$$

(c) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\,dP$

Here is the substitution and converted limits and don't get too excited about the substitution. It's a little messy in the case, but that can happen on occasion.

$$u = 2 + \sec(3P) \qquad du = 3\sec(3P)\tan(3P)\,dP \quad \Rightarrow \quad \sec(3P)\tan(3P)\,dP = \frac{1}{3}du$$
$$P = \frac{\pi}{12} \qquad \Rightarrow \qquad u = 2 + \sec\left(\frac{\pi}{4}\right) = 2 + \sqrt{2}$$
$$P = \frac{\pi}{9} \qquad \Rightarrow \qquad u = 2 + \sec\left(\frac{\pi}{3}\right) = 4$$

Here is the integral,

$$\begin{aligned}\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\,dP &= \frac{1}{3}\int_{2+\sqrt{2}}^{4} u^{-\frac{1}{3}}\,du \\ &= \frac{1}{2}u^{\frac{2}{3}}\Big|_{2+\sqrt{2}}^{4} \\ &= \frac{1}{2}\left(4^{\frac{2}{3}} - \left(2+\sqrt{2}\right)^{\frac{2}{3}}\right)\end{aligned}$$

So, not only was the substitution messy, but we also have a messy answer, but again that's life on occasion.

(d) $\displaystyle\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx$

This problem not as bad as it looks. Here is the substitution and converted limits.

$$u = \sin x \qquad du = \cos x\,dx$$
$$x = \frac{\pi}{2} \qquad \Rightarrow \qquad u = \sin\frac{\pi}{2} = 1$$
$$x = -\pi \qquad \Rightarrow \qquad u = \sin(-\pi) = 0$$

The cosine in the very front of the integrand will get substituted away in the differential and so this integrand actually simplifies down significantly. Here is the integral.

$$\begin{aligned}\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx &= \int_0^1 \cos u\,du \\ &= \sin(u)\Big|_0^1 \\ &= \sin(1) - \sin(0) \\ &= \sin(1)\end{aligned}$$

Don't get excited about these kinds of answers. On occasion we will end up with trig function evaluations like this.

(e) $\displaystyle\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw$

This is also a tricky substitution (at least until you see it). Here it is,

$$u = \frac{2}{w} \qquad du = -\frac{2}{w^2}dw \qquad \Rightarrow \qquad \frac{1}{w^2}dw = -\frac{1}{2}du$$

$$w = 2 \quad \Rightarrow \quad u = 1$$

$$w = \frac{1}{50} \quad \Rightarrow \quad u = 100$$

Here is the integral.

$$\begin{aligned}\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw &= -\frac{1}{2}\int_{100}^{1} \mathbf{e}^{u}\,du \\ &= -\frac{1}{2}\mathbf{e}^{u}\Big|_{100}^{1} \\ &= -\frac{1}{2}\left(\mathbf{e}^{1} - \mathbf{e}^{100}\right)\end{aligned}$$

In this last set of examples we saw some tricky substitutions and messy limits, but these are a fact of life with some substitution problems and so we need to be prepared for dealing with them when they happen.

6 Applications of Integrals

The previous chapter dealt exclusively with the computation of definite and indefinite integrals as well as some discussion of their properties and interpretations. It is now time to start looking at some applications of integrals. Note as well that we should probably say applications of **definite** integrals as that is really what we'll be looking at in this section.

In addition, we should note that there are a lot of different applications of (definite) integrals out there. We will look at the ones that can easily be done with the knowledge we have at our disposal at this point. Once we have covered the next chapter, Integration Techniques, we will be able to take a look at a few more applications of integrals. At this point we would not be able to compute many of the integrals that arise in those later applications.

In this chapter we'll take a look at using integrals to compute the average value of a function and the work required to move an object over a given distance. In addition we will take a look at a couple of geometric applications of integrals. In particular we will use integrals to compute the area that is between two curves and note that this application should not be too surprising given one of the major interpretations of the definite integral. We will also see how to compute the volume of some solids. We will compute the volume of solids of revolution, *i.e.* a solid obtained by rotating a curve about a given axis. In addition, we will compute the volume of some slightly more general solids in which the cross sections can be easily described with nice 2D geometric formulas (*i.e.* rectangles, triangles, circles, *etc.*).

6.1 Average Function Value

The first application of integrals that we'll take a look at is the average value of a function. The following fact tells us how to compute this.

Average Function Value

The average value of a continuous function $f(x)$ over the interval $[a, b]$ is given by,

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

To see a justification of this formula see the Proof of Various Integral Properties section of the Extras appendix.

Let's work a couple of quick examples.

Example 1

Determine the average value of each of the following functions on the given interval.

(a) $f(t) = t^2 - 5t + 6\cos(\pi t)$ on $\left[-1, \frac{5}{2}\right]$

(b) $R(z) = \sin(2z)\,\mathbf{e}^{1-\cos(2z)}$ on $[-\pi, \pi]$

Solution

(a) $f(t) = t^2 - 5t + 6\cos(\pi t)$ on $\left[-1, \frac{5}{2}\right]$

There's really not a whole lot to do in this problem other than just use the formula.

$$\begin{aligned} f_{avg} &= \frac{1}{\frac{5}{2} - (-1)}\int_{-1}^{\frac{5}{2}} t^2 - 5t + 6\cos(\pi t)\,dt \\ &= \frac{2}{7}\left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + \frac{6}{\pi}\sin(\pi t)\right)\Bigg|_{-1}^{\frac{5}{2}} \\ &= \frac{12}{7\pi} - \frac{13}{6} \\ &= -1.620993 \end{aligned}$$

You caught the substitution needed for the third term right?

So, the average value of this function of the given interval is -1.620993.

(b) $R(z) = \sin(2z)\,\mathbf{e}^{1-\cos(2z)}$ on $[-\pi, \pi]$

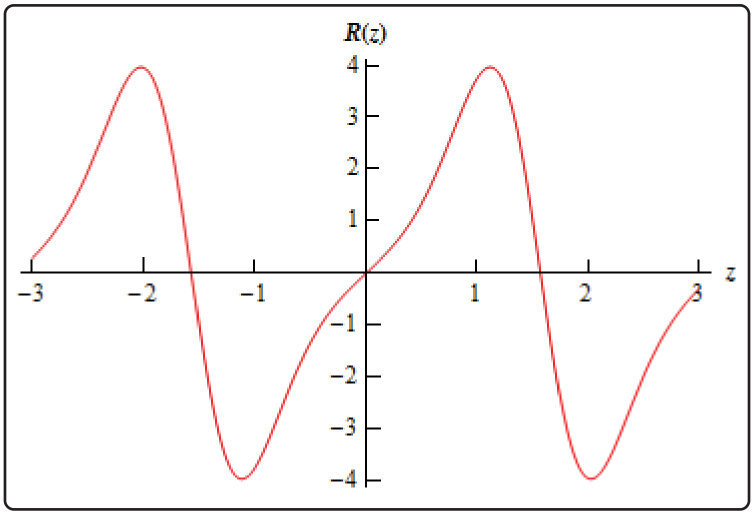
Again, not much to do here other than use the formula. Note that the integral will need the following substitution.

$$u = 1 - \cos(2z)$$

Here is the average value of this function,

$$\begin{aligned} R_{avg} &= \frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} \sin(2z)\,\mathbf{e}^{1-\cos(2z)}\,dz \\ &= \frac{1}{4\pi}\mathbf{e}^{1-\cos(2z)}\Big|_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

So, in this case the average function value is zero. Do not get excited about getting zero here. It will happen on occasion. In fact, if you look at the graph of the function on this interval it's not too hard to see that this is the correct answer.



There is also a theorem that is related to the average function value.

The Mean Value Theorem for Integrals

If $f(x)$ is a continuous function on $[a, b]$ then there is a number c in $[a, b]$ such that,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Note that this is very similar to the Mean Value Theorem that we saw in the Derivatives Applications chapter. See the Proof of Various Integral Properties section of the Extras appendix for the proof.

Note that one way to think of this theorem is the following. First rewrite the result as,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

and from this we can see that this theorem is telling us that there is a number $a < c < b$ such that $f_{avg} = f(c)$. Or, in other words, if $f(x)$ is a continuous function then somewhere in $[a, b]$ the function will take on its average value.

Let's take a quick look at an example using this theorem.

Example 2

Determine the number c that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^2 + 3x + 2$ on the interval $[1, 4]$.

Solution

First let's notice that the function is a polynomial and so is continuous on the given interval. This means that we can use the Mean Value Theorem. So, let's do that.

$$\begin{aligned}\int_1^4 x^2 + 3x + 2\,dx &= \left(c^2 + 3c + 2\right)(4-1)\\ \left.\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right)\right|_1^4 &= 3\left(c^2 + 3c + 2\right)\\ \frac{99}{2} &= 3c^2 + 9c + 6\\ 0 &= 3c^2 + 9c - \frac{87}{2}\end{aligned}$$

This is a quadratic equation that we can solve. Using the quadratic formula we get the following two solutions,

$$c = \frac{-3 + \sqrt{67}}{2} = 2.593$$
$$c = \frac{-3 - \sqrt{67}}{2} = -5.593$$

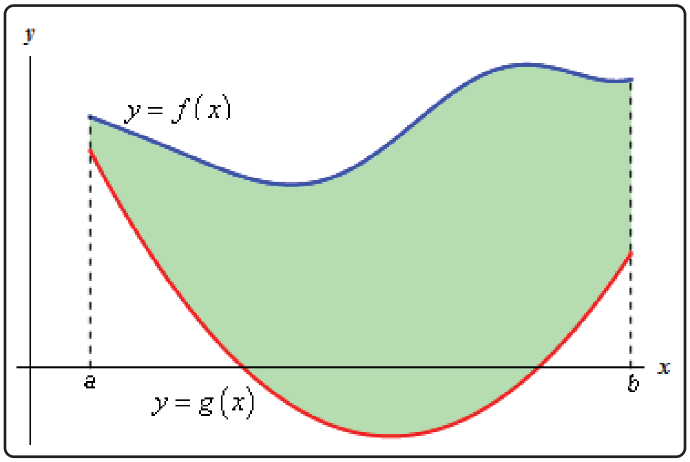
Clearly the second number is not in the interval and so that isn't the one that we're after. The first however is in the interval and so that's the number we want.

Note that it is possible for both numbers to be in the interval so don't expect only one to be in the interval.

6.2 Area Between Curves

In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

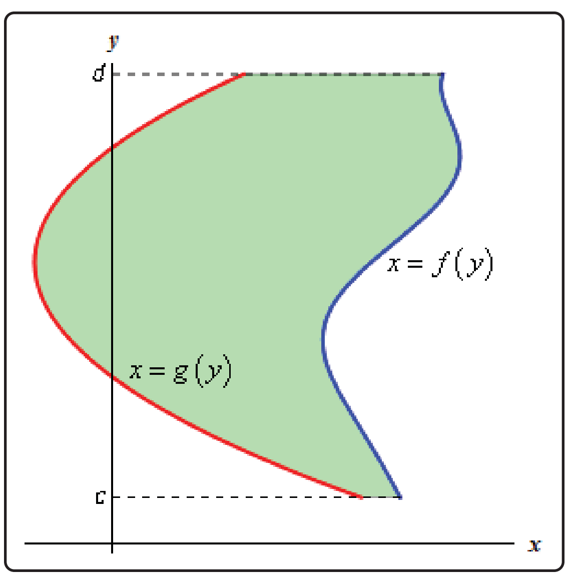
In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



In the Area and Volume Formulas section of the Extras appendix we derived the following formula for the area in this case.

$$A = \int_a^b f(x) - g(x)\, dx \tag{6.1}$$

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$.



In this case the formula is,

$$A = \int_{c}^{d} f(y) - g(y)\, dy \tag{6.2}$$

Now Equation 6.1 and Equation 6.2 are perfectly serviceable formulas, however, it is sometimes easy to forget that these always require the first function to be the larger of the two functions. So, instead of these formulas we will instead use the following "word" formulas to make sure that we remember that the area is always the "larger" function minus the "smaller" function.

In the first case we will use,

Area Between Curves, Case 1

$$A = \int_{a}^{b} \left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right) dx, \qquad a \le x \le b \tag{6.3}$$

In the second case we will use,

Area Between Curves, Case 2

$$A = \int_{c}^{d} \left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right) dy, \qquad c \le y \le d \tag{6.4}$$

Using these formulas will always force us to think about what is going on with each problem and to make sure that we've got the correct order of functions when we go to use the formula.

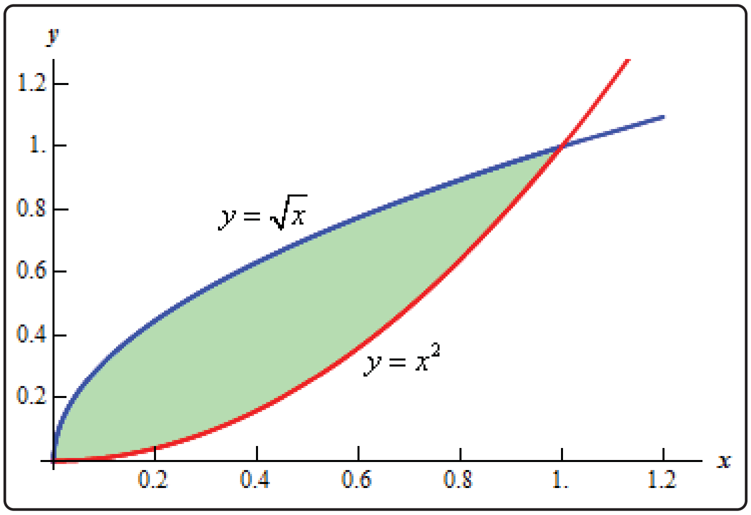
Let's work an example.

Example 1

Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

Solution

First of all, just what do we mean by "area enclosed by". This means that the region we're interested in must have one of the two curves on every boundary of the region. So, here is a graph of the two functions with the enclosed region shaded.



Note that we don't take any part of the region to the right of the rightmost intersection point of these two graphs. In this region there is no boundary on the right side and so this region is not part of the enclosed area. Remember that one of the given functions must be on the boundary of the enclosed region.

Also, from this graph it's clear that the upper function will be dependent on the range of x's that we use. Because of this you should always sketch of a graph of the region. Without a sketch it's often easy to mistake which of the two functions is the larger. In this case most would probably say that $y = x^2$ is the upper function and they would be right for the vast majority of the x's. However, in this case it is the lower of the two functions.

The limits of integration for this will be the intersection points of the two curves. In this case it's pretty easy to see that they will intersect at $x = 0$ and $x = 1$ so these are the limits of integration.

So, the integral that we'll need to compute to find the area is,

$$\begin{aligned} A &= \int_a^b \left(\begin{array}{c} \text{upper} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{lower} \\ \text{function} \end{array} \right) dx \\ &= \int_0^1 \sqrt{x} - x^2 \, dx \\ &= \left(\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right) \Big|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

Before moving on to the next example, there are a couple of important things to note.

First, in almost all of these problems a graph is pretty much required. Often the bounding region, which will give the limits of integration, is difficult to determine without a graph.

Also, it can often be difficult to determine which of the functions is the upper function and which is the lower function without a graph. This is especially true in cases like the last example where the answer to that question actually depended upon the range of x's that we were using.

Finally, unlike the area under a curve that we looked at in the previous chapter the area between two curves will always be positive. If we get a negative number or zero we can be sure that we've made a mistake somewhere and will need to go back and find it.

Note as well that sometimes instead of saying region enclosed by we will say region bounded by. They mean the same thing.

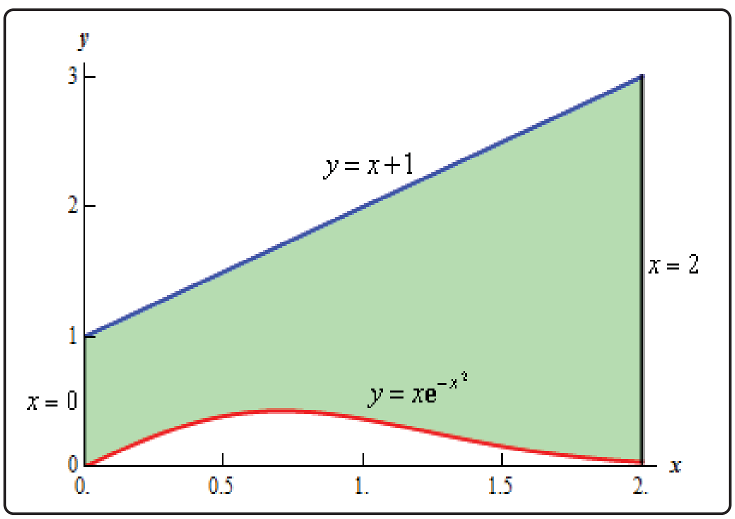
Let's work some more examples.

Example 2

Determine the area of the region bounded by $y = x\mathbf{e}^{-x^2}$, $y = x + 1$, $x = 2$, and the y-axis.

Solution

In this case the last two pieces of information, $x = 2$ and the y-axis, tell us the right and left boundaries of the region. Also, recall that the y-axis is given by the line $x = 0$. Here is the graph with the enclosed region shaded in.



Here, unlike the first example, the two curves don't meet. Instead we rely on two vertical lines to bound the left and right sides of the region as we noted above

Here is the integral that will give the area.

$$\begin{aligned} A &= \int_a^b \left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right) dx \\ &= \int_0^2 x + 1 - x\mathbf{e}^{-x^2}\,dx \\ &= \left(\frac{1}{2}x^2 + x + \frac{1}{2}\mathbf{e}^{-x^2}\right)\Bigg|_0^2 \\ &= \frac{7}{2} + \frac{\mathbf{e}^{-4}}{2} = 3.5092 \end{aligned}$$

Example 3

Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.

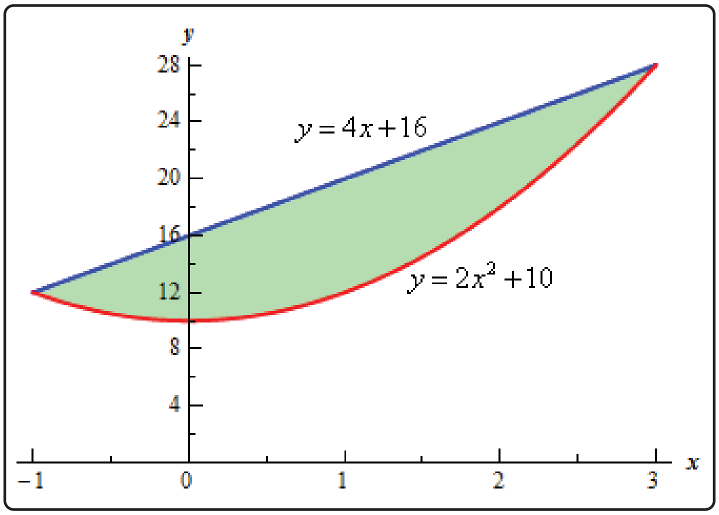
Solution

In this case the intersection points (which we'll need eventually) are not going to be easily identified from the graph so let's go ahead and get them now. Note that for most of these problems you'll not be able to accurately identify the intersection points from the graph and so you'll need to be able to determine them by hand. In this case we can get the intersection points by setting the two equations equal.

$$\begin{aligned} 2x^2 + 10 &= 4x + 16 \\ 2x^2 - 4x - 6 &= 0 \\ 2(x+1)(x-3) &= 0 \end{aligned}$$

So, it looks like the two curves will intersect at $x = -1$ and $x = 3$. If we need them we can get the y values corresponding to each of these by plugging the values back into either of the equations. We'll leave it to you to verify that the coordinates of the two intersection points on the graph are $(-1, 12)$ and $(3, 28)$.

Note as well that if you aren't good at graphing knowing the intersection points can help in at least getting the graph started. Here is a graph of the region.



With the graph we can now identify the upper and lower function and so we can now find the enclosed area.

$$\begin{aligned} A &= \int_a^b \left(\begin{array}{c} \text{upper} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{lower} \\ \text{function} \end{array} \right) dx \\ &= \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right) \, dx \\ &= \int_{-1}^{3} -2x^2 + 4x + 6 \, dx \\ &= \left(-\frac{2}{3}x^3 + 2x^2 + 6x \right) \Big|_{-1}^{3} \\ &= \frac{64}{3} \end{aligned}$$

Be careful with parenthesis in these problems. One of the more common mistakes students make with these problems is to neglect parenthesis on the second term.

Example 4

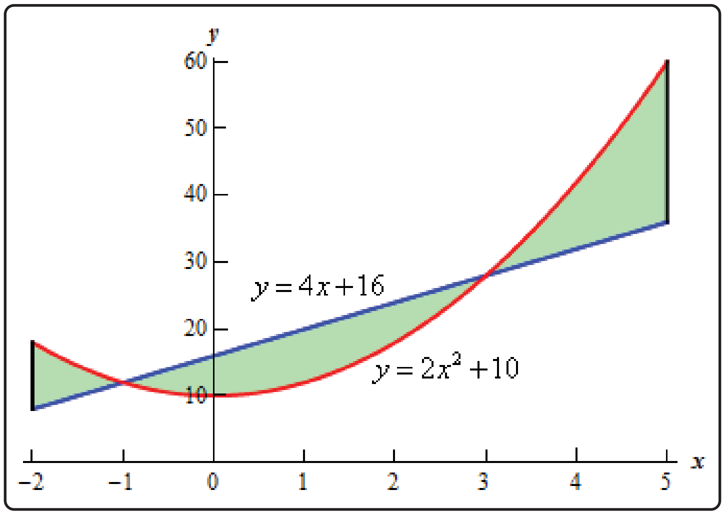
Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$.

Solution

So, the functions used in this problem are identical to the functions from the first problem. The difference is that we've extended the bounded region out from the intersection points.

Since these are the same functions we used in the previous example we won't bother finding the intersection points again.

Here is a graph of this region.



Okay, we have a small problem here. Our formula requires that one function always be the upper function and the other function always be the lower function and we clearly do not have that here. However, this actually isn't the problem that it might at first appear to be. There are three regions in which one function is always the upper function and the other is always the lower function. So, all that we need to do is find the area of each of the three regions, which we can do, and then add them all up.

Here is the area.

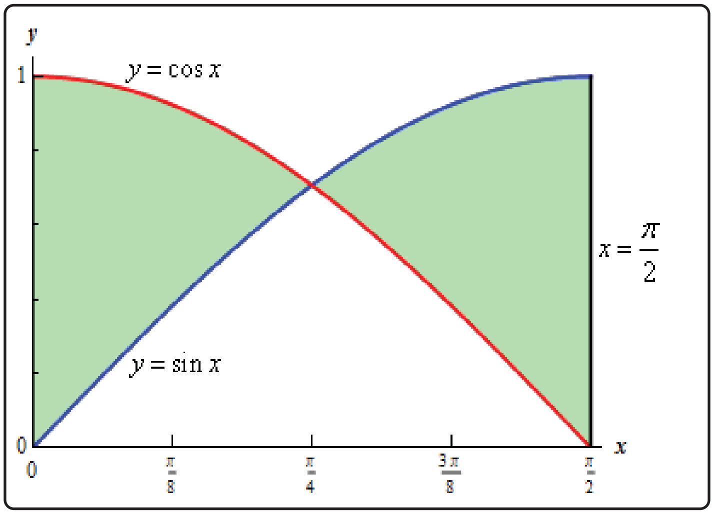
$$
\begin{aligned}
A &= \int_{-2}^{-1} 2x^2 + 10 - (4x + 16)\,dx + \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right)\,dx \\
&\qquad\qquad + \int_{3}^{5} 2x^2 + 10 - (4x + 16)\,dx \\
&= \int_{-2}^{-1} 2x^2 - 4x - 6\,dx + \int_{-1}^{3} -2x^2 + 4x + 6\,dx + \int_{3}^{5} 2x^2 - 4x - 6\,dx \\
&= \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{-2}^{-1} + \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Bigg|_{-1}^{3} + \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{3}^{5} \\
&= \frac{14}{3} + \frac{64}{3} + \frac{64}{3} \\
&= \frac{142}{3}
\end{aligned}
$$

Example 5

Determine the area of the region enclosed by $y = \sin(x)$, $y = \cos(x)$, $x = \frac{\pi}{2}$, and the y-axis.

Solution

First let's get a graph of the region.



So, we have another situation where we will need to do two integrals to get the area. The intersection point will be where

$$\sin(x) = \cos(x)$$

in the interval. We'll leave it to you to verify that this will be $x = \frac{\pi}{4}$. The area is then,

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x)\,dx + \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)\,dx \\ &= \left(\sin(x) + \cos(x)\right)\Big|_0^{\frac{\pi}{4}} + \left(-\cos(x) - \sin(x)\right)\Big|_{\pi/4}^{\pi/2} \\ &= \sqrt{2} - 1 + \left(\sqrt{2} - 1\right) \\ &= 2\sqrt{2} - 2 = 0.828427 \end{aligned}$$

We will need to be careful with this next example.

Example 6

Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$.

Solution

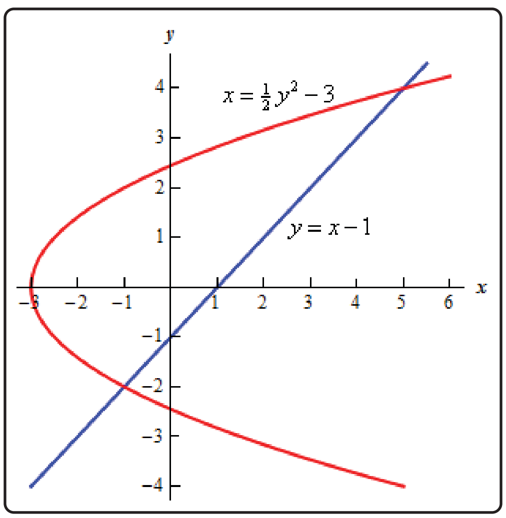
Don't let the first equation get you upset. We will have to deal with these kinds of equations occasionally so we'll need to get used to dealing with them.

As always, it will help if we have the intersection points for the two curves. In this case we'll get the intersection points by solving the second equation for x and then setting them equal. Here is that work,

$$\begin{aligned} y + 1 &= \frac{1}{2}y^2 - 3 \\ 2y + 2 &= y^2 - 6 \\ 0 &= y^2 - 2y - 8 \\ 0 &= (y-4)(y+2) \end{aligned}$$

So, it looks like the two curves will intersect at $y = -2$ and $y = 4$ or if we need the full coordinates they will be : $(-1, -2)$ and $(5, 4)$.

Here is a sketch of the two curves.



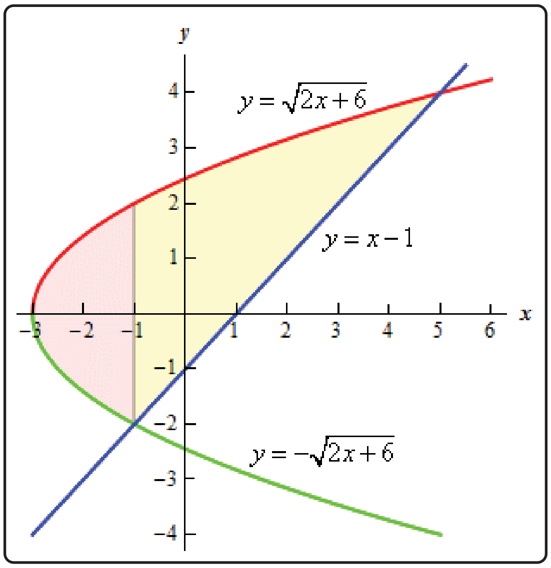
Now, we will have a serious problem at this point if we aren't careful. To this point we've been using an upper function and a lower function. To do that here notice that there are actually two portions of the region that will have different lower functions. In the range $[-3, -1]$ the parabola is actually both the upper and the lower function.

To use the formula that we've been using to this point we need to solve the parabola for y. This gives,

$$y = \pm\sqrt{2x+6}$$

where the "$+$" gives the upper portion of the parabola and the "-" gives the lower portion.

Here is a sketch of the complete area with each region shaded that we'd need if we were going to use the first formula.



The integrals for the area would then be,

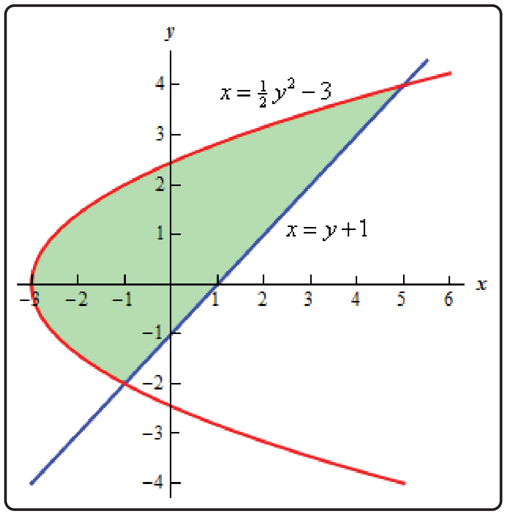
$$\begin{aligned} A &= \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right)\,dx + \int_{-1}^{5} \sqrt{2x+6} - (x-1)\,dx \\ &= \int_{-3}^{-1} 2\sqrt{2x+6}\,dx + \int_{-1}^{5} \sqrt{2x+6} - x + 1\,dx \\ &= \int_{-3}^{-1} 2\sqrt{2x+6}\,dx + \int_{-1}^{5} \sqrt{2x+6}\,dx + \int_{-1}^{5} -x+1\,dx \\ &= \frac{2}{3}u^{\frac{3}{2}}\bigg|_{0}^{4} + \frac{1}{3}u^{\frac{3}{2}}\bigg|_{4}^{16} + \left(-\frac{1}{2}x^2 + x\right)\bigg|_{-1}^{5} \\ &= 18 \end{aligned}$$

While these integrals aren't terribly difficult they are more difficult than they need to be.

Recall that there is another formula for determining the area. It is,

$$A = \int_{c}^{d} \left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right)\,dy, \qquad c \le y \le d$$

and in our case we do have one function that is always on the left and the other is always on the right. So, in this case this is definitely the way to go. Note that we will need to rewrite the equation of the line since it will need to be in the form $x = f(y)$ but that is easy enough to do. Here is the graph for using this formula.



The area is,

$$\begin{aligned} A &= \int_c^d \left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right) dy \\ &= \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy \\ &= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\, dy \\ &= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right)\Big|_{-2}^{4} \\ &= 18 \end{aligned}$$

This is the same that we got using the first formula and this was definitely easier than the first method.

So, in this last example we've seen a case where we could use either formula to find the area. However, the second was definitely easier.

Students often come into a calculus class with the idea that the only easy way to work with functions is to use them in the form $y = f(x)$. However, as we've seen in this previous example there are definitely times when it will be easier to work with functions in the form $x = f(y)$. In fact, there are going to be occasions when this will be the only way in which a problem can be worked so make sure that you can deal with functions in this form.

Let's take a look at one more example to make sure we can deal with functions in this form.

Example 7

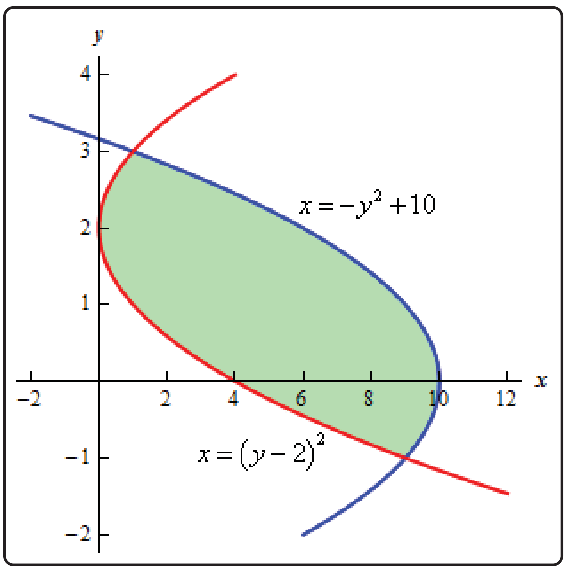
Determine the area of the region bounded by $x = -y^2 + 10$ and $x = (y-2)^2$.

Solution

First, we will need intersection points.

$$\begin{aligned} -y^2 + 10 &= (y-2)^2 \\ -y^2 + 10 &= y^2 - 4y + 4 \\ 0 &= 2y^2 - 4y - 6 \\ 0 &= 2(y+1)(y-3) \end{aligned}$$

The intersection points are $y = -1$ and $y = 3$. Here is a sketch of the region.



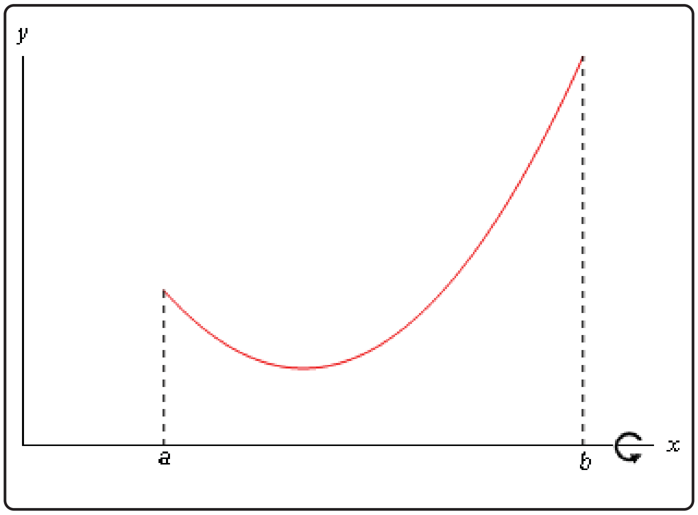
This is definitely a region where the second area formula will be easier. If we used the first formula there would be three different regions that we'd have to look at.

The area in this case is,

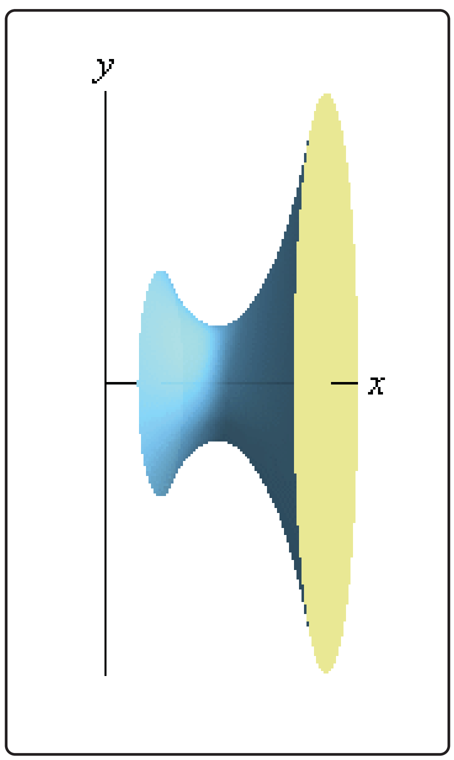
$$\begin{aligned}
A &= \int_{c}^{d} \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy \\
&= \int_{-1}^{3} -y^2 + 10 - (y-2)^2 \, dy \\
&= \int_{-1}^{3} -2y^2 + 4y + 6 \, dy \\
&= \left(-\frac{2}{3}y^3 + 2y^2 + 6y \right) \Bigg|_{-1}^{3} = \frac{64}{3}
\end{aligned}$$

6.3 Solids of Revolution / Method of Rings

In this section we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y = f(x)$, on an interval $[a, b]$.



We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this discussion let's rotate the curve about the x-axis, although it could be any vertical or horizontal axis. Doing this for the curve above gives the following three dimensional region.



What we want to do over the course of the next two sections is to determine the volume of this object.

In the Area and Volume Formulas section of the Extras appendix we derived the following formulas for the volume of this solid.

Volume Formulas

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

where, $A(x)$ and $A(y)$ are the cross-sectional area functions of the solid. There are many ways to get the cross-sectional area and we'll see two (or three depending on how you look at it) over the next two sections. Whether we will use $A(x)$ or $A(y)$ will depend upon the method and the axis of rotation used for each problem.

One of the easier methods for getting the cross-sectional area is to cut the object perpendicular to the axis of rotation. Doing this the cross section will be either a solid disk if the object is solid (as our above example is) or a ring if we've hollowed out a portion of the solid (we will see this eventually).

In the case that we get a solid disk the area is,

$$A = \pi(\text{radius})^2$$

where the radius will depend upon the function and the axis of rotation.

In the case that we get a ring the area is,

Area of Ring

$$A = \pi\left(\begin{pmatrix}\text{outer}\\\text{radius}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\\text{radius}\end{pmatrix}^2\right)$$

where again both of the radii will depend on the functions given and the axis of rotation. Note as well that in the case of a solid disk we can think of the inner radius as zero and we'll arrive at the correct formula for a solid disk and so this is a much more general formula to use.

Also, in both cases, whether the area is a function of x or a function of y will depend upon the axis of rotation as we will see.

This method is often called the **method of disks** or the **method of rings**.

Let's do an example.

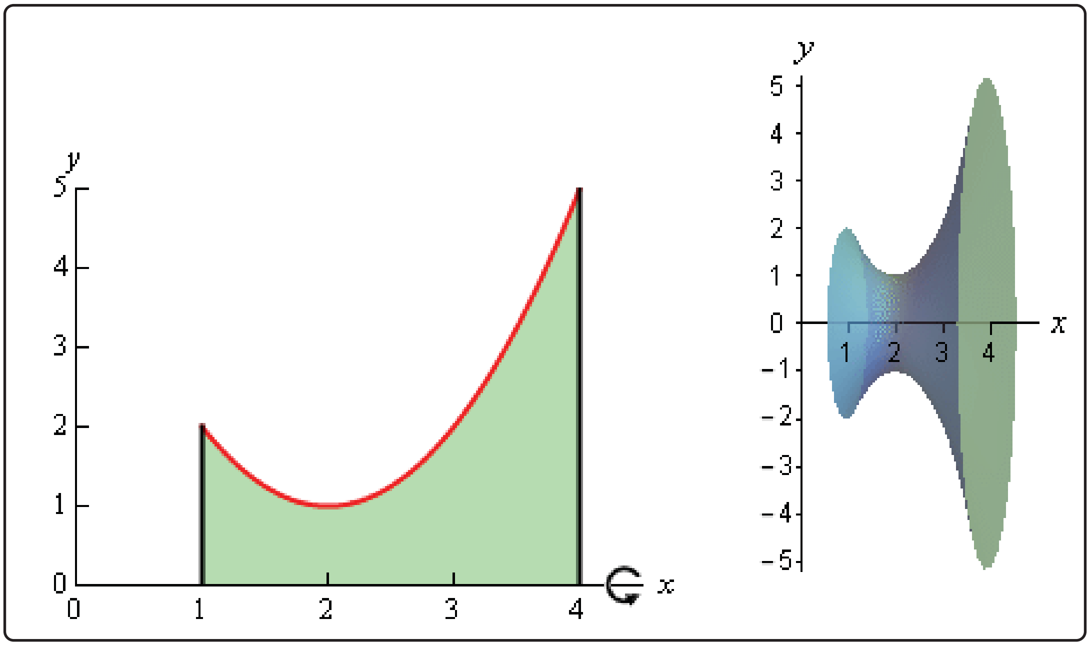
Example 1

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and the x-axis about the x-axis.

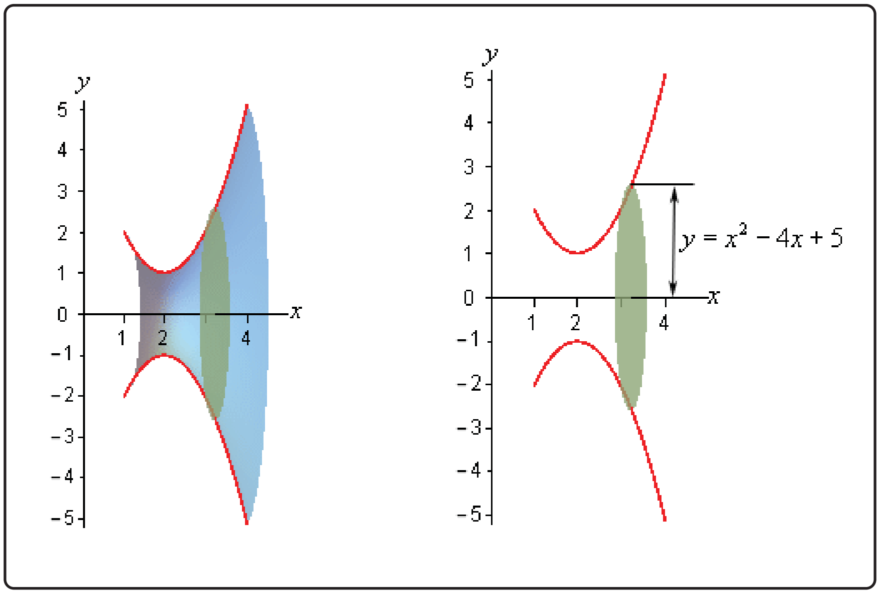
Solution

The first thing to do is get a sketch of the bounding region and the solid obtained by rotating the region about the x-axis. We don't need a picture perfect sketch of the curves we just need something that will allow us to get a feel for what the bounded region looks like so we can get a quick sketch of the solid. With that in mind we can note that the first equation is just a parabola with vertex $(2, 1)$ (you do remember how to get the vertex of a parabola right?) and opens upward and so we don't really need to put a lot of time into sketching it.

Here are both of these sketches.



Okay, to get a cross section we cut the solid at any x. Below are a couple of sketches showing a typical cross section. The sketch on the right shows a cut away of the object with a typical cross section without the caps. The sketch on the left shows just the curve we're rotating as well as its mirror image along the bottom of the solid.



In this case the radius is simply the distance from the x-axis to the curve and this is nothing more than the function value at that particular x as shown above. The cross-sectional area is then,

$$A(x) = \pi\left(x^2 - 4x + 5\right)^2 = \pi\left(x^4 - 8x^3 + 26x^2 - 40x + 25\right)$$

Next, we need to determine the limits of integration. Working from left to right the first cross section will occur at $x = 1$ and the last cross section will occur at $x = 4$. These are the limits of integration.

The volume of this solid is then,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \pi\int_1^4 x^4 - 8x^3 + 26x^2 - 40x + 25\,dx \\ &= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Bigg|_1^4 \\ &= \frac{78\pi}{5} \end{aligned}$$

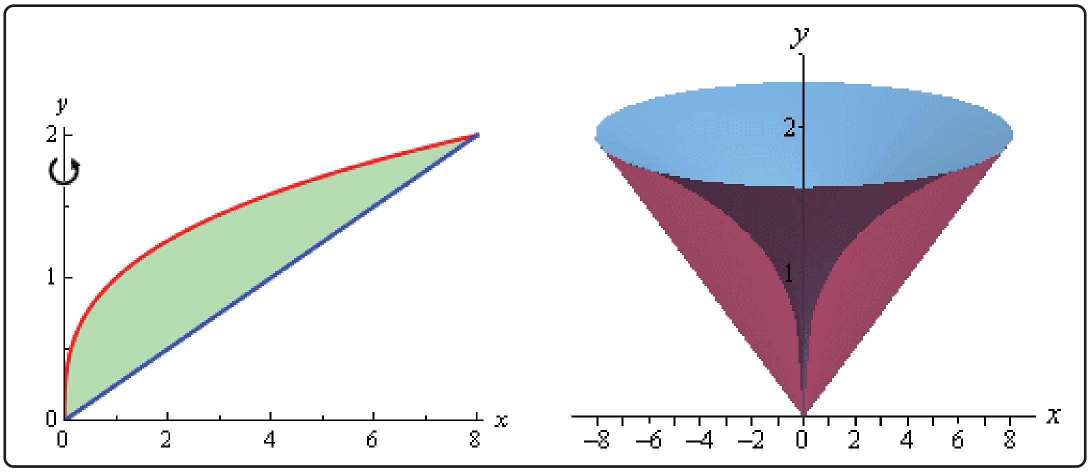
In the above example the object was a solid object, but the more interesting objects are those that are not solid so let's take a look at one of those.

Example 2

Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant about the y-axis.

Solution

First, let's get a graph of the bounding region and a graph of the object. Remember that we only want the portion of the bounding region that lies in the first quadrant. There is a portion of the bounding region that is in the third quadrant as well, but we don't want that for this problem.

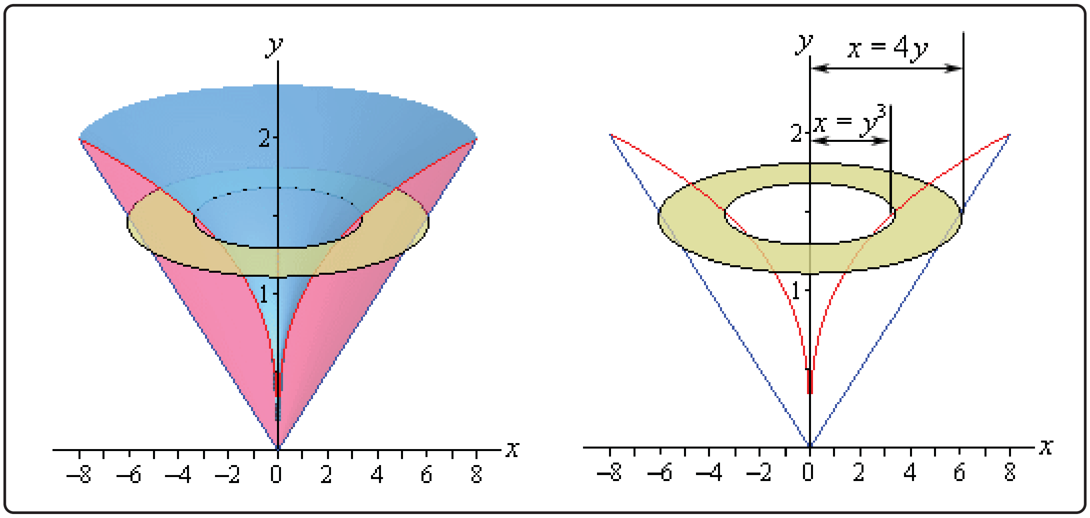


There are a couple of things to note with this problem. First, we are only looking for the volume of the "walls" of this solid, not the complete interior as we did in the last example.

Next, we will get our cross section by cutting the object perpendicular to the axis of rotation. The cross section will be a ring (remember we are only looking at the walls) for this example and it will be horizontal at some y. This means that the inner and outer radius for the ring will be x values and so we will need to rewrite our functions into the form $x = f(y)$. Here are the functions written in the correct form for this example.

$$y = \sqrt[3]{x} \qquad \Rightarrow \qquad x = y^3$$
$$y = \frac{x}{4} \qquad \Rightarrow \qquad x = 4y$$

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.



The inner radius in this case is the distance from the y-axis to the inner curve while the outer radius is the distance from the y-axis to the outer curve. Both of these are then x distances and so are given by the equations of the curves as shown above.

The cross-sectional area is then,

$$A\left(y\right) = \pi\left(\left(4y\right)^2 - \left(y^3\right)^2\right) = \pi\left(16y^2 - y^6\right)$$

Working from the bottom of the solid to the top we can see that the first cross-section will occur at $y = 0$ and the last cross-section will occur at $y = 2$. These will be the limits of integration. The volume is then,

$$\begin{aligned} V &= \int_c^d A\left(y\right)\,dy \\ &= \pi\int_0^2 16y^2 - y^6\,dy \\ &= \pi\left(\frac{16}{3}y^3 - \frac{1}{7}y^7\right)\Big|_0^2 \\ &= \frac{512\pi}{21} \end{aligned}$$

With these two examples out of the way we can now make a generalization about this method. If we rotate about a horizontal axis (the x-axis for example) then the cross-sectional area will be a function of x. Likewise, if we rotate about a vertical axis (the y-axis for example) then the cross-sectional area will be a function of y.

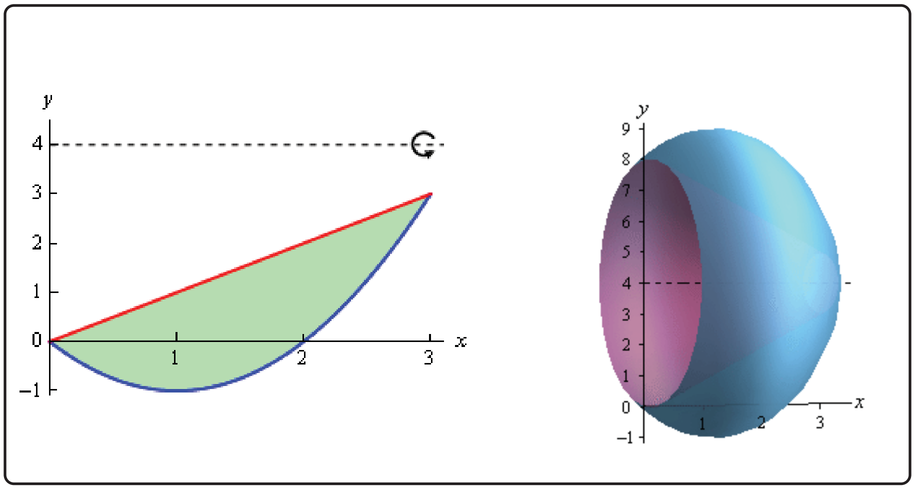
The remaining two examples in this section will make sure that we don't get too used to the idea of always rotating about the x or y-axis.

Example 3

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

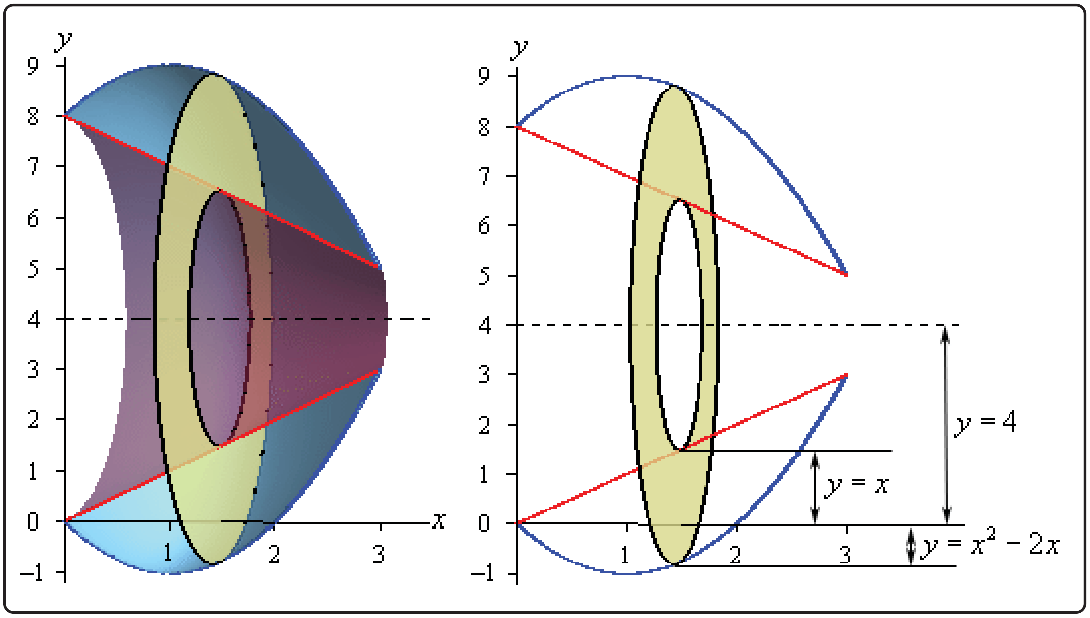
Solution

First let's get the bounding region and the solid graphed.



Again, we are going to be looking for the volume of the walls of this object. Also, since we are rotating about a horizontal axis we know that the cross-sectional area will be a function of x.

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.



Now, we're going to have to be careful here in determining the inner and outer radius as they aren't going to be quite as simple they were in the previous two examples.

Let's start with the inner radius as this one is a little clearer. First, the inner radius is NOT x. The distance from the x-axis to the inner edge of the ring is x, but we want the radius and that is the distance from the axis of rotation to the inner edge of the ring. So, we know that the distance from the axis of rotation to the x-axis is 4 and the distance from the x-axis to the inner ring is x. The inner radius must then be the difference between these two. Or,

$$\text{inner radius} = 4 - x$$

The outer radius works the same way. The outer radius is,

$$\text{outer radius} = 4 - \left(x^2 - 2x\right) = -x^2 + 2x + 4$$

Note that given the location of the typical ring in the sketch above the formula for the outer radius may not look quite right but it is in fact correct. As sketched the outer edge of the ring is below the x-axis and at this point the value of the function will be negative and so when we do the subtraction in the formula for the outer radius we'll actually be subtracting off a negative number which has the net effect of adding this distance onto 4 and that gives the correct outer radius. Likewise, if the outer edge is above the x-axis, the function value will be positive and so we'll be doing an honest subtraction here and again we'll get the correct radius in this case.

The cross-sectional area for this case is,

$$A\left(x\right) = \pi\left(\left(-x^2 + 2x + 4\right)^2 - \left(4 - x\right)^2\right) = \pi\left(x^4 - 4x^3 - 5x^2 + 24x\right)$$

The first ring will occur at $x = 0$ and the last ring will occur at $x = 3$ and so these are our limits of integration. The volume is then,

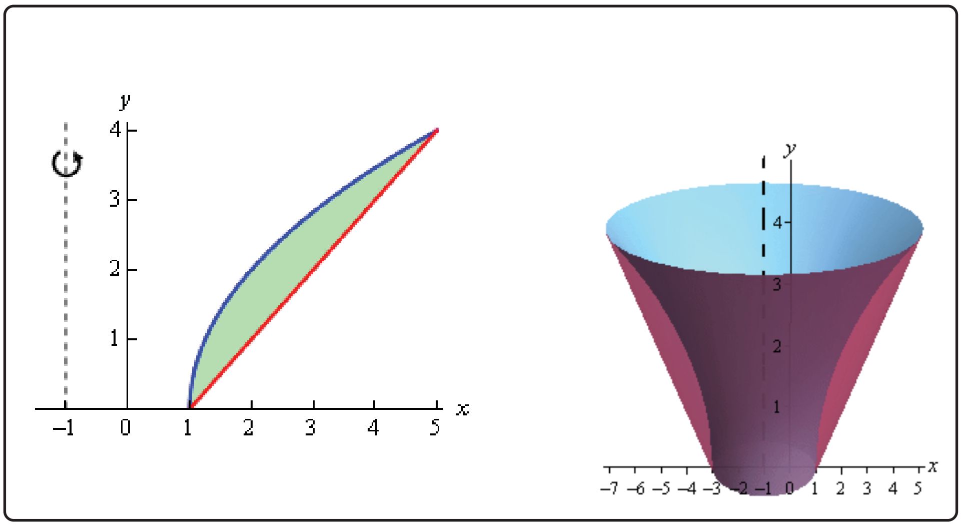
$$\begin{aligned} V &= \int_a^b A(x)\, dx \\ &= \pi \int_0^3 x^4 - 4x^3 - 5x^2 + 24x\, dx \\ &= \pi \left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2 \right)\Bigg|_0^3 \\ &= \frac{153\pi}{5} \end{aligned}$$

Example 4

Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = -1$.

Solution

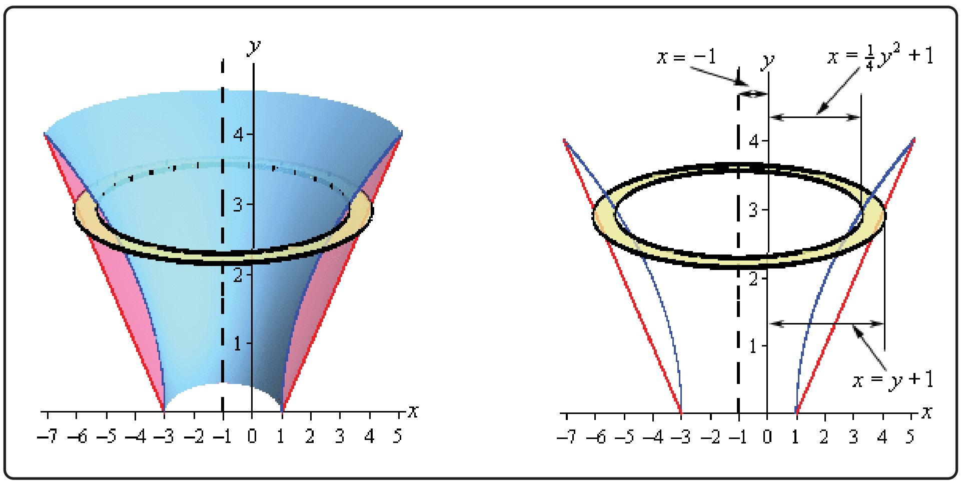
As with the previous examples, let's first graph the bounded region and the solid.



Now, let's notice that since we are rotating about a vertical axis and so the cross-sectional area will be a function of y. This also means that we are going to have to rewrite the functions to also get them in terms of y.

$$y = 2\sqrt{x-1} \qquad \Rightarrow \qquad x = \frac{y^2}{4} + 1$$
$$y = x - 1 \qquad \Rightarrow \qquad x = y + 1$$

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.



The inner and outer radius for this case is both similar and different from the previous example. This example is similar in the sense that the radii are not just the functions. In this example the functions are the distances from the y-axis to the edges of the rings. The center of the ring however is a distance of 1 from the y-axis. This means that the distance from the center to the edges is a distance from the axis of rotation to the y-axis (a distance of 1) and then from the y-axis to the edge of the rings.

So, the radii are then the functions plus 1 and that is what makes this example different from the previous example. Here we had to add the distance to the function value whereas in the previous example we needed to subtract the function from this distance. Note that without sketches the radii on these problems can be difficult to get.

So, in summary, we've got the following for the inner and outer radius for this example.

$$\text{outer radius} = y + 1 + 1 = y + 2$$
$$\text{inner radius} = \frac{y^2}{4} + 1 + 1 = \frac{y^2}{4} + 2$$

The cross-sectional area is then,

$$A(y) = \pi\left((y+2)^2 - \left(\frac{y^2}{4}+2\right)^2\right) = \pi\left(4y - \frac{y^4}{16}\right)$$

The first ring will occur at $y = 0$ and the final ring will occur at $y = 4$ and so these will be our limits of integration.

The volume is,

$$\begin{aligned} V &= \int_c^d A(y)\, dy \\ &= \pi\int_0^4 4y - \frac{y^4}{16}\, dy \\ &= \pi\left(2y^2 - \frac{1}{80}y^5\right)\Big|_0^4 \\ &= \frac{96\pi}{5} \end{aligned}$$

6.4 Solids of Revolution / Method of Cylinders

In the previous section we started looking at finding volumes of solids of revolution. In that section we took cross sections that were rings or disks, found the cross-sectional area and then used the following formulas to find the volume of the solid.

$$V = \int_a^b A(x)\, dx \qquad\qquad V = \int_c^d A(y)\, dy$$

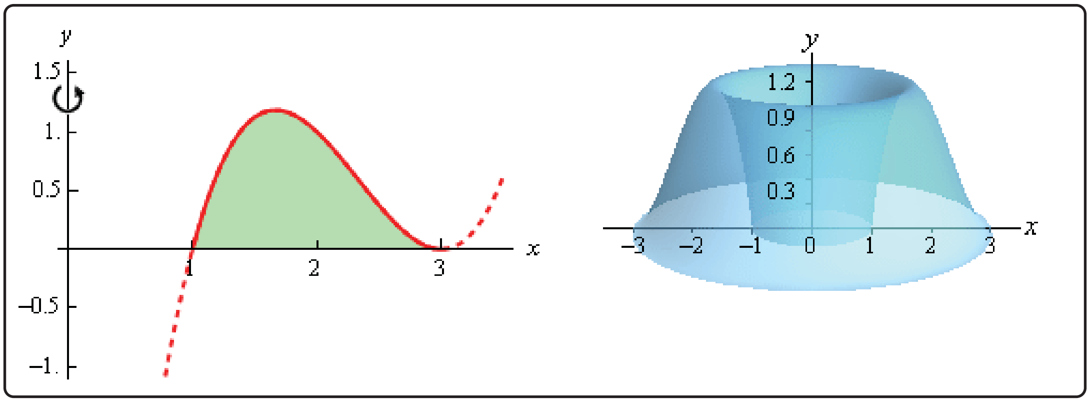
In the previous section we only used cross sections that were in the shape of a disk or a ring. This however does not always need to be the case. We can use any shape for the cross sections as long as it can be expanded or contracted to completely cover the solid we're looking at. This is a good thing because as our first example will show us we can't always use rings/disks.

Example 1

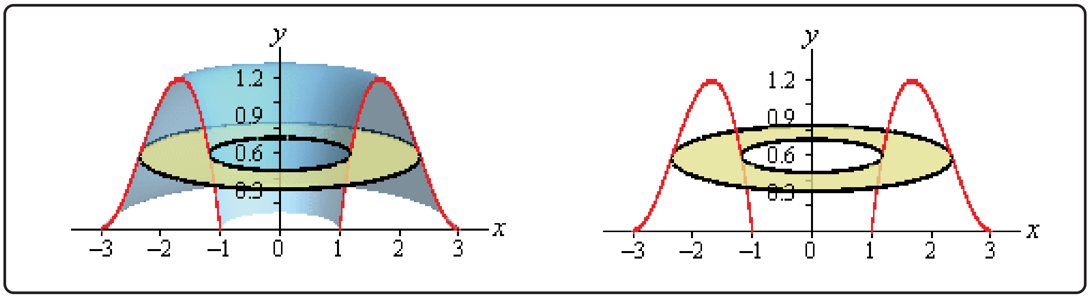
Determine the volume of the solid obtained by rotating the region bounded by $y = (x-1)(x-3)^2$ and the x-axis about the y-axis.

Solution

As we did in the previous section, let's first graph the bounded region and solid. Note that the bounded region here is the shaded portion shown. The curve is extended out a little past this for the purposes of illustrating what the curve looks like.



So, we've basically got something that's roughly doughnut shaped. If we were to use rings on this solid here is what a typical ring would look like.



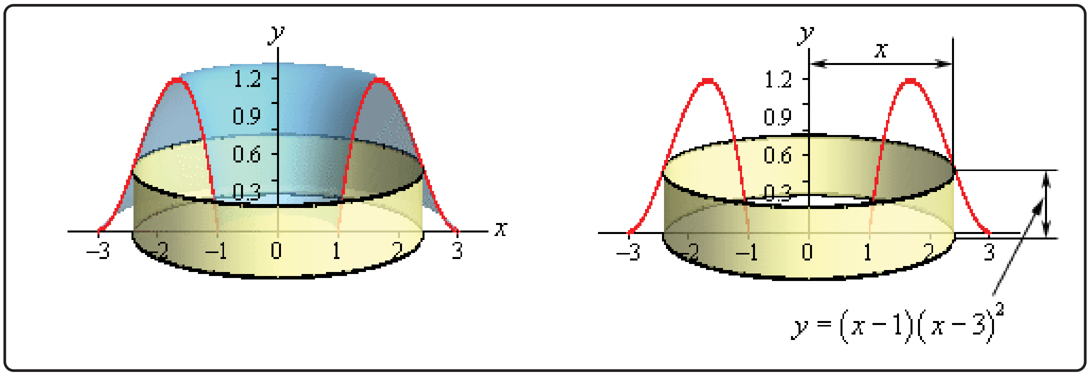
This leads to several problems. First, both the inner and outer radius are defined by the same function. This, in itself, can be dealt with on occasion as we saw in a example in the Area Between Curves section. However, this usually means more work than other methods so it's often not the best approach.

This leads to the second problem we got here. In order to use rings we would need to put this function into the form $x = f(y)$. That is NOT easy in general for a cubic polynomial and in other cases may not even be possible to do. Even when it is possible to do this the resulting equation is often significantly messier than the original which can also cause problems.

The last problem with rings in this case is not so much a problem as it's just added work. If we were to use rings the limit would be y limits and this means that we will need to know how high the graph goes. To this point the limits of integration have always been intersection points that were fairly easy to find. However, in this case the highest point is not an intersection point, but instead a relative maximum. We spent several sections in the Applications of Derivatives chapter talking about how to find maximum values of functions. However, finding them can, on occasion, take some work.

So, we've seen three problems with rings in this case that will either increase our work load or outright prevent us from using rings.

What we need to do is to find a different way to cut the solid that will give us a cross-sectional area that we can work with. One way to do this is to think of our solid as a lump of cookie dough and instead of cutting it perpendicular to the axis of rotation we could instead center a cylindrical cookie cutter on the axis of rotation and push this down into the solid. Doing this would give the following picture,



Doing this gives us a cylinder or shell in the object and we can easily find its surface area. The surface area of this cylinder is,

$$\begin{aligned} A(x) &= 2\pi\,(\text{radius})\,(\text{height}) \\ &= 2\pi\,(x)\left((x-1)(x-3)^2\right) \\ &= 2\pi\left(x^4 - 7x^3 + 15x^2 - 9x\right) \end{aligned}$$

Notice as well that as we increase the radius of the cylinder we will completely cover the solid and so we can use this in our formula to find the volume of this solid. All we need are limits of integration. The first cylinder will cut into the solid at $x = 1$ and as we increase x to $x = 3$ we will completely cover both sides of the solid since expanding the cylinder in one direction will automatically expand it in the other direction as well.

The volume of this solid is then,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= 2\pi\int_1^3 x^4 - 7x^3 + 15x^2 - 9x\,dx \\ &= 2\pi\left(\frac{1}{5}x^5 - \frac{7}{4}x^4 + 5x^3 - \frac{9}{2}x^2\right)\Big|_1^3 \\ &= \frac{24\pi}{5} \end{aligned}$$

The method used in the last example is called the **method of cylinders** or **method of shells**. The formula for the area in all cases will be,

Area of Cylinder

$$A = 2\pi\,(\text{radius})\,(\text{height})$$

There are a couple of important differences between this method and the method of rings/disks that we should note before moving on. First, rotation about a vertical axis will give an area that is a function of x and rotation about a horizontal axis will give an area that is a function of y. This is exactly opposite of the method of rings/disks.

Second, we don't take the complete range of x or y for the limits of integration as we did in the previous section. Instead we take a range of x or y that will cover one side of the solid. As we noted in the first example if we expand out the radius to cover one side we will automatically expand in the other direction as well to cover the other side.

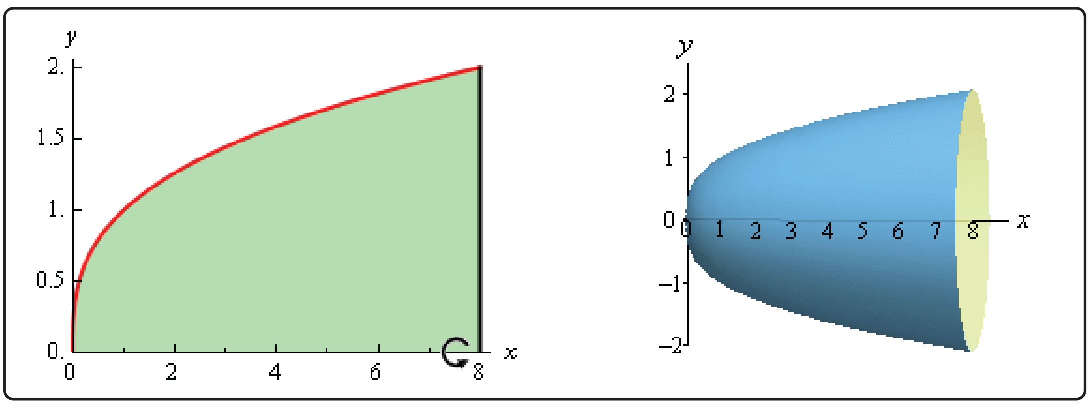
Let's take a look at another example.

Example 2

Determine the volume of the solid obtained by rotating the region bounded by $y = \sqrt[3]{x}$, $x = 8$ and the x-axis about the x-axis.

Solution

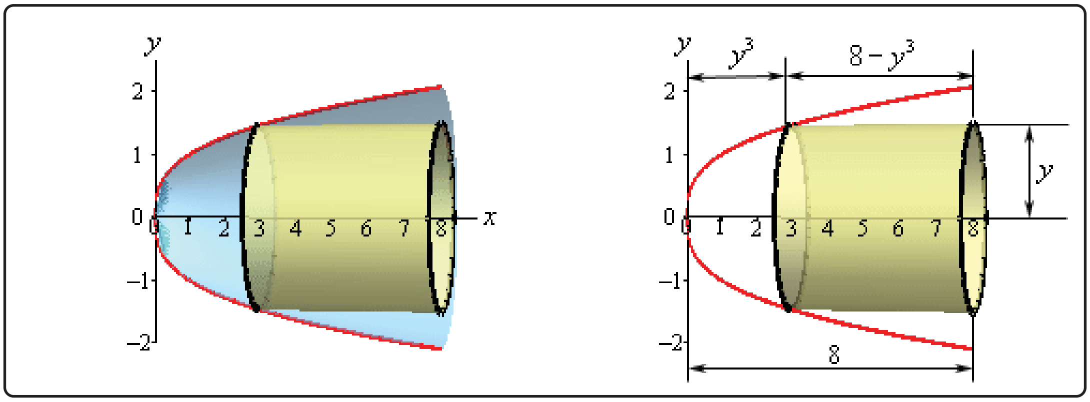
First let's get a graph of the bounded region and the solid.



Okay, we are rotating about a horizontal axis. This means that the area will be a function of y and so our equation will also need to be written in $x = f(y)$ form.

$$y = \sqrt[3]{x} \qquad \Rightarrow \qquad x = y^3$$

As we did in the ring/disk section let's take a couple of looks at a typical cylinder. The sketch on the left shows a typical cylinder with the back half of the object also in the sketch to give the right sketch some context. The sketch on the right contains a typical cylinder and only the curves that define the edge of the solid.



In this case the width of the cylinder is not the function value as it was in the previous example. In this case the function value is the distance between the edge of the cylinder and the y-axis. The distance from the edge out to the line is $x = 8$ and so the width is then $8 - y^3$. The cross-sectional area in this case is,

$$\begin{aligned} A(y) &= 2\pi\,(\text{radius})\,(\text{width}) \\ &= 2\pi\,(y)\left(8 - y^3\right) \\ &= 2\pi\left(8y - y^4\right) \end{aligned}$$

The first cylinder will cut into the solid at $y = 0$ and the final cylinder will cut in at $y = 2$ and so these are our limits of integration.

The volume of this solid is,

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= 2\pi\int_0^2 8y - y^4\,dy \\ &= 2\pi\left(4y^2 - \frac{1}{5}y^5\right)\Big|_0^2 \\ &= \frac{96\pi}{5} \end{aligned}$$

The remaining examples in this section will have axis of rotation about axis other than the x and

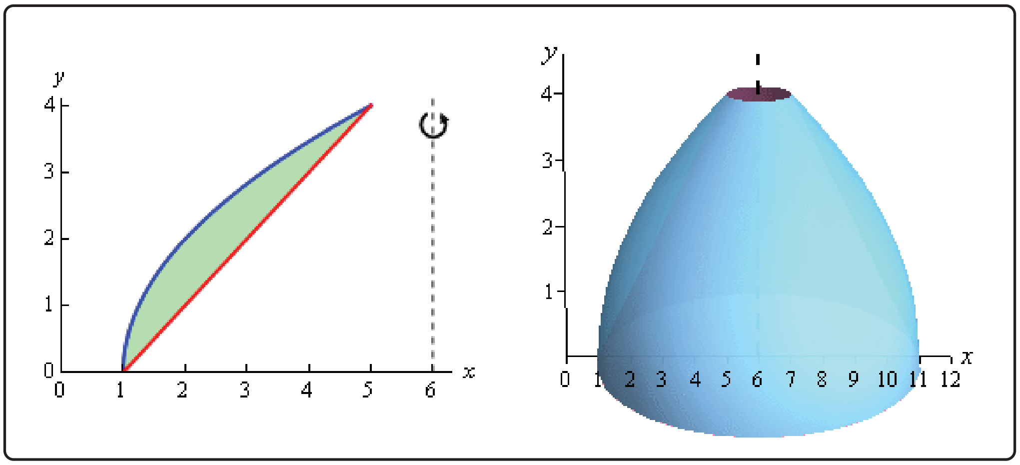
y-axis. As with the method of rings/disks we will need to be a little careful with these.

Example 3

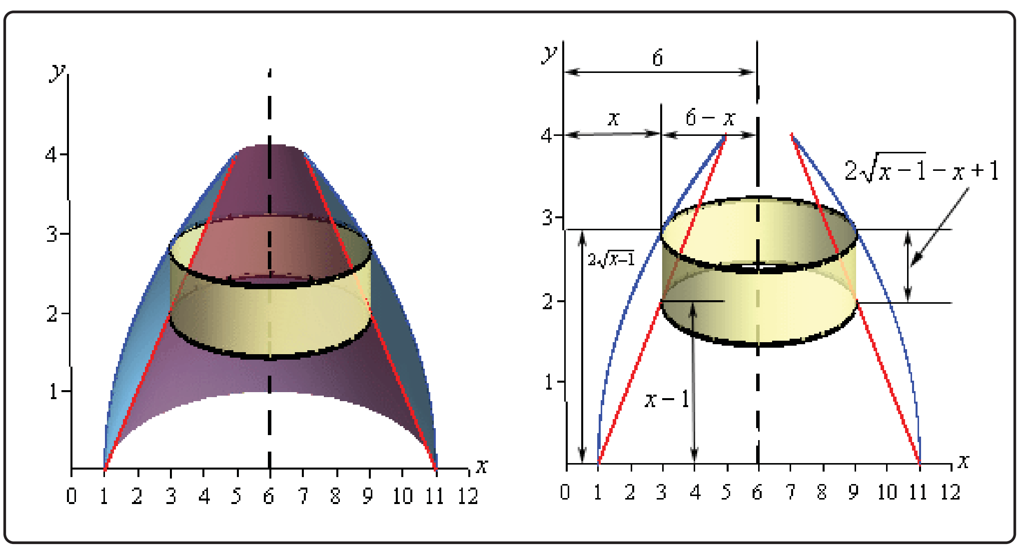
Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = 6$.

Solution

Here's a graph of the bounded region and solid.



Here are our sketches of a typical cylinder. Again, the sketch on the left is here to provide some context for the sketch on the right.



Okay, there is a lot going on in the sketch to the left. First notice that the radius is not just an x or y as it was in the previous two cases. In this case x is the distance from the *y*-axis to the edge of the cylinder and we need the distance from the axis of rotation to the edge of the cylinder. That means that the radius of this cylinder is $6 - x$.

Secondly, the height of the cylinder is the difference of the two functions in this case.

The cross-sectional area is then,

$$\begin{aligned} A(x) &= 2\pi\,(\text{radius})\,(\text{height}) \\ &= 2\pi\,(6-x)\left(2\sqrt{x-1} - x + 1\right) \\ &= 2\pi\left(x^2 - 7x + 6 + 12\sqrt{x-1} - 2x\sqrt{x-1}\right) \end{aligned}$$

Now the first cylinder will cut into the solid at $x = 1$ and the final cylinder will cut into the solid at $x = 5$ so there are our limits.

Here is the volume.

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= 2\pi\int_1^5 x^2 - 7x + 6 + 12\sqrt{x-1} - 2x\sqrt{x-1}\,dx \\ &= 2\pi\left(\frac{1}{3}x^3 - \frac{7}{2}x^2 + 6x + 8(x-1)^{\frac{3}{2}} - \frac{4}{3}(x-1)^{\frac{3}{2}} - \frac{4}{5}(x-1)^{\frac{5}{2}}\right)\Bigg|_1^5 \\ &= 2\pi\left(\frac{136}{15}\right) \\ &= \frac{272\pi}{15} \end{aligned}$$

The integration of the last term is a little tricky so let's do that here. It will use the substitution,

$$u = x - 1 \qquad du = dx \qquad x = u + 1$$

$$\begin{aligned} \int 2x\sqrt{x-1}\,dx &= 2\int (u+1)\,u^{\frac{1}{2}}\,du \\ &= 2\int u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du \\ &= 2\left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + c \\ &= \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + c \end{aligned}$$

We saw one of these kinds of substitutions back in the substitution section.

Example 4

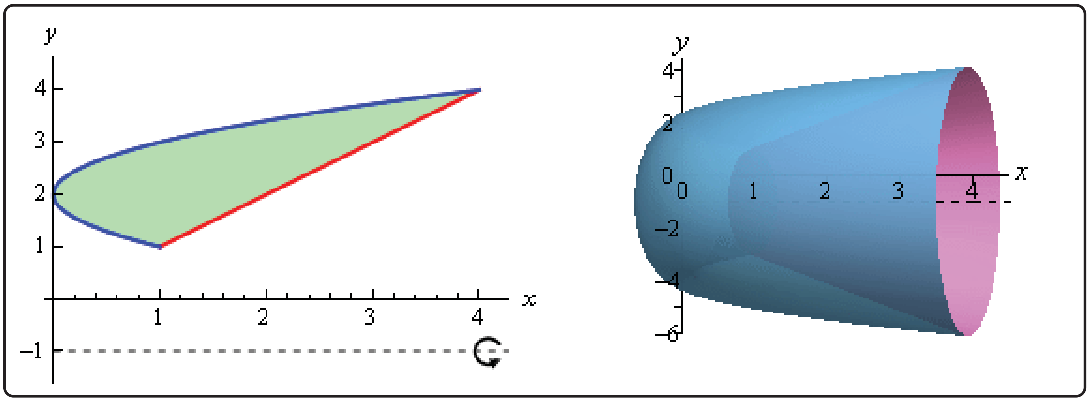
Determine the volume of the solid obtained by rotating the region bounded by $x = (y-2)^2$ and $y = x$ about the line $y = -1$.

Solution

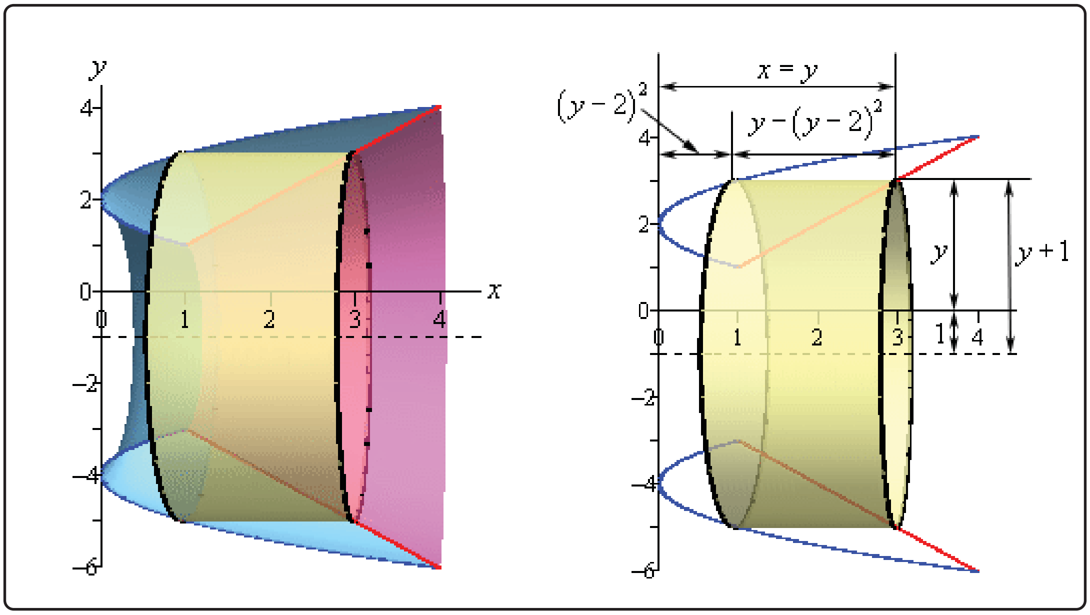
We should first get the intersection points there.

$$\begin{aligned} y &= (y-2)^2 \\ y &= y^2 - 4y + 4 \\ 0 &= y^2 - 5y + 4 \\ 0 &= (y-4)(y-1) \end{aligned}$$

So, the two curves will intersect at $y = 1$ and $y = 4$. Here is a sketch of the bounded region and the solid.



Here are our sketches of a typical cylinder. The sketch on the left is here to provide some context for the sketch on the right.



Here's the cross-sectional area for this cylinder.

$$\begin{aligned} A(y) &= 2\pi\,(\text{radius})\,(\text{width}) \\ &= 2\pi\,(y+1)\left(y-(y-2)^2\right) \\ &= 2\pi\left(-y^3+4y^2+y-4\right) \end{aligned}$$

The first cylinder will cut into the solid at $y = 1$ and the final cylinder will cut in at $y = 4$. The volume is then,

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= 2\pi\int_1^4 -y^3+4y^2+y-4\,dy \\ &= 2\pi\left(-\frac{1}{4}y^4+\frac{4}{3}y^3+\frac{1}{2}y^2-4y\right)\bigg|_1^4 \\ &= \frac{63\pi}{2} \end{aligned}$$

6.5 More Volume Problems

In this section we're going to take a look at some more volume problems. However, the problems we'll be looking at here will not be solids of revolution as we looked at in the previous two sections. There are many solids out there that cannot be generated as solids of revolution, or at least not easily and so we need to take a look at how to do some of these problems.

Now, having said that these will not be solids of revolutions they will still be worked in pretty much the same manner. For each solid we'll need to determine the cross-sectional area, either $A\left(x\right)$ or $A\left(y\right)$, and they use the formulas we used in the previous two sections,

Volume Formulas

$$V = \int_{a}^{b}{{A\left(x \right)\,dx}} \qquad\qquad V = \int_{c}^{d}{{A\left(y \right)\,dy}}$$

The "hard" part of these problems will be determining what the cross-sectional area for each solid is. Each problem will be different and so each cross-sectional area will be determined by different means.

Also, before we proceed with any examples we need to acknowledge that the integrals in this section might look a little tricky at first. There are going to be very few numbers in these problems. All of the examples in this section are going to be more general derivation of volume formulas for certain solids. As such we'll be working with things like circles of radius r and we'll not be giving a specific value of r and we'll have heights of h instead of specific heights, *etc*.

All the letters in the integrals are going to make the integrals look a little tricky, but all you have to remember is that the r's and the h's are just letters being used to represent a fixed quantity for the problem, *i.e.* it is a constant. So, when we integrate we only need to worry about the letter in the differential as that is the variable we're actually integrating with respect to. All other letters in the integral should be thought of as constants. If you have trouble doing that, just think about what you'd do if the r was a 2 or the h was a 3 for example.

Let's start with a simple example that we don't actually need to do an integral that will illustrate how these problems work in general and will get us used to seeing multiple letters in integrals.

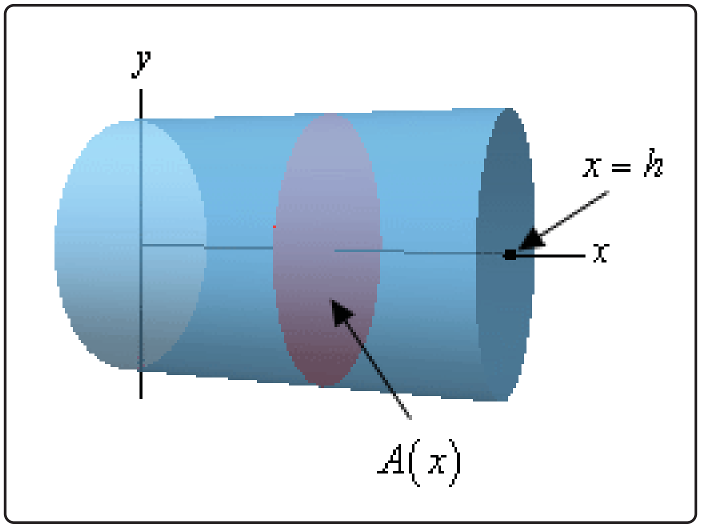
Example 1

Find the volume of a cylinder of radius r and height h.

Solution

Now, as we mentioned before starting this example we really don't need to use an integral to find this volume, but it is a good example to illustrate the method we'll need to use for these types of problems.

We'll start off with the sketch of the cylinder below.



We'll center the cylinder on the x-axis and the cylinder will start at $x = 0$ and end at $x = h$ as shown. Note that we're only choosing this particular set up to get an integral in terms of x and to make the limits nice to deal with. There are many other orientations that we could use.

What we need here is to get a formula for the cross-sectional area at any x. In this case the cross-sectional area is constant and will be a disk of radius r. Therefore, for any x we'll have the following cross-sectional area,

$$A(x) = \pi r^2$$

Next the limits for the integral will be $0 \le x \le h$ since that is the range of x in which the cylinder lives. Here is the integral for the volume,

$$V = \int_0^h \pi r^2 \, dx = \pi r^2 \int_0^h dx = \pi r^2 x \Big|_0^h = \pi r^2 h$$

So, we get the expected formula.

Also, recall we are using r to represent the radius of the cylinder. While r can clearly take different values it will never change once we start the problem. Cylinders do not change their radius in the middle of a problem and so as we move along the center of the cylinder (*i.e.* the x-axis) r is a fixed number and won't change. In other words, it is a constant that will not change as we change the x. Therefore, because we integrated with respect to x the r will be a constant as far as the integral is concerned. The r can then be pulled out of the integral as shown (although that's not required, we just did it to make the point). At this point we're just integrating dx and we know how to do that.

When we evaluate the integral remember that the limits are x values and so we plug into the x and NOT the r. Again, remember that r is just a letter that is being used to represent the radius of the cylinder and, once we start the integration, is assumed to be a fixed constant.

As noted before we started this example if you're having trouble with the r just think of what you'd do if there was a 2 there instead of an r. In this problem, because we're integrating with respect to x, both the 2 and the r will behave in the same manner. Note however that you should NEVER actually replace the r with a 2 as that WILL lead to a wrong answer. You should just think of what you would do IF the r was a 2.

So, to work these problems we'll first need to get a sketch of the solid with a set of x and y axes to help us see what's going on. At the very least we'll need the sketch to get the limits of the integral, but we will often need it to see just what the cross-sectional area is. Once we have the sketch we'll need to determine a formula for the cross-sectional area and then do the integral.

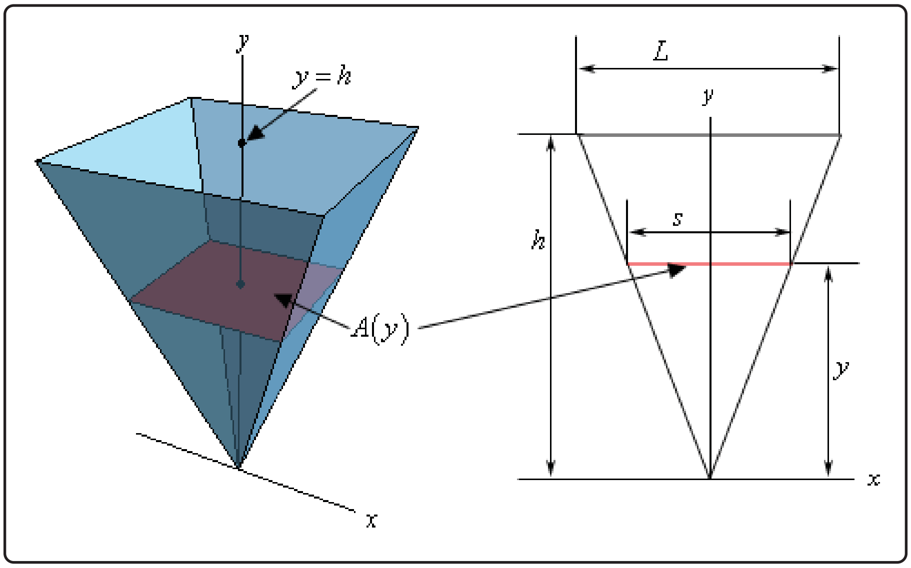
Let's work a couple of more complicated examples. In these examples the main issue is going to be determining what the cross-sectional areas are.

Example 2

Find the volume of a pyramid whose base is a square with sides of length L and whose height is h.

Solution

Let's start off with a sketch of the pyramid. In this case we'll center the pyramid on the y-axis and to make the equations easier we are going to position the point of the pyramid at the origin.



Now, as shown here the cross-sectional area will be a function of y and it will also be a square with sides of length s. The area of the square is easy, but we'll need to get the length of the side in terms of y. To determine this, consider the figure on the right above. If we look at the pyramid directly from the front we'll see that we have two similar triangles and we know that the ratio of any two sides of similar triangles must be equal. In other words, we know that,

$$\frac{s}{L} = \frac{y}{h} \qquad \Rightarrow \qquad s = \frac{y}{h}L = \frac{L}{h}y$$

So, the cross-sectional area is then,

$$A\left(y\right) = s^2 = \frac{L^2}{h^2}y^2$$

The limit for the integral will be $0 \le y \le h$ and the volume will be,

$$V = \int_0^h \frac{L^2}{h^2}y^2\,dy = \frac{L^2}{h^2}\int_0^h y^2\,dy = \frac{L^2}{h^2}\left(\frac{1}{3}y^3\right)\Big|_0^h = \frac{1}{3}L^2 h$$

Again, do not get excited about the L and the h in the integral. Once we start the problem if we change y they will not change and so they are constants as far as the integral is concerned and can get pulled out of the integral. Also, remember that when we evaluate will only plug the limits into the variable we integrated with respect to, y in this case.

Before we proceed with some more complicated examples we should once again remind you to not get excited about the other letters in the integrals. If we're integrating with respect to x or y then all other letters in the formula that represent fixed quantities (*i.e.* radius, height, length of a side, *etc.*) are just constants and can be treated as such when doing the integral.

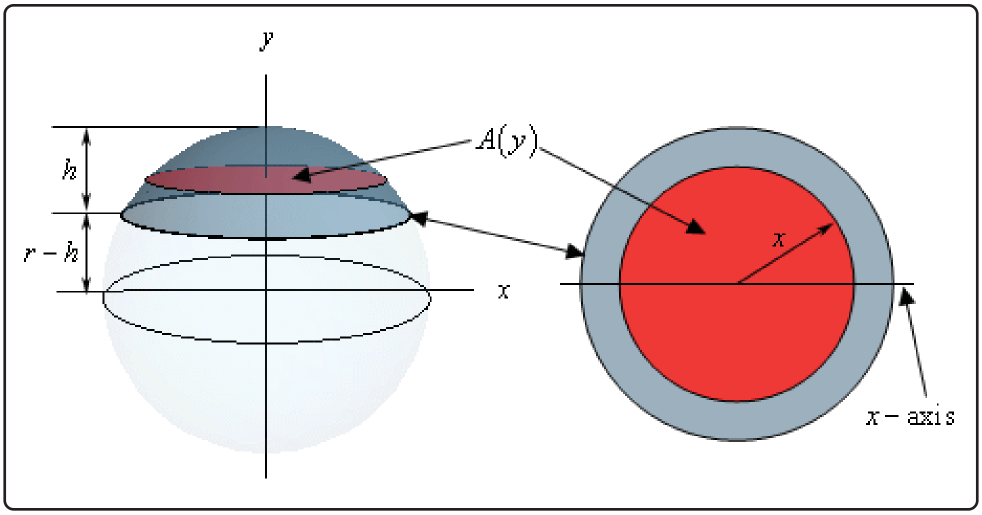
Now let's do some more examples.

Example 3

For a sphere of radius r find the volume of the cap of height h.

Solution

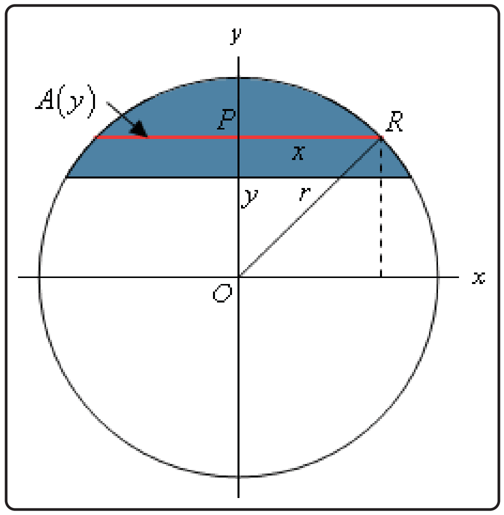
A sketch is probably best to illustrate what we're after here.



We are after the top portion of the sphere and the height of this is portion is h. In this case we'll use a cross-sectional area that starts at the bottom of the cap, which is at $y = r - h$, and moves up towards the top, which is at $y = r$. So, each cross-section will be a disk of radius x. It is a little easier to see that the radius will be x if we look at it from the top as shown in the sketch to the right above. The area of this disk is then,

$$\pi x^2$$

This is a problem however as we need the cross-sectional area in terms of y. So, what we really need to determine what x^2 will be for any given y at the cross-section. To get this let's look at the sphere from the front.



In particular look at the triangle POR. Because the point R lies on the sphere itself we can see that the length of the hypotenuse of this triangle (the line OR) is r, the radius of the sphere. The line PR has a length of x and the line OP has length y so by the Pythagorean Theorem we know,

$$x^2 + y^2 = r^2 \qquad \Rightarrow \qquad x^2 = r^2 - y^2$$

So, we now know what x^2 will be for any given y and so the cross-sectional area is,

$$A(y) = \pi\left(r^2 - y^2\right)$$

As we noted earlier the limits on y will be $r - h \le y \le r$ and so the volume is,

$$\begin{aligned} V &= \int_{r-h}^{r} \pi\left(r^2 - y^2\right)\,dy \\ &= \pi\left(r^2 y - \frac{1}{3}y^3\right)\Bigg|_{r-h}^{r} \\ &= \pi\left(h^2 r - \frac{1}{3}h^3\right) = \pi h^2\left(r - \frac{1}{3}h\right) \end{aligned}$$

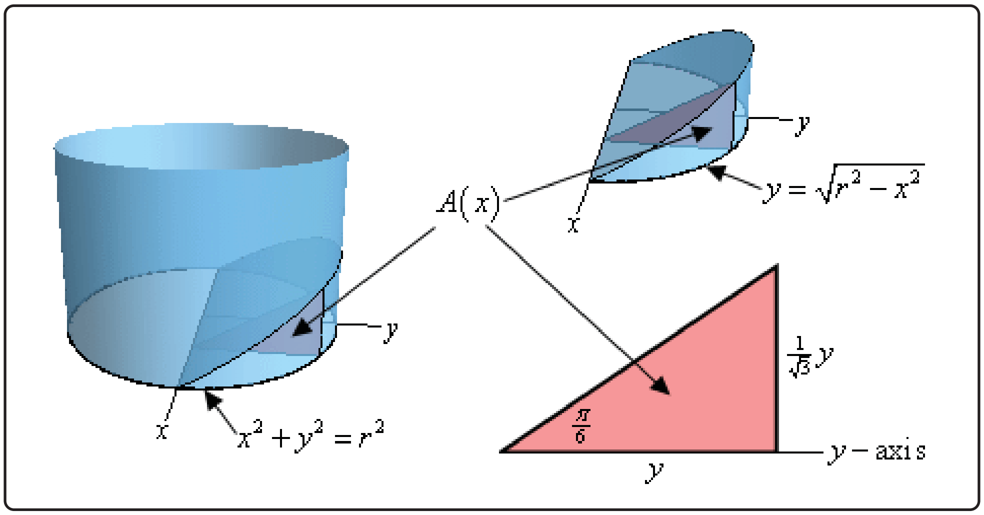
In the previous example we again saw an r in the integral. However, unlike the previous two examples it was not multiplied times the x or the y and so could not be pulled out of the integral. In this case it was like we were integrating $4 - y^2$ and we know how to integrate that. So, in this case we need to treat the r^2 like the 4 and so when we integrate that we'll pick up a y.

Example 4

Find the volume of a wedge cut out of a cylinder of radius r if the angle between the top and bottom of the wedge is $\frac{\pi}{6}$.

Solution

We should really start off with a sketch of just what we're looking for here.



On the left we see how the wedge is being cut out of the cylinder. The base of the cylinder is the circle give by $x^2 + y^2 = r^2$ and the angle between this circle and the top of the wedge is $\frac{\pi}{6}$. The sketch in the upper right position is the actual wedge itself. Given the orientation of the axes here we get the portion of the circle with positive y and so we can write the equation of the circle as $y = \sqrt{r^2 - x^2}$ since we only need the positive y values. Note as well that this is the reason for the way we oriented the axes here.

We get positive y's and we can write the equation of the circle as a function only of x's.

Now, as we can see in the two sketches of the wedge the cross-sectional area will be a right triangle and the area will be a function of x as we move from the back of the cylinder, at $x = -r$, to the front of the cylinder, at $x = r$.

The right angle of the triangle will be on the circle itself while the point on the x-axis will have an interior angle of $\frac{\pi}{6}$. The base of the triangle will have a length of y and using a little right triangle trig we see that the height of the rectangle is,

$$\text{height} = y \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} y$$

So, we now know the base and height of our triangle, in terms of y, and we have an equation for y in terms of x and so we can see that the area of the triangle, *i.e.* the cross-sectional area is,

$$A(x) = \frac{1}{2}(y)\left(\frac{1}{\sqrt{3}}y\right) = \frac{1}{2}\sqrt{r^2-x^2}\left(\frac{1}{\sqrt{3}}\sqrt{r^2-x^2}\right) = \frac{1}{2\sqrt{3}}\left(r^2-x^2\right)$$

The limits on x are $-r \le x \le r$ and so the volume is then,

$$V = \int_{-r}^{r} \frac{1}{2\sqrt{3}}\left(r^2-x^2\right)dx = \frac{1}{2\sqrt{3}}\left(r^2x - \frac{1}{3}x^3\right)\Bigg|_{-r}^{r} = \frac{2r^3}{3\sqrt{3}}$$

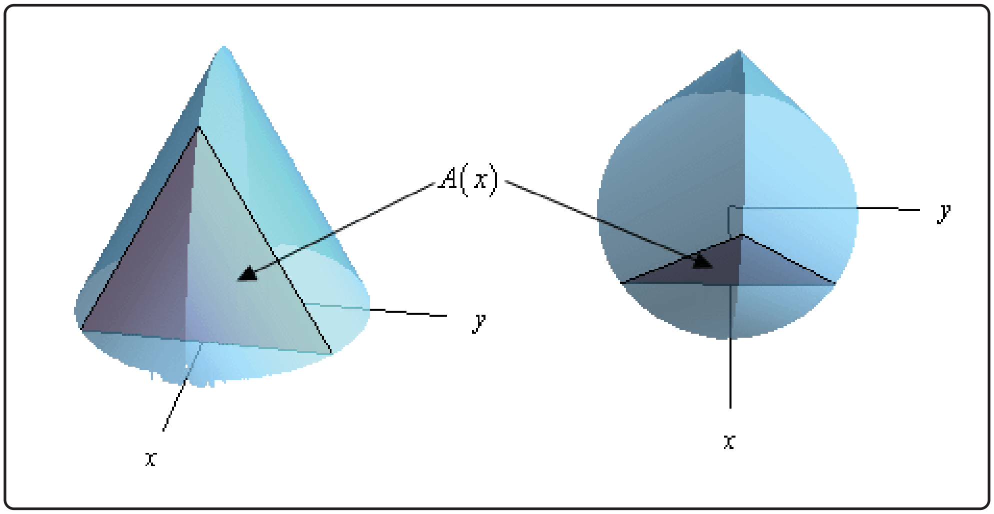
The next example is very similar to the previous one except it can be a little difficult to visualize the solid itself.

Example 5

Find the volume of the solid whose base is a disk of radius r and whose cross-sections are equilateral triangles.

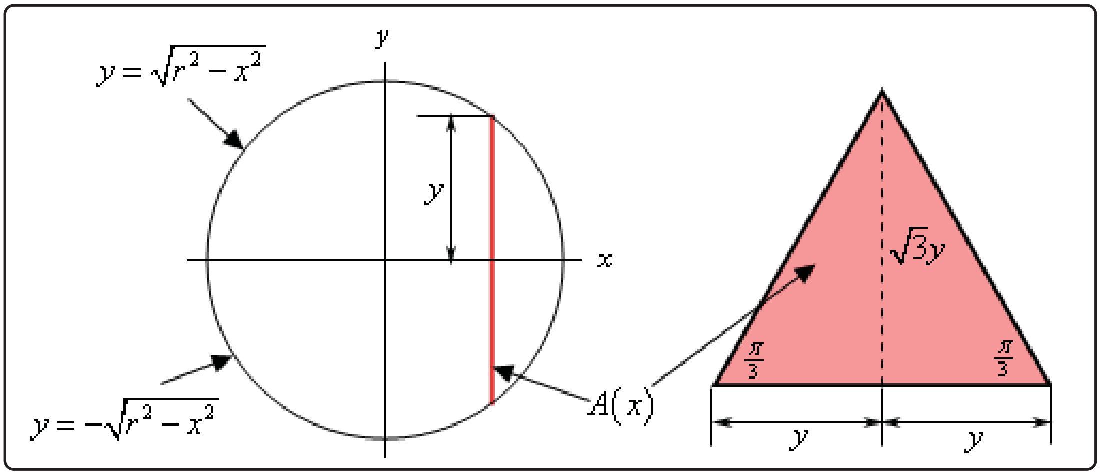
Solution

Let's start off with a couple of sketches of this solid. The sketch on the left is from the "front" of the solid and the sketch on the right is more from the top of the solid.



The base of this solid is the disk of radius r and we move from the back of the disk at $x = -r$ to the front of the disk at $x = r$ we form equilateral triangles to form the solid. A sample equilateral triangle, which is also the cross-sectional area, is shown above to hopefully make it a little clearer how the solid is formed.

Now, let's get a formula for the cross-sectional area. Let's start with the two sketches below.



In the left hand sketch we are looking at the solid from directly above and notice that we "reoriented" the sketch a little to put the x and y-axis in the "normal" orientation. The solid vertical line in this sketch is the cross-sectional area. From this we can see that the cross-section occurs at a given x and the top half will have a length of y where the value of y will be the y-coordinate of the point on the circle and so is,

$$y = \sqrt{r^2 - x^2}$$

Also, because the cross-section is an equilateral triangle that is centered on the x-axis the bottom half will also have a length of y. Thus, the base of the cross-section must have a length of $2y$.

The sketch to the right is of one of the cross-sections. As noted above the base of the triangle has a length of $2y$. Also note that because it is an equilateral triangle the angles are all $\frac{\pi}{3}$. If we divide the cross-section in two (as shown with the dashed line) we now have two right triangles and using right triangle trig we can see that the length of the dashed line is,

$$\text{dashed line} = y \tan\left(\frac{\pi}{3}\right) = \sqrt{3}\, y$$

Therefore, the height of the cross-section is $\sqrt{3}\, y$. Because the cross-section is a triangle we know that that it's area must then be,

$$A(x) = \frac{1}{2}(2y)\left(\sqrt{3}\, y\right) = \frac{1}{2}\left(2\sqrt{r^2 - x^2}\right)\left(\sqrt{3}\sqrt{r^2 - x^2}\right) = \sqrt{3}\left(r^2 - x^2\right)$$

Note that we used the cross-sectional area in terms of x because each of the cross-sections is perpendicular to the x-axis and this pretty much forces us to integrate with respect to x.

The volume of the solid is then,

$$V(x) = \int_{-r}^{r} \sqrt{3}\left(r^2 - x^2\right)\,dx = \sqrt{3}\left(r^2 x - \frac{1}{3}x^3\right)\Big|_{-r}^{r} = \frac{4}{\sqrt{3}}r^3$$

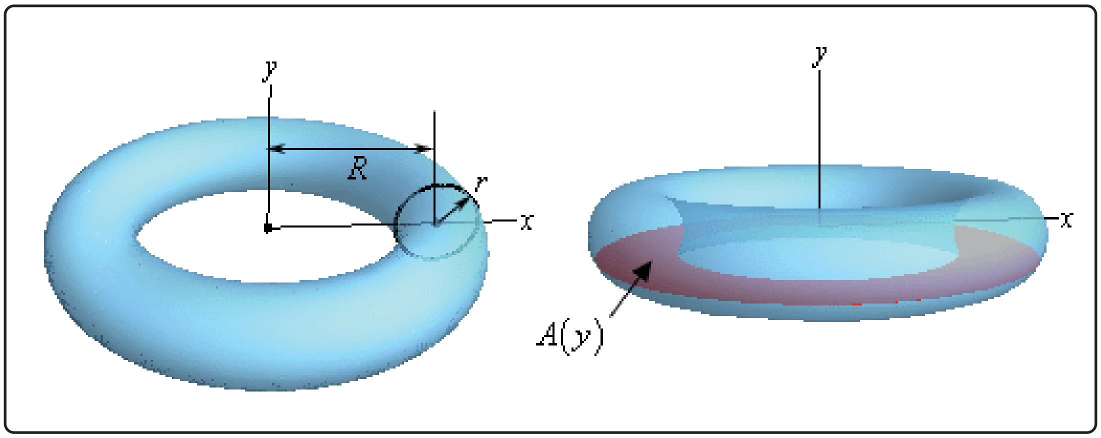
The final example we're going to work here is a little tricky both in seeing how to set it up and in doing the integral.

Example 6

Find the volume of a torus with radii r and R.

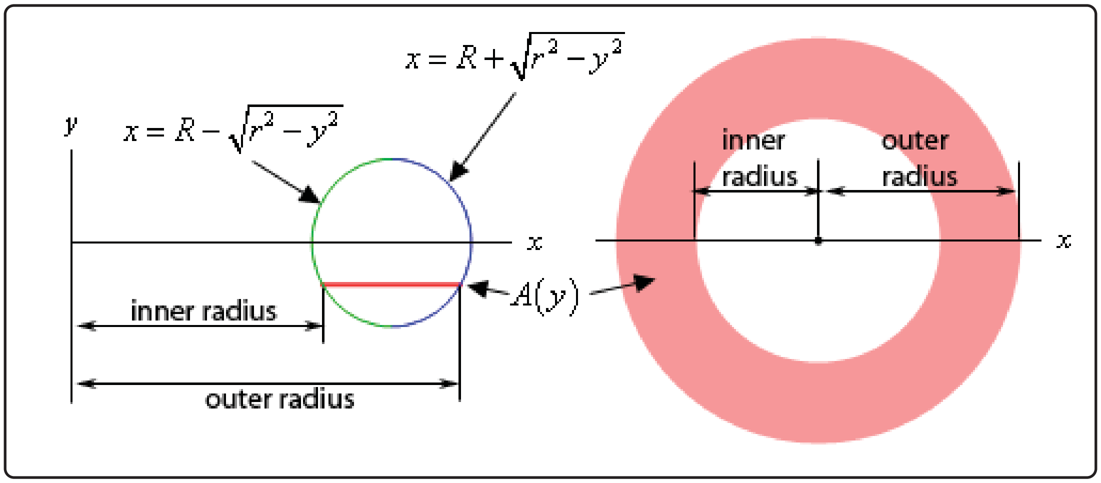
Solution

First, just what is a torus? A torus is a donut shaped solid that is generated by rotating the circle of radius r and centered at (R, 0) about the y-axis. This is shown in the sketch to the left below.



One of the trickiest parts of this problem is seeing what the cross-sectional area needs to be. There is an obvious one. Most people would probably think of using the circle of radius r that we're rotating about the y-axis as the cross-section. It is definitely one of the more obvious choices, however setting up an integral using this is not so easy.

So, what we'll do is use a cross-section as shown in the sketch to the right above. If we cut the torus perpendicular to the y-axis we'll get a cross-section of a ring and finding the area of that shouldn't be too bad. To do that let's take a look at the two sketches below.



The sketch to the left is a sketch of the full cross-section. The sketch to the right is more important however. This is a sketch of the circle that we are rotating about the y-axis. Included is a line representing where the cross-sectional area would be in the torus.

Notice that the inner radius will always be the left portion of the circle and the outer radius will always be the right portion of the circle. Now, we know that the equation of this is,

$$(x - R)^2 + y^2 = r^2$$

and so if we solve for x we can get the equations for the left and right sides as shown above in the sketch. This however means that we also now have equations for the inner and outer radii.

inner radius : $x = R - \sqrt{r^2 - y^2}$ outer radius : $x = R + \sqrt{r^2 - y^2}$

The cross-sectional area is then,

$$\begin{aligned} A(y) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi\left[\left(R + \sqrt{r^2 - y^2}\right)^2 - \left(R - \sqrt{r^2 - y^2}\right)^2\right] \\ &= \pi\left[R^2 + 2R\sqrt{r^2 - y^2} + r^2 - y^2 - \left(R^2 - 2R\sqrt{r^2 - y^2} + r^2 - y^2\right)\right] \\ &= 4\pi R\sqrt{r^2 - y^2} \end{aligned}$$

Next, the lowest cross-section will occur at $y = -r$ and the highest cross-section will occur at $y = r$ and so the limits for the integral will be $-r \le y \le r$.

The integral giving the volume is then,

$$V = \int_{-r}^{r} 4\pi R\sqrt{r^2 - y^2}\, dy = 2\int_{0}^{r} 4\pi R\sqrt{r^2 - y^2}\, dy = 8\pi R\int_{0}^{r} \sqrt{r^2 - y^2}\, dy$$

Note that we used the fact that because the integrand is an even function and we're integrating over $[-r, r]$ we could change the lower limit to zero and double the value of the integral. We saw this fact back in the Computing Definite Integrals section.

We've now reached the second really tricky part of this example. With the knowledge that we've currently got at this point this integral is not possible to do. It requires something called a trig substitution and that is a topic for Calculus II. Luckily enough for us, and this is the tricky part, in this case we can actually determine the integral's value using what we know about integrals.

Just for a second let's think about a different problem. Let's suppose we wanted to use an integral to determine the area under the portion of the circle of radius r and centered at the origin that is in the first quadrant. There are a couple of ways to do this, but to match what we're doing here let's do the following.

We know that the equation of the circle is $x^2 + y^2 = r^2$ and if we solve for x the equation of the circle in the first (and fourth for that matter) quadrant is,

$$x = \sqrt{r^2 - y^2}$$

If we want an integral for the area in the first quadrant we can think of this area as the region between the curve $x = \sqrt{r^2 - y^2}$ and the y-axis for $0 \le y \le r$ and this is,

$$A = \int_0^r \sqrt{r^2 - y^2}\, dy$$

In other words, this integral represents one quarter of the area of a circle of radius r and from basic geometric formulas we now know that this integral must have the value,

$$A = \int_0^r \sqrt{r^2 - y^2}\, dy = \frac{1}{4}\pi r^2$$

So, putting all this together the volume of the torus is then,

$$V = 8R\pi \int_0^r \sqrt{r^2 - y^2}\, dy = 8\pi R \left(\frac{1}{4}\pi r^2\right) = 2R\pi^2 r^2$$

6.6 Work

This is the final application of integral that we'll be looking at in this course. In this section we will be looking at the amount of work that is done by a force in moving an object.

In a first course in Physics you typically look at the work that a constant force, F, does when moving an object over a distance of d. In these cases the work is,

$$W = Fd$$

However, most forces are not constant and will depend upon where exactly the force is acting. So, let's suppose that the force at any x is given by $F\left(x\right)$. Then the work done by the force in moving an object from $x = a$ to $x = b$ is given by,

$$W = \int_a^b F\left(x\right)\,dx$$

To see a justification of this formula see the Proof of Various Integral Properties section of the Extras appendix.

Notice that if the force is constant we get the correct formula for a constant force.

$$\begin{aligned} W &= \int_a^b F\,dx \\ &= Fx\Big|_a^b \\ &= F\left(b - a\right) \end{aligned}$$

where $b - a$ is simply the distance moved, or d.

So, let's take a look at a couple of examples of non-constant forces.

Example 1

A spring has a natural length of 20 cm. A 40 N force is required to stretch (and hold the spring) to a length of 30 cm. How much work is done in stretching the spring from 35 cm to 38 cm?

Solution

This example will require Hooke's Law to determine the force. Hooke's Law tells us that the force required to stretch a spring a distance of x meters from its natural length is,

$$F\left(x\right) = k\,x$$

where $k > 0$ is called the spring constant. It is important to remember that the x in this formula is the distance the spring is stretched from its natural length and not the actual length of the spring.

So, the first thing that we need to do is determine the spring constant for this spring. We can do that using the initial information. A force of 40 N is required to stretch the spring

$$30 \text{ cm} - 20 \text{ cm} = 10 \text{ cm} = 0.1 \text{ m}$$

from its natural length. Using Hooke's Law we have,

$$40 = 0.10k \qquad \Rightarrow \qquad k = 400$$

So, according to Hooke's Law the force required to hold this spring x meters from its natural length is,

$$F(x) = 400x$$

We want to know the work required to stretch the spring from 35cm to 38cm. First, we need to convert these into distances from the natural length in meters. Doing that gives us x's of 0.15m and 0.18m.

The work is then,

$$\begin{aligned} W &= \int_{0.15}^{0.18} 400x \, dx \\ &= 200x^2 \Big|_{0.15}^{0.18} \\ &= 1.98 \text{ J} \end{aligned}$$

Example 2

We have a cable that weighs 2 lbs/ft attached to a bucket filled with coal that weighs 800 lbs. The bucket is initially at the bottom of a 500 ft mine shaft. Answer each of the following about this.

(a) Determine the amount of work required to lift the bucket to the midpoint of the shaft.

(b) Determine the amount of work required to lift the bucket from the midpoint of the shaft to the top of the shaft.

(c) Determine the amount of work required to lift the bucket all the way up the shaft.

Solution

Before answering either part we first need to determine the force. In this case the force will be the weight of the bucket and cable at any point in the shaft.

To determine a formula for this we will first need to set a convention for x. For this problem we will set x to be the amount of cable that has been pulled up. So at the bottom of the shaft $x = 0$, at the midpoint of the shaft $x = 250$ and at the top of the shaft $x = 500$. Also, at any point in the shaft there is $500 - x$ feet of cable still in the shaft.

The force then for any x is then nothing more than the weight of the cable and bucket at that point. This is,

$$\begin{aligned} F(x) &= \text{weight of cable} + \text{ weight of bucket/coal} \\ &= 2(500 - x) + 800 \\ &= 1800 - 2x \end{aligned}$$

We can now answer the questions.

(a) Determine the amount of work required to lift the bucket to the midpoint of the shaft.

In this case we want to know the work required to move the cable and bucket/coal from $x = 0$ to $x = 250$. The work required is,

$$\begin{aligned} W &= \int_0^{250} F(x)\, dx \\ &= \int_0^{250} 1800 - 2x\, dx \\ &= \left(1800x - x^2\right)\Big|_0^{250} \\ &= 387500 \text{ ft - lb} \end{aligned}$$

(b) Determine the amount of work required to lift the bucket from the midpoint of the shaft to the top of the shaft.

In this case we want to move the cable and bucket/coal from $x = 250$ to $x = 500$. The work required is,

$$\begin{aligned} W &= \int_{250}^{500} F(x)\, dx \\ &= \int_{250}^{500} 1800 - 2x\, dx \\ &= \left(1800x - x^2\right)\Big|_{250}^{500} \\ &= 262500 \text{ ft - lb} \end{aligned}$$

(c) Determine the amount of work required to lift the bucket all the way up the shaft.

In this case the work is,

$$\begin{aligned} W &= \int_0^{500} F(x)\,dx \\ &= \int_0^{500} 1800 - 2x\,dx \\ &= \left(1800x - x^2\right)\Big|_0^{500} \\ &= 650000 \text{ ft - lb} \end{aligned}$$

Note that we could have instead just added the results from the first two parts and we would have gotten the same answer to the third part.

Example 3

A 20 ft cable weighs 80 lbs and hangs from the ceiling of a building without touching the floor. Determine the work that must be done to lift the bottom end of the chain all the way up until it touches the ceiling.

Solution

First, we need to determine the weight per foot of the cable. This is easy enough to get,

$$\frac{80 \text{ lbs}}{20 \text{ ft}} = 4 \text{ lb/ft}$$

Next, let x be the distance from the ceiling to any point on the cable. Using this convention we can see that the portion of the cable in the range $10 < x \le 20$ will actually be lifted. The portion of the cable in the range $0 \le x \le 10$ will not be lifted at all since once the bottom of the cable has been lifted up to the ceiling the cable will be doubled up and each portion will have a length of 10 ft. So, the upper 10 foot portion of the cable will never be lifted while the lower 10 ft portion will be lifted.

Now, the very bottom of the cable, $x = 20$, will be lifted 10 feet to get to the midpoint and then a further 10 feet to get to the ceiling. A point 2 feet from the bottom of the cable, $x = 18$ will lift 8 feet to get to the midpoint and then a further 8 feet until it reaches its final position (if it is 2 feet from the bottom then its final position will be 2 feet from the ceiling). Continuing on in this fashion we can see that for any point on the lower half of the cable, *i.e.* $10 \le x \le 20$ it will be lifted a total of $2(x - 10)$.

As with the previous example the force required to lift any point of the cable in this range is simply the distance that point will be lifted times the weight/foot of the cable. So, the force is then,

$$\begin{aligned} F(x) &= (\text{distance lifted})\,(\text{weight per foot of cable}) \\ &= 2(x-10)(4) \\ &= 8(x-10) \end{aligned}$$

The work required is now,

$$\begin{aligned} W &= \int_{10}^{20} 8(x-10)\,dx \\ &= \left(4x^2 - 80x\right)\Big|_{10}^{20} \\ &= 400 \text{ ft - lb} \end{aligned}$$

Provided we can find the force, $F(x)$, for a given problem then using the above method for determining the work is (generally) pretty simple. However, there are some problems where this approach won't easily work. Let's take a look at one of those kinds of problems.

Example 4

A tank in the shape of an inverted cone has a height of 15 meters and a base radius of 4 meters and is filled with water to a depth of 12 meters. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the density of the water is 1000 kg/m^3.

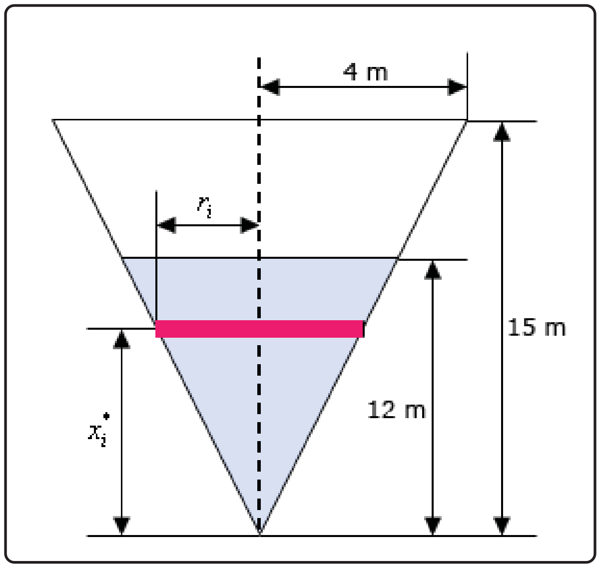
Solution

Okay, in this case we cannot just determine a force function, $F(x)$ that will work for us. So, we are going to need to approach this from a different standpoint.

Let's first set $x = 0$ to be the lower end of the tank/cone and $x = 15$ to be the top of the tank/cone. With this *definition* of our x's we can now see that the water in the tank will correspond to the interval $[0, 12]$.

So, let's start off by dividing $[0, 12]$ into n subintervals each of width Δx and let's also let x_i^* be any point from the i^{th} subinterval where $i = 1, 2, \ldots n$. Now, for each subinterval we will approximate the water in the tank corresponding to that interval as a cylinder of radius r_i and height Δx.

Here is a quick sketch of the tank. Note that the sketch really isn't to scale and we are looking at the tank from directly in front so we can see all the various quantities that we need to work with.



The red strip in the sketch represents the "cylinder" of water in the i^{th} subinterval. A quick application of similar triangles will allow us to relate r_i to x_i^* (which we'll need in a bit) as follows.

$$\frac{r_i}{x_i^*} = \frac{4}{15} \qquad \Rightarrow \qquad r_i = \frac{4}{15} x_i^*$$

Okay, the mass, m_i, of the volume of water, V_i, for the i^{th} subinterval is simply,

$$m_i = \text{density} \times V_i$$

We know the density of the water (it was given in the problem statement) and because we are approximating the water in the i^{th} subinterval as a cylinder we can easily approximate the volume as,

$$V_i \approx \pi(\text{radius})^2 (\text{height})$$

We can now approximate the mass of water in the i^{th} subinterval,

$$m_i \approx (1000)\left[\pi r_i^2 \Delta x\right] = 1000\pi\left(\frac{4}{15}x_i^*\right)^2 \Delta x = \frac{640}{9}\pi(x_i^*)^2\Delta x$$

To raise this volume of water we need to overcome the force of gravity that is acting on the volume and that is, $F = m_i g$, where $g = 9.8\,m/s^2$ is the gravitational acceleration. The force to raise the volume of water in the i^{th} subinterval is then approximately,

$$F_i = m_i g \approx (9.8)\,\frac{640}{9}\pi(x_i^*)^2\Delta x$$

Next, in order to reach to the top of the tank the water in the i^{th} subinterval will need to travel approximately $15 - x_i^*$ to reach the top of the tank. Because the volume of the water in the

i^{th} subinterval is constant the force needed to raise the water through any distance is also a constant force.

Therefore, the work to move the volume of water in the i^{th} subinterval to the top of the tank, *i.e.* raise it a distance of $15 - x_i^*$, is then approximately,

$$W_i \approx F_i \left(15 - x_i^*\right) = (9.8)\,\frac{640}{9}\pi \left(x_i^*\right)^2 \left(15 - x_i^*\right) \Delta x$$

The total amount of work required to raise all the water to the top of the tank is then approximately the sum of each of the W_i for $i = 1, 2, \ldots n$. Or,

$$W \approx \sum_{i=1}^{n} (9.8)\,\frac{640}{9}\pi \left(x_i^*\right)^2 \left(15 - x_i^*\right) \Delta x$$

To get the actual amount of work we simply need to take $n \to \infty$. *I.e.* compute the following limit,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} (9.8)\,\frac{640}{9}\pi \left(x_i^*\right)^2 \left(15 - x_i^*\right) \Delta x$$

This limit of a summation should look somewhat familiar to you. It's probably been some time, but recalling the definition of the definite integral we can see that this is nothing more than the following definite integral,

$$\begin{aligned} W &= \int_0^{12} (9.8)\,\frac{640}{9}\pi x^2 \left(15 - x\right)\,dx = (9.8)\,\frac{640}{9}\pi \int_0^{12} 15x^2 - x^3\,dx \\ &= (9.8)\,\frac{640}{9}\pi \left(5x^3 - \frac{1}{4}x^4\right)\Bigg|_0^{12} = 7,566,362.543\text{ J} \end{aligned}$$

As we've seen in the previous example we sometimes need to compute "incremental" work and then use that to determine the actual integral we need to compute. This idea does arise on occasion and we shouldn't forget it!

7 Integration Techniques

By this point we've now looked at basic integration techniques. We've seen how to integrate most of the “basic” functions we're liable to run into : polynomials, roots, trig, exponential, logarithm and inverse trig functions to name a few. In addition, we've seen how to do basic u-substitutions allowing us to integrate some more complicated functions.

We've also taken a look at some basic applications of (definite) integrals. However, as was noted at the time, there are applications of (definite) integrals that will, on occasion, have integrals that need more than just a basic u-substitution. So, before we can take a look at those applications we'll need to first talk about some more involved integration techniques.

Before getting into the new techniques we first need to make it clear that in this chapter it is assumed at you are comfortable with basic integration, including u-substitutions. Many of the problems in this chapter will not have a lot, if any, discussion of the basic integration work under the assumption that you are comfortable enough with the basic work that discussion in simply not needed. In addition, we will usually, although not always, give the substitution that we're using for the u-substitution but we will generally not show the actual substitution work. Again, this is under the assumption that you are comfortable enough with basic u-substitutions that you can fill in the details if you need to.

The reason for skipping the discussion of the basic integration work and/or not showing the full substitution work is so we can concentrate our discussion on the particular method that we are covering in that particular section. This is not to “punish” you but simply to acknowledge that we only have so much time in which to discuss the material and just can't afford to spend a lot of time basically re-lecturing basic integration material. We realize that, for many of you, this is the start of your Calculus II course and so you may have had some time off and may well have some “rust” on your basic integration skills. This is a warning to start scraping that rust off. If you need do scrape some rust off you can check out the practice problems for some practice problems covering basic integration to refresh your memory on how basic integration works.

It is also very important for you to understand that most of the problems we'll be looking at in this chapter will involve u-substitutions in one way or another. In fact, many of the techniques in this chapter are really just substitutions. The only difference is that either they need a fair amount of work to get to the point where the substitutions can be used or they will involve substitutions used in ways that we've not seen to this point. So, again, if you have some rust on your u-substitution skills you'll need to get it scraped off so you can do the work in this chapter.

In addition, we will be doing indefinite integrals almost exclusively in most of the sections in this chapter. There are a few sections were we'll be doing some definite integrals but for the most part we'll keep the problems in most of the sections shorter by just doing indefinite integrals. It is assumed that if you were given a definite integral you could do the extra evaluation steps needed to finish the definite integral. Having said that, there are a few sections were definite integrals are done either because there are some subtleties that need to be dealt with for definite integrals or because the topic at hand, the last few sections in particular, involve only definite integrals.

So, with all that out of the way, here is a quick rundown of the new integration techniques we'll take a look at in this section.

Probably the most important technique, in this sense that it will be the most commonly seen technique out of this class, is integration by parts. This is the one new technique in this chapter that is not just u-substitutions done in new ways. Integration by Parts will involve u-substitutions at various steps the process on occasion but it will not be just a new way of doing a u-substitution.

As noted a lot of the techniques in this chapter are really just u-substitutions except they will need some manipulation of the integrand prior to actually doing the substitution. The techniques using this idea will include integrating some, but not all, products and quotients of trig functions, some integrands involving roots or quadratics that can't be done without manipulation of the integrand or "different" u-substitutions that we are used to. We'll also see how to use partial fractions to write some integrands involving rational expressions into a form that we can actually do the integral.

We'll also take a look at something called trig substitutions. This is probably the one technique that is usually considered the most difficult, or at the least, the longest method. As we'll see a trig substitution is really a substitution but it is not a traditional u-substitution. However, having said that, if you understand how basic u-substitutions work it will help greatly when it comes to working with trig substitutions as the basic concepts are the same.

Next we'll be taking a look at a new kind of integral, Improper Integrals. This topic will address how to deal with definite integrals for which one or both of the limits of integration will be an infinity. In addition, we'll see how we can, on occasion, deal with discontinuities in the integrand (we'll focus on division by zero in the integrand).

We'll close out the chapter with a quick section on approximating the value of definite integrals.

We will leave this introduction with a warning. It is with this chapter that you will find that you can't just memorize your way through the class anymore. We will acknowledge that up to this point it is possible, for the most part, to just memorize your way through the class. You may not get the highest grades through just memorization as there are some topics that require a fair amount of understanding of the topic, but you can survive up to this point if your really good at memorization.

Integration by Parts is a really good example of this warning. While you will need to memorize/know the basic integration by parts formula simply memorizing that will not help you to actually use integration by parts on the problem. You will need to actually understand how integration by parts works and how to "assign" various portions of the integrand to the various portions of the integration parts formula.

Also while there are some basic formulas we can, and do on occasion, give for some of the methods there are also situations that just don't fit into those formulas and so again you'll really need to understand how to do those methods in order to work problems for which basic formulas just won't work. Or, again, you can't just memorize your way out of most the methods taught in this chapter. Memorization may allow you to get through the basic problems but will not help all that much with more complicated problems.

Finally, we also need to warn you about seeing "patterns" and just assuming that all the problems will fall into those patterns. Integration by Parts is, again, a good example of this. There are some "patterns" that seem to show up because a lot of the problems we do in that section do fall into the patterns. The problem is that there are also some problems for which the "patterns" simply don't work and yet they still require integration by parts. If you get so locked into "patterns" you'll find it all but impossible to do some problems because they simply don't fall into those patterns.

This is not to say that recognizing that patterns in always a bad thing. Patterns do, on occasion, show up and they can be useful to understand/know as a possible solution method. However, you also need to always remember that there are problems that just don't fit easily into the patterns. This is also a warning that will be valid in other chapters in a typical Calculus II course as well. Again, patterns aren't bad per se, you just need to be careful to not always assume that every problem will fall into the patterns.

7.1 Integration by Parts

Let's start off with this section with a couple of integrals that we should already be able to do to get us started. First let's take a look at the following.

$$\int \mathbf{e}^{x}\,dx = \mathbf{e}^{x} + c$$

So, that was simple enough. Now, let's take a look at,

$$\int x\mathbf{e}^{x^2}\,dx$$

To do this integral we'll use the following substitution.

$$u = x^2 \qquad du = 2x\,dx \qquad \Rightarrow \qquad x\,dx = \frac{1}{2}\,du$$

$$\int x\mathbf{e}^{x^2}\,dx = \frac{1}{2}\int \mathbf{e}^{u}\,du = \frac{1}{2}\mathbf{e}^{u} + c = \frac{1}{2}\mathbf{e}^{x^2} + c$$

Again, simple enough to do provided you remember how to do substitutions. By the way make sure that you can do these kinds of substitutions quickly and easily. From this point on we are going to be doing these kinds of substitutions in our head. If you have to stop and write these out with every problem you will find that it will take you significantly longer to do these problems.

Now, let's look at the integral that we really want to do.

$$\int x\mathbf{e}^{6x}\,dx$$

If we just had an x by itself or $\mathbf{e}^{6x}$ by itself we could do the integral easily enough. Likewise, if the integrand was $x\mathbf{e}^{6x^2}$ we could do the integral with a substitution. Unfortunately, however, neither of these are options. So, at this point we don't have the knowledge to do this integral.

To do this integral we will need to use integration by parts so let's derive the integration by parts formula. We'll start with the product rule.

$$\left(f(x)\,g(x)\right)' = f'(x)\,g(x) + f(x)\,g'(x)$$

Now, integrate both sides of this.

$$\int \left(f(x)\,g(x)\right)'\,dx = \int f'(x)\,g(x) + f(x)\,g'(x)\,dx$$

The left side is easy enough to integrate (we know that integrating a derivative just "undoes" the derivative) and we'll split up the right side of the integral.

$$f(x)g(x) = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

Note that technically we should have had a constant of integration show up on the left side after doing the integration. We can drop it at this point since other constants of integration will be showing up down the road and they would just end up absorbing this one.

Finally, rewrite the formula as follows and we arrive at the integration by parts formula.

Integration by Parts (formal)

$$\int f(x)\, g'(x)\, dx = f(x)g(x) - \int f'(x)\, g(x)\, dx$$

This is not the easiest formula to use however. So, let's do a couple of substitutions.

$$u = f(x) \qquad\qquad v = g(x)$$
$$du = f'(x)\, dx \qquad\qquad dv = g(x)\, dx$$

Both of these are just the standard Calculus I substitutions that hopefully you are used to by now. Don't get excited by the fact that we are using two substitutions here. They will work the same way.

Using these substitutions gives us the formula that most people think of as the integration by parts formula.

Integration By Parts (simplified)

$$\int u\, dv = uv - \int v\, du$$

To use this formula, we will need to identify u and dv, compute du and v and then use the formula. Note as well that computing v is very easy. All we need to do is integrate dv.

$$v = \int dv$$

One of the more complicated things about using this formula is you need to be able to correctly identify both the u and the dv. It won't always be clear what the correct choices are and we will, on occasion, make the wrong choice. This is not something to worry about. If we make the wrong choice, we can always go back and try a different set of choices.

This does lead to the obvious question of how do we know if we made the correct choice for u and dv? The answer is actually pretty simple. We made the correct choices for u and dv if, after using the integration by parts formula the new integral (the one on the right of the formula) is one we can actually integrate.

So, let's take a look at the integral above that we mentioned we wanted to do.

Example 1

Evaluate the following integral.

$$\int x\mathbf{e}^{6x}\,dx$$

Solution

So, on some level, the problem here is the x that is in front of the exponential. If that wasn't there we could do the integral. Notice as well that in doing integration by parts anything that we choose for u will be differentiated. So, it seems that choosing $u = x$ will be a good choice since upon differentiating the x will drop out.

Now that we've chosen u we know that dv will be everything else that remains. So, here are the choices for u and dv as well as du and v.

$$u = x \qquad\qquad dv = \mathbf{e}^{6x}\,dx$$

$$du = dx \qquad\qquad v = \int \mathbf{e}^{6x}\,dx = \frac{1}{6}\mathbf{e}^{6x}$$

The integral is then,

$$\begin{aligned}\int x\mathbf{e}^{6x}\,dx &= \frac{x}{6}\mathbf{e}^{6x} - \int \frac{1}{6}\mathbf{e}^{6x}\,dx \\ &= \frac{x}{6}\mathbf{e}^{6x} - \frac{1}{36}\mathbf{e}^{6x} + c\end{aligned}$$

Once we have done the last integral in the problem we will add in the constant of integration to get our final answer.

Note as well that, as noted above, we know we made made a correct choice for u and dv when we got a new integral that we can actually evaluate after applying the integration by parts formula.

Next, let's take a look at integration by parts for definite integrals. The integration by parts formula for definite integrals is,

Integration By Parts, Definite Integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Note that the $uv|_a^b$ in the first term is just the standard integral evaluation notation that you should be familiar with at this point. All we do is evaluate the term, *uv* in this case, at b then subtract off the evaluation of the term at a.

At some level we don't really need a formula here because we know that when doing definite integrals all we need to do is evaluate the indefinite integral and then do the evaluation. In fact, this is probably going to be slightly easier as we don't need to track evaluating each term this way.

Let's take a quick look at a definite integral using integration by parts.

Example 2

Evaluate the following integral.

$$\int_{-1}^{2} x\mathbf{e}^{6x}\,dx$$

Solution

This is the same integral that we looked at in the first example so we'll use the same u and dv to get,

$$\begin{aligned}\int_{-1}^{2} x\mathbf{e}^{6x}\,dx &= \frac{x}{6}\mathbf{e}^{6x}\Big|_{-1}^{2} - \frac{1}{6}\int_{-1}^{2}\mathbf{e}^{6x}\,dx\\ &= \frac{x}{6}\mathbf{e}^{6x}\Big|_{-1}^{2} - \frac{1}{36}\mathbf{e}^{6x}\Big|_{-1}^{2}\\ &= \frac{11}{36}\mathbf{e}^{12} + \frac{7}{36}\mathbf{e}^{-6}\end{aligned}$$

As noted above we could just as easily used the result from the first example to do the evaluation. We know, from the first example that,

$$\int x\mathbf{e}^{6x}\,dx = \frac{x}{6}\mathbf{e}^{6x} - \frac{1}{36}\mathbf{e}^{6x} + c$$

Using this we can quickly proceed to the evaluation of the definite integral as follows,

$$\begin{aligned}\int_{-1}^{2} x\mathbf{e}^{6x}\,dx &= \left(\frac{x}{6}\mathbf{e}^{6x} - \frac{1}{36}\mathbf{e}^{6x}\right)\Big|_{-1}^{2}\\ &= \left(\frac{1}{3}\mathbf{e}^{12} - \frac{1}{36}\mathbf{e}^{12}\right) - \left(-\frac{1}{6}\mathbf{e}^{-6} - \frac{1}{36}\mathbf{e}^{-6}\right)\\ &= \frac{11}{36}\mathbf{e}^{12} + \frac{7}{36}\mathbf{e}^{-6}\end{aligned}$$

Either method of evaluating definite integrals with integration by part is pretty simple so which one you choose to use is pretty much up to you.

Since we need to be able to do the indefinite integral in order to do the definite integral and doing the definite integral amounts to nothing more than evaluating the indefinite integral at a couple of points we will concentrate on doing indefinite integrals in the rest of this section. In fact, throughout most of this chapter this will be the case. We will be doing far more indefinite integrals than definite integrals.

Let's take a look at some more examples.

Example 3

Evaluate the following integral.

$$\int (3t+5)\cos\left(\frac{t}{4}\right)\,dt$$

Solution

There are two ways to proceed with this example. For many, the first thing that they try is multiplying the cosine through the parenthesis, splitting up the integral and then doing integration by parts on the first integral.

While that is a perfectly acceptable way of doing the problem it's more work than we really need to do. Instead of splitting the integral up let's instead use the following choices for u and dv.

$$u = 3t+5 \qquad dv = \cos\left(\frac{t}{4}\right)\,dt$$
$$du = 3\,dt \qquad v = 4\sin\left(\frac{t}{4}\right)$$

The integral is then,

$$\begin{aligned}\int (3t+5)\cos\left(\frac{t}{4}\right)\,dt &= 4\,(3t+5)\sin\left(\frac{t}{4}\right) - 12\int \sin\left(\frac{t}{4}\right)\,dt \\ &= 4\,(3t+5)\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + c\end{aligned}$$

Notice that we pulled any constants out of the integral when we used the integration by parts formula. We will usually do this in order to simplify the integral a little.

Example 4

Evaluate the following integral.

$$\int w^2 \sin(10w)\, dw$$

Solution

For this example, we'll use the following choices for u and dv.

$$u = w^2 \qquad dv = \sin(10w)\, dw$$
$$du = 2w\, dw \qquad v = -\frac{1}{10}\cos(10w)$$

The integral is then,

$$\int w^2 \sin(10w)\, dw = -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\int w\cos(10w)\, dw$$

In this example, unlike the previous examples, the new integral will also require integration by parts. For this second integral we will use the following choices.

$$u = w \qquad dv = \cos(10w)\, dw$$
$$du = dw \qquad v = \frac{1}{10}\sin(10w)$$

So, the integral becomes,

$$\begin{aligned}\int w^2 \sin(10w)\, dw &= -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\left(\frac{w}{10}\sin(10w) - \frac{1}{10}\int \sin(10w)\, dw\right)\\ &= -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\left(\frac{w}{10}\sin(10w) + \frac{1}{100}\cos(10w)\right) + c\\ &= -\frac{w^2}{10}\cos(10w) + \frac{w}{50}\sin(10w) + \frac{1}{500}\cos(10w) + c\end{aligned}$$

Be careful with the coefficient on the integral for the second application of integration by parts. Since the integral is multiplied by $\frac{1}{5}$ we need to make sure that the results of actually doing the integral are also multiplied by $\frac{1}{5}$. Forgetting to do this is one of the more common mistakes with integration by parts problems.

As this last example has shown us, we will sometimes need more than one application of integration by parts to completely evaluate an integral. This is something that will happen so don't get excited about it when it does.

In this next example we need to acknowledge an important point about integration techniques.

Some integrals can be done in using several different techniques. That is the case with the integral in the next example.

Example 5

Evaluate the following integral

$$\int x\sqrt{x+1}\,dx$$

(a) Using Integration by Parts.

(b) Using a standard Calculus I substitution.

Solution

(a) Using Integration by Parts.

First notice that there are no trig functions or exponentials in this integral. While a good many integration by parts integrals will involve trig functions and/or exponentials not all of them will so don't get too locked into the idea of expecting them to show up.

In this case we'll use the following choices for u and dv.

$$u = x \qquad\qquad dv = \sqrt{x+1}\,dx$$
$$du = dx \qquad\qquad v = \frac{2}{3}(x+1)^{\frac{3}{2}}$$

The integral is then,

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}\,dx \\ &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + c\end{aligned}$$

(b) Using a standard Calculus I substitution.

Now let's do the integral with a substitution. We can use the following substitution.

$$u = x+1 \qquad x = u-1 \qquad du = dx$$

Notice that we'll actually use the substitution twice, once for the quantity under the square root and once for the x in front of the square root. The integral is then,

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \int (u-1)\sqrt{u}\,du\\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du\\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c\\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c\end{aligned}$$

So, we used two different integration techniques in this example and we got two different answers. The obvious question then should be : Did we do something wrong?

It turns out that, we didn't do anything wrong. We need to remember the following fact from Calculus I.

$$\text{If } f'(x) = g'(x) \quad \text{then} \quad f(x) = g(x) + c$$

In other words, if two functions have the same derivative then they will differ by no more than a constant. So, how does this apply to the above problem? First define the following,

$$f'(x) = g'(x) = x\sqrt{x+1}$$

Then we can compute $f(x)$ and $g(x)$ by integrating as follows,

$$f(x) = \int f'(x)\,dx \qquad\qquad g(x) = \int g'(x)\,dx$$

We'll use integration by parts for the first integral and the substitution for the second integral. Then according to the fact $f(x)$ and $g(x)$ should differ by no more than a constant. Let's verify this and see if this is the case. We can verify that they differ by no more than a constant if we take a look at the difference of the two and do a little algebraic manipulation and simplification.

$$\begin{aligned}\left(\tfrac{2}{3}x(x+1)^{\frac{3}{2}} - \tfrac{4}{15}(x+1)^{\frac{5}{2}}\right) - \left(\tfrac{2}{5}(x+1)^{\frac{5}{2}} - \tfrac{2}{3}(x+1)^{\frac{3}{2}}\right)&\\ = (x+1)^{\frac{3}{2}}\left(\tfrac{2}{3}x - \tfrac{4}{15}(x+1) - \tfrac{2}{5}(x+1) + \tfrac{2}{3}\right)&\\ = (x+1)^{\frac{3}{2}}(0)&\\ = 0&\end{aligned}$$

So, in this case it turns out the two functions are exactly the same function since the difference is zero. Note that this won't always happen. Sometimes the difference will yield a nonzero constant. For an example of this check out the Constant of Integration section in the Calculus I notes.

So just what have we learned? First, there will, on occasion, be more than one method for evaluating an integral. Secondly, we saw that different methods will often lead to different answers. Last, even though the answers are different it can be shown, sometimes with a lot of work, that they differ by no more than a constant.

When we are faced with an integral the first thing that we'll need to decide is if there is more than one way to do the integral. If there is more than one way we'll then need to determine which method we should use. The general rule of thumb that I use in my classes is that you should use the method that *you* find easiest. This may not be the method that others find easiest, but that doesn't make it the wrong method.

One of the more common mistakes with integration by parts is for people to get too locked into perceived patterns. For instance, all of the previous examples used the basic pattern of taking u to be the polynomial that sat in front of another function and then letting dv be the other function. This will not always happen so we need to be careful and not get locked into any patterns that we think we see.

Let's take a look at some integrals that don't fit into the above pattern.

Example 6

Evaluate the following integral.

$$\int \ln(x)\,dx$$

Solution

So, unlike any of the other integral we've done to this point there is only a single function in the integral and no polynomial sitting in front of the logarithm.

The first choice of many people here is to try and fit this into the pattern from above and make the following choices for u and dv.

$$u = 1 \qquad\qquad dv = \ln(x)\,dx$$

This leads to a real problem however since that means v must be,

$$v = \int \ln(x)\,dx$$

In other words, we would need to know the answer ahead of time in order to actually do the problem. So, this choice simply won't work.

Therefore, if the logarithm doesn't belong in the dv it must belong instead in the u. So, let's use the following choices instead

$$\begin{aligned} u &= \ln(x) & \qquad dv &= \,dx \\ du &= \frac{1}{x}dx & \qquad v &= x \end{aligned}$$

The integral is then,

$$\begin{aligned}\int \ln(x)\,dx &= x\ln(x) - \int \frac{1}{x}\,x\,dx\\ &= x\ln(x) - \int dx\\ &= x\ln(x) - x + c\end{aligned}$$

Example 7

Evaluate the following integral.

$$\int x^5\sqrt{x^3+1}\,dx$$

Solution

So, if we again try to use the pattern from the first few examples for this integral our choices for u and dv would probably be the following.

$$u = x^5 \qquad\qquad dv = \sqrt{x^3+1}\,dx$$

However, as with the previous example this won't work since we can't easily compute v.

$$v = \int \sqrt{x^3+1}\,dx$$

This is not an easy integral to do. However, notice that if we had an x^2 in the integral along with the root we could very easily do the integral with a substitution. Also notice that we do have a lot of x's floating around in the original integral. So instead of putting all the x's (outside of the root) in the u let's split them up as follows.

$$\begin{aligned} u &= x^3 & dv &= x^2\sqrt{x^3+1}\,dx\\ du &= 3x^2dx & v &= \frac{2}{9}\left(x^3+1\right)^{\frac{3}{2}}\end{aligned}$$

We can now easily compute v and after using integration by parts we get,

$$\begin{aligned}\int x^5\sqrt{x^3+1}\,dx &= \frac{2}{9}x^3\left(x^3+1\right)^{\frac{3}{2}} - \frac{2}{3}\int x^2\left(x^3+1\right)^{\frac{3}{2}}\,dx\\ &= \frac{2}{9}x^3\left(x^3+1\right)^{\frac{3}{2}} - \frac{4}{45}\left(x^3+1\right)^{\frac{5}{2}} + c\end{aligned}$$

So, in the previous two examples we saw cases that didn't quite fit into any perceived pattern that we might have gotten from the first couple of examples. This is always something that we need to be on the lookout for with integration by parts.

Let's take a look at another example that also illustrates another integration technique that sometimes arises out of integration by parts problems.

Example 8

Evaluate the following integral.

$$\int \mathbf{e}^{\theta} \cos(\theta)\, d\theta$$

Solution

Okay, to this point we've always picked u in such a way that upon differentiating it would make that portion go away or at the very least put it the integral into a form that would make it easier to deal with. In this case no matter which part we make u it will never go away in the differentiation process.

It doesn't much matter which we choose to be u so we'll choose in the following way. Note however that we could choose the other way as well and we'll get the same result in the end.

$$u = \cos(\theta) \qquad dv = \mathbf{e}^{\theta}\, d\theta$$
$$du = -\sin(\theta)\, d\theta \qquad v = \mathbf{e}^{\theta}$$

The integral is then,

$$\int \mathbf{e}^{\theta} \cos(\theta)\, d\theta = \mathbf{e}^{\theta} \cos(\theta) + \int \mathbf{e}^{\theta} \sin(\theta)\, d\theta$$

So, it looks like we'll do integration by parts again. Here are our choices this time.

$$u = \sin(\theta) \qquad dv = \mathbf{e}^{\theta}\, d\theta$$
$$du = \cos(\theta)\, d\theta \qquad v = \mathbf{e}^{\theta}$$

The integral is now,

$$\int \mathbf{e}^{\theta} \cos(\theta)\, d\theta = \mathbf{e}^{\theta} \cos(\theta) + \mathbf{e}^{\theta} \sin(\theta) - \int \mathbf{e}^{\theta} \cos(\theta)\, d\theta$$

Now, at this point it looks like we're just running in circles. However, notice that we now have the same integral on both sides and on the right side it's got a minus sign in front of it. This means that we can add the integral to both sides to get,

$$2\int \mathbf{e}^{\theta} \cos(\theta)\, d\theta = \mathbf{e}^{\theta} \cos(\theta) + \mathbf{e}^{\theta} \sin(\theta)$$

All we need to do now is divide by 2 and we're done. The integral is,

$$\int \mathbf{e}^{\theta} \cos(\theta)\, d\theta = \frac{1}{2}\left(\mathbf{e}^{\theta}\cos(\theta) + \mathbf{e}^{\theta}\sin(\theta)\right) + c$$

Notice that after dividing by the two we add in the constant of integration at that point.

This idea of integrating until you get the same integral on both sides of the equal sign and then simply solving for the integral is kind of nice to remember. It doesn't show up all that often, but when it does it may be the only way to actually do the integral.

Note as well that this is really just Algebra, admittedly done in a way that you may not be used to seeing it, but it is really just Algebra.

At this stage of your mathematical career everyone can solve,

$$x = 3 - x \qquad \rightarrow \qquad x = \frac{3}{2}$$

We are still solving an "equation". The only difference is that instead of solving for an x in we are solving for an integral and instead of a nice constant, "3" in the above Algebra problem, we've got a "messier" function.

We've got one more example to do. As we will see some problems could require us to do integration by parts numerous times and there is a short hand method that will allow us to do multiple applications of integration by parts quickly and easily.

Example 9

Evaluate the following integral.

$$\int x^4 \mathbf{e}^{\frac{x}{2}}\, dx$$

Solution

We start off by choosing u and dv as we always would. However, instead of computing du and v we put these into the following table. We then differentiate down the column corresponding to u until we hit zero. In the column corresponding to dv we integrate once for each entry in the first column. There is also a third column which we will explain in a bit and it always starts with a ""+"" and then alternates signs as shown.

x^4	$\mathbf{e}^{\frac{x}{2}}$	$+$
$4x^3$	$2\mathbf{e}^{\frac{x}{2}}$	$-$
$12x^2$	$4\mathbf{e}^{\frac{x}{2}}$	$+$
$24x$	$8\mathbf{e}^{\frac{x}{2}}$	$-$
24	$16\mathbf{e}^{\frac{x}{2}}$	$+$
0	$32\mathbf{e}^{\frac{x}{2}}$	$-$

Now, multiply along the diagonals shown in the table. In front of each product put the sign in the third column that corresponds to the "u" term for that product. In this case this would give,

$$\begin{aligned}\int x^4\mathbf{e}^{\frac{x}{2}}\,dx &= \left(x^4\right)\left(2\mathbf{e}^{\frac{x}{2}}\right) - \left(4x^3\right)\left(4\mathbf{e}^{\frac{x}{2}}\right) + \left(12x^2\right)\left(8\mathbf{e}^{\frac{x}{2}}\right) - \left(24x\right)\left(16\mathbf{e}^{\frac{x}{2}}\right) + \left(24\right)\left(32\mathbf{e}^{\frac{x}{2}}\right)\\ &= 2x^4\mathbf{e}^{\frac{x}{2}} - 16x^3\mathbf{e}^{\frac{x}{2}} + 96x^2\mathbf{e}^{\frac{x}{2}} - 384x\mathbf{e}^{\frac{x}{2}} + 768\mathbf{e}^{\frac{x}{2}} + c\end{aligned}$$

We've got the integral. This is much easier than writing down all the various u's and dv's that we'd have to do otherwise.

So, in this section we've seen how to do integration by parts. In your later math classes this is liable to be one of the more frequent integration techniques that you'll encounter.

It is important to not get too locked into patterns that you may think you've seen. In most cases any pattern that you think you've seen can (and will be) violated at some point in time. Be careful!

7.2 Integrals Involving Trig Functions

In this section we are going to look at quite a few integrals involving trig functions and some of the techniques we can use to help us evaluate them. Let's start off with an integral that we should already be able to do.

$$\begin{aligned}\int \cos(x)\sin^5(x)\,dx &= \int u^5\,du \qquad \text{using the substitution } u = \sin(x)\\ &= \frac{1}{6}\sin^6(x) + c\end{aligned}$$

This integral is easy to do with a substitution because the presence of the cosine, however, what about the following integral.

Example 1

Evaluate the following integral.

$$\int \sin^5(x)\,dx$$

Solution

This integral no longer has the cosine in it that would allow us to use the substitution that we used above. Therefore, that substitution won't work and we are going to have to find another way of doing this integral.

Let's first notice that we could write the integral as follows,

$$\int \sin^5(x)\,dx = \int \sin^4(x)\,\sin(x)\,dx = \int \left(\sin^2(x)\right)^2\,\sin(x)\,dx$$

Now recall the trig identity,

$$\cos^2(x) + \sin^2(x) = 1 \qquad \Rightarrow \qquad \sin^2(x) = 1 - \cos^2(x)$$

With this identity the integral can be written as,

$$\int \sin^5(x)\,dx = \int \left(1 - \cos^2(x)\right)^2\,\sin(x)\,dx$$

and we can now use the substitution $u = \cos x$. Doing this gives us,

$$\begin{aligned}\int \sin^5(x)\,dx &= -\int \left(1-u^2\right)^2\,du\\ &= -\int 1 - 2u^2 + u^4\,du\\ &= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + c\\ &= -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + c\end{aligned}$$

So, with a little rewriting on the integrand we were able to reduce this to a fairly simple substitution.

Notice that we were able to do the rewrite that we did in the previous example because the exponent on the sine was odd. In these cases all that we need to do is strip out one of the sines. The exponent on the remaining sines will then be even and we can easily convert the remaining sines to cosines using the identity,

$$\cos^2(x) + \sin^2(x) = 1 \tag{7.1}$$

If the exponent on the sines had been even this would have been difficult to do. We could strip out a sine, but the remaining sines would then have an odd exponent and while we could convert them to cosines the resulting integral would often be even more difficult than the original integral in most cases.

Let's take a look at another example.

Example 2

Evaluate the following integral.

$$\int \sin^6(x)\cos^3(x)\,dx$$

Solution

So, in this case we've got both sines and cosines in the problem and in this case the exponent on the sine is even while the exponent on the cosine is odd. So, we can use a similar technique in this integral. This time we'll strip out a cosine and convert the rest to sines.

$$\begin{aligned}
\int \sin^6(x)\cos^3(x)\,dx &= \int \sin^6(x)\cos^2(x)\,\cos(x)\,dx \\
&= \int \sin^6(x)\left(1-\sin^2(x)\right)\,\cos(x)\,dx \qquad\qquad u = \sin(x) \\
&= \int u^6\left(1-u^2\right)\,du \\
&= \int u^6 - u^8\,du \\
&= \frac{1}{7}\sin^7(x) - \frac{1}{9}\sin^9(x) + c
\end{aligned}$$

At this point let's pause for a second to summarize what we've learned so far about integrating powers of sine and cosine.

$$\int \sin^n(x)\cos^m(x)\,dx \tag{7.2}$$

In this integral if the exponent on the sines (n) is odd we can strip out one sine, convert the rest to cosines using Equation 7.1 and then use the substitution $u = \cos(x)$. Likewise, if the exponent on the cosines (m) is odd we can strip out one cosine and convert the rest to sines and the use the substitution $u = \sin(x)$.

Of course, if both exponents are odd then we can use either method. However, in these cases it's usually easier to convert the term with the smaller exponent.

The one case we haven't looked at is what happens if both of the exponents are even? In this case the technique we used in the first couple of examples simply won't work and in fact there really isn't any one set method for doing these integrals. Each integral is different and in some cases there will be more than one way to do the integral.

With that being said most, if not all, of integrals involving products of sines and cosines in which both exponents are even can be done using one or more of the following formulas to rewrite the integrand.

$$\begin{aligned}\cos^2(x) &= \frac{1}{2}\left(1+\cos(2x)\right)\\ \sin^2(x) &= \frac{1}{2}\left(1-\cos(2x)\right)\\ \sin(x)\cos(x) &= \frac{1}{2}\sin(2x)\end{aligned}$$

The first two formulas are the standard half angle formula from a trig class written in a form that will be more convenient for us to use. The last is the standard double angle formula for sine, again with a small rewrite.

Let's take a look at an example.

Example 3

Evaluate the following integral.

$$\int \sin^2(x)\cos^2(x)\,dx$$

Solution

As noted above there are often more than one way to do integrals in which both of the exponents are even. This integral is an example of that. There are at least two solution techniques for this problem. We will do both solutions starting with what is probably the longer of the two, but it's also the one that many people see first.

Solution 1

In this solution we will use the two half angle formulas above and just substitute them into the integral.

$$\begin{aligned}\int \sin^2(x)\cos^2(x)\,dx &= \int \frac{1}{2}\left(1-\cos(2x)\right)\left(\frac{1}{2}\right)\left(1+\cos(2x)\right)dx \\ &= \frac{1}{4}\int 1-\cos^2(2x)\,dx\end{aligned}$$

So, we still have an integral that can't be completely done, however notice that we have managed to reduce the integral down to just one term causing problems (a cosine with an even power) rather than two terms causing problems.

In fact to eliminate the remaining problem term all that we need to do is reuse the first half angle formula given above.

$$\begin{aligned}\int \sin^2(x)\cos^2(x)\,dx &= \frac{1}{4}\int 1-\frac{1}{2}\left(1+\cos(4x)\right)dx \\ &= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos(4x)\,dx \\ &= \frac{1}{4}\left(\frac{1}{2}x-\frac{1}{8}\sin(4x)\right)+c \\ &= \frac{1}{8}x-\frac{1}{32}\sin(4x)+c\end{aligned}$$

So, this solution required a total of three trig identities to complete.

Solution 2

In this solution we will use the double angle formula to help simplify the integral as follows.

$$\begin{aligned}\int \sin^2(x)\cos^2(x)\,dx &= \int \left(\sin(x)\cos(x)\right)^2 dx \\ &= \int \left(\frac{1}{2}\sin(2x)\right)^2 dx \\ &= \frac{1}{4}\int \sin^2(2x)\,dx\end{aligned}$$

Now, we use the half angle formula for sine to reduce to an integral that we can do.

$$\begin{aligned}\int \sin^2(x)\cos^2(x)\,dx &= \frac{1}{8}\int 1-\cos(4x)\,dx \\ &= \frac{1}{8}x-\frac{1}{32}\sin(4x)+c\end{aligned}$$

This method required only two trig identities to complete.

Notice that the difference between these two methods is more one of "messiness". The second method is not appreciably easier (other than needing one less trig identity) it is just not as messy and that will often translate into an "easier" process.

In the previous example we saw two different solution methods that gave the same answer. Note that this will not always happen. In fact, more often than not we will get different answers. However, as we discussed in the Integration by Parts section, the two answers will differ by no more than a constant.

In general, when we have products of sines and cosines in which both exponents are even we will need to use a series of half angle and/or double angle formulas to reduce the integral into a form that we can integrate. Also, the larger the exponents the more we'll need to use these formulas and hence the messier the problem.

Sometimes in the process of reducing integrals in which both exponents are even we will run across products of sine and cosine in which the arguments are different. These will require one of the following formulas to reduce the products to integrals that we can do.

$$\begin{aligned} \sin(\alpha)\cos(\beta) &= \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right] \\ \sin(\alpha)\sin(\beta) &= \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right] \\ \cos(\alpha)\cos(\beta) &= \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right] \end{aligned}$$

Let's take a look at an example of one of these kinds of integrals.

Example 4

Evaluate the following integral.

$$\int \cos(15x)\cos(4x)\,dx$$

Solution

This integral requires the last formula listed above.

$$\begin{aligned} \int \cos(15x)\cos(4x)\,dx &= \frac{1}{2}\int \cos(11x)+\cos(19x)\,dx \\ &= \frac{1}{2}\left(\frac{1}{11}\sin(11x)+\frac{1}{19}\sin(19x)\right)+c \end{aligned}$$

Okay, at this point we've covered pretty much all the possible cases involving products of sines and cosines. It's now time to look at integrals that involve products of secants and tangents.

This time, let's do a little analysis of the possibilities before we just jump into examples. The general integral will be,

$$\int \sec^n(x)\tan^m(x)\,dx \tag{7.3}$$

The first thing to notice is that we can easily convert even powers of secants to tangents and even powers of tangents to secants by using a formula similar to Equation 7.1. In fact, the formula can be derived from Equation 7.1 so let's do that.

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} &= \frac{1}{\cos^2(x)} \\ \tan^2(x) + 1 &= \sec^2(x) \end{aligned} \tag{7.4}$$

Now, we're going to want to deal with Equation 7.3 similarly to how we dealt with Equation 7.2. We'll want to eventually use one of the following substitutions.

$$\begin{aligned} u &= \tan(x) & du &= \sec^2(x)\,dx \\ u &= \sec(x) & du &= \sec(x)\tan(x)\,dx \end{aligned}$$

So, if we use the substitution $u = \tan(x)$ we will need two secants left for the substitution to work. This means that if the exponent on the secant (n) is even we can strip two out and then convert the remaining secants to tangents using Equation 7.4.

Next, if we want to use the substitution $u = \sec(x)$ we will need one secant and one tangent left over in order to use the substitution. This means that if the exponent on the tangent (m) is odd and we have at least one secant in the integrand we can strip out one of the tangents along with one of the secants of course. The tangent will then have an even exponent and so we can use Equation 7.4 to convert the rest of the tangents to secants. Note that this method does require that we have at least one secant in the integral as well. If there aren't any secants then we'll need to do something different.

If the exponent on the secant is even and the exponent on the tangent is odd then we can use either case. Again, it will be easier to convert the term with the smallest exponent.

Let's take a look at a couple of examples.

Example 5

Evaluate the following integral.

$$\int \sec^9(x)\tan^5(x)\,dx$$

Solution

First note that since the exponent on the secant isn't even we can't use the substitution $u = \tan(x)$. However, the exponent on the tangent is odd and we've got a secant in the integral and so we will be able to use the substitution $u = \sec(x)$. This means stripping out a single tangent (along with a secant) and converting the remaining tangents to secants using Equation 7.4.

Here's the work for this integral.

$$\begin{aligned}\int \sec^9(x)\tan^5(x)\,dx &= \int \sec^8(x)\tan^4(x)\,\tan(x)\sec(x)\,dx\\ &= \int \sec^8(x)\left(\sec^2(x)-1\right)^2\,\tan(x)\sec(x)\,dx \qquad u=\sec(x)\\ &= \int u^8\left(u^2-1\right)^2\,du\\ &= \int u^{12}-2u^{10}+u^8\,du\\ &= \frac{1}{13}\sec^{13}(x)-\frac{2}{11}\sec^{11}(x)+\frac{1}{9}\sec^9(x)+c\end{aligned}$$

Example 6

Evaluate the following integral.

$$\int \sec^4(x)\tan^6(x)\,dx$$

Solution

So, in this example the exponent on the tangent is even so the substitution $u=\sec(x)$ won't work. The exponent on the secant is even and so we can use the substitution $u=\tan(x)$ for this integral. That means that we need to strip out two secants and convert the rest to tangents. Here is the work for this integral.

$$\begin{aligned}\int \sec^4(x)\tan^6(x)\,dx &= \int \sec^2(x)\tan^6(x)\sec^2(x)\,dx\\ &= \int \left(\tan^2(x)+1\right)\tan^6(x)\sec^2(x)\,dx \qquad u=\tan(x)\\ &= \int \left(u^2+1\right)u^6\,du\\ &= \int u^8+u^6\,du\\ &= \frac{1}{9}\tan^9(x)+\frac{1}{7}\tan^7(x)+c\end{aligned}$$

Both of the previous examples fit very nicely into the patterns discussed above and so were not all that difficult to work. However, there are a couple of exceptions to the patterns above and in these cases there is no single method that will work for every problem. Each integral will be different and may require different solution methods in order to evaluate the integral.

Let's first take a look at a couple of integrals that have odd exponents on the tangents, but no secants. In these cases we can't use the substitution $u = \sec(x)$ since it requires there to be at least one secant in the integral.

Example 7

Evaluate the following integral.

$$\int \tan(x)\,dx$$

Solution

To do this integral all we need to do is recall the definition of tangent in terms of sine and cosine and then this integral is nothing more than a Calculus I substitution.

$$\begin{aligned}\int \tan(x)\,dx &= \int \frac{\sin(x)}{\cos(x)}\,dx & u &= \cos(x)\\ &= -\int \frac{1}{u}\,du & & \\ &= -\ln\left|\cos(x)\right| + c & r\ln(x) &= \ln\left(x^r\right)\\ &= \ln\left|\cos(x)\right|^{-1} + c & & \\ &= \ln\left|\sec(x)\right| + c & & \end{aligned}$$

Note that for many folks,

$$\int \tan(x)\,dx = -\ln\left|\cos(x)\right| + c$$

We went a step or two further with some simplification. The simplification was done solely to eliminate the minus sign that was in front of the logarithm. This does not have to be done in general, but it is always easy to lose minus signs and in this case it was easy to eliminate it without introducing any real complexity to the answer and so we did.

Example 8

Evaluate the following integral.

$$\int \tan^3(x)\,dx$$

Solution

The trick to this one is do the following manipulation of the integrand.

$$\begin{aligned}\int \tan^3(x)\,dx &= \int \tan(x)\tan^2(x)\,dx\\ &= \int \tan(x)\left(\sec^2(x)-1\right)\,dx\\ &= \int \tan(x)\sec^2(x)\,dx - \int \tan(x)\,dx\end{aligned}$$

We can now use the substitution $u = \tan x$ on the first integral and the results from the previous example on the second integral.

The integral is then,

$$\int \tan^3(x)\,dx = \frac{1}{2}\tan^2(x) - \ln\left|\sec(x)\right| + c$$

Note that all odd powers of tangent (with the exception of the first power) can be integrated using the same method we used in the previous example. For instance,

$$\int \tan^5(x)\,dx = \int \tan^3(x)\left(\sec^2(x)-1\right)\,dx = \int \tan^3(x)\sec^2(x)\,dx - \int \tan^3(x)\,dx$$

So, a quick substitution ($u = \tan(x)$) will give us the first integral and the second integral will always be the previous odd power.

Now let's take a look at a couple of examples in which the exponent on the secant is odd and the exponent on the tangent is even. In these cases the substitutions used above won't work.

It should also be noted that both of the following two integrals are integrals that we'll be seeing on occasion in later sections of this chapter and in later chapters. Because of this it wouldn't be a bad idea to make a note of these results so you'll have them ready when you need them later.

Example 9

Evaluate the following integral.

$$\int \sec(x)\,dx$$

Solution

This one isn't too bad once you see what you've got to do. By itself the integral can't be done. However, if we manipulate the integrand as follows we can do it.

$$\begin{aligned}\int \sec(x)\,dx &= \int \frac{\sec(x)\big(\sec(x)+\tan(x)\big)}{\sec(x)+\tan(x)}\,dx \\ &= \int \frac{\sec^2(x)+\tan(x)\sec(x)}{\sec(x)+\tan(x)}\,dx\end{aligned}$$

In this form we can do the integral using the substitution $u = \sec(x) + \tan(x)$. Doing this gives,

$$\int \sec(x)\,dx = \ln\left|\sec(x)+\tan(x)\right| + c$$

The idea used in the above example is a nice idea to keep in mind. Multiplying the numerator and denominator of a term by the same term above can, on occasion, put the integral into a form that can be integrated. Note that this method won't always work and even when it does it won't always be clear what you need to multiply the numerator and denominator by. However, when it does work and you can figure out what term you need it can greatly simplify the integral.

Here's the next example.

Example 10

Evaluate the following integral.

$$\int \sec^3(x)\,dx$$

Solution

This one is different from any of the other integrals that we've done in this section. The first step to doing this integral is to perform integration by parts using the following choices for u and dv.

$$\begin{aligned} u &= \sec(x) & dv &= \sec^2(x)\,dx \\ du &= \sec(x)\tan(x)\,dx & v &= \tan(x)\end{aligned}$$

Note that using integration by parts on this problem is not an obvious choice, but it does work very nicely here. After doing integration by parts we have,

$$\int \sec^3(x)\,dx = \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$$

Now the new integral also has an odd exponent on the secant and an even exponent on the tangent and so the previous examples of products of secants and tangents still won't do us any good.

To do this integral we'll first write the tangents in the integral in terms of secants. Again, this is not necessarily an obvious choice but it's what we need to do in this case.

$$\begin{aligned}\int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \sec(x)\left(\sec^2(x) - 1\right)dx \\ &= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx\end{aligned}$$

Now, we can use the results from the previous example to do the second integral and notice that the first integral is exactly the integral we're being asked to evaluate with a minus sign in front. So, add it to both sides to get,

$$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln\left|\sec(x) + \tan(x)\right|$$

Finally divide by two and we're done.

$$\int \sec^3(x)\,dx = \frac{1}{2}\left(\sec(x)\tan(x) + \ln\left|\sec(x) + \tan(x)\right|\right) + c$$

Again, note that we've again used the idea of integrating the right side until the original integral shows up and then moving this to the left side and dividing by its coefficient to complete the evaluation. We first saw this in the Integration by Parts section and noted at the time that this was a nice technique to remember. Here is another example of this technique.

Now that we've looked at products of secants and tangents let's also acknowledge that because we can relate cosecants and cotangents by

$$1 + \cot^2(x) = \csc^2(x)$$

all of the work that we did for products of secants and tangents will also work for products of cosecants and cotangents. We'll leave it to you to verify that.

There is one final topic to be discussed in this section before moving on.

To this point we've looked only at products of sines and cosines and products of secants and tangents. However, the methods used to do these integrals can also be used on some quotients involving sines and cosines and quotients involving secants and tangents (and hence quotients involving cosecants and cotangents).

Let's take a quick look at an example of this.

Example 11

Evaluate the following integral.

$$\int \frac{\sin^7(x)}{\cos^4(x)}\,dx$$

Solution

If this were a product of sines and cosines we would know what to do. We would strip out a sine (since the exponent on the sine is odd) and convert the rest of the sines to cosines. The same idea will work in this case. We'll strip out a sine from the numerator and convert the rest to cosines as follows,

$$\begin{aligned}\int \frac{\sin^7(x)}{\cos^4(x)}\,dx &= \int \frac{\sin^6(x)}{\cos^4(x)}\sin(x)\,dx\\ &= \int \frac{\left(\sin^2(x)\right)^3}{\cos^4(x)}\sin(x)\,dx\\ &= \int \frac{\left(1-\cos^2(x)\right)^3}{\cos^4(x)}\sin(x)\,dx\end{aligned}$$

At this point all we need to do is use the substitution $u = \cos(x)$ and we're done.

$$\begin{aligned}\int \frac{\sin^7(x)}{\cos^4(x)}\,dx &= -\int \frac{\left(1-u^2\right)^3}{u^4}\,du\\ &= -\int u^{-4} - 3u^{-2} + 3 - u^2\,du\\ &= -\left(-\frac{1}{3}\frac{1}{u^3} + 3\frac{1}{u} + 3u - \frac{1}{3}u^3\right) + c\\ &= \frac{1}{3\cos^3(x)} - \frac{3}{\cos(x)} - 3\cos(x) + \frac{1}{3}\cos^3(x) + c\end{aligned}$$

So, under the right circumstances, we can use the ideas developed to help us deal with products of trig functions to deal with quotients of trig functions. The natural question then, is just what are the right circumstances?

First notice that if the quotient had been reversed as in this integral,

$$\int \frac{\cos^4(x)}{\sin^7(x)}\,dx$$

we wouldn't have been able to strip out a sine.

$$\int \frac{\cos^4(x)}{\sin^7(x)}\,dx = \int \frac{\cos^4(x)}{\sin^6(x)}\,\frac{1}{\sin(x)}\,dx$$

In this case the "stripped out" sine remains in the denominator and it won't do us any good for the substitution $u = \cos(x)$ since this substitution requires a sine in the numerator of the quotient. Also note that, while we could convert the sines to cosines, the resulting integral would still be a fairly difficult integral.

So, we can use the methods we applied to products of trig functions to quotients of trig functions provided the term that needs parts stripped out in is the numerator of the quotient.

7.3 Trig Substitutions

As we have done in the last couple of sections, let's start off with a couple of integrals that we should already be able to do with a standard substitution.

$$\int x\sqrt{25x^2-4}\,dx = \frac{1}{75}\left(25x^2-4\right)^{\frac{3}{2}} + c \qquad \int \frac{x}{\sqrt{25x^2-4}}\,dx = \frac{1}{25}\sqrt{25x^2-4} + c$$

Both of these used the substitution $u = 25x^2 - 4$ and at this point should be pretty easy for you to do. However, let's take a look at the following integral.

Example 1

Evaluate the following integral.

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx$$

Solution

In this case the substitution $u = 25x^2 - 4$ will not work (we don't have the $x\,dx$ in the numerator the substitution needs) and so we're going to have to do something different for this integral.

It would be nice if we could reduce the two terms in the root down to a single term somehow. The following substitution will do that for us.

$$x = \frac{2}{5}\sec(\theta)$$

Do not worry about where this came from at this point. As we work the problem you will see that it works and that if we have a similar type of square root in the problem we can always use a similar substitution.

Before we actually do the substitution however let's verify the claim that this will allow us to reduce the two terms in the root to a single term.

$$\sqrt{25x^2-4} = \sqrt{25\left(\frac{4}{25}\right)\sec^2(\theta) - 4} = \sqrt{4\left(\sec^2(\theta)-1\right)} = 2\sqrt{\sec^2(\theta)-1}$$

Now reduce the two terms to a single term all we need to do is recall the relationship,

$$\tan^2(\theta) + 1 = \sec^2(\theta) \qquad \Rightarrow \qquad \sec^2(\theta) - 1 = \tan^2(\theta)$$

Using this fact the square root becomes,

$$\sqrt{25x^2-4} = 2\sqrt{\tan^2(\theta)} = 2\left|\tan(\theta)\right|$$

So, not only were we able to reduce the two terms to a single term in the process we were able to easily eliminate the root as well!

Note, however, the presence of the absolute value bars. These are important. Recall that

$$\sqrt{x^2} = |x|$$

There should always be absolute value bars at this stage. If we knew that $\tan(\theta)$ was always positive or always negative we could eliminate the absolute value bars using,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Without limits we won't be able to determine if $\tan(\theta)$ is positive or negative, however, we will need to eliminate them in order to do the integral. Therefore, since we are doing an indefinite integral we will assume that $\tan(\theta)$ will be positive and so we can drop the absolute value bars. This gives,

$$\sqrt{25x^2 - 4} = 2\tan(\theta)$$

So, we were able to reduce the two terms under the root to a single term with this substitution and in the process eliminate the root as well. Eliminating the root is a nice side effect of this substitution as the problem will now become somewhat easier to do.

Let's now do the substitution and see what we get. In doing the substitution don't forget that we'll also need to substitute for the dx. This is easy enough to get from the substitution.

$$x = \frac{2}{5}\sec(\theta) \qquad \Rightarrow \qquad dx = \frac{2}{5}\sec(\theta)\tan(\theta)\,d\theta$$

Using this substitution the integral becomes,

$$\begin{aligned} \int \frac{\sqrt{25x^2 - 4}}{x}\,dx &= \int \frac{2\tan(\theta)}{\frac{2}{5}\sec(\theta)} \left(\frac{2}{5}\sec(\theta)\tan(\theta)\right) d\theta \\ &= 2\int \tan^2(\theta)\,d\theta \end{aligned}$$

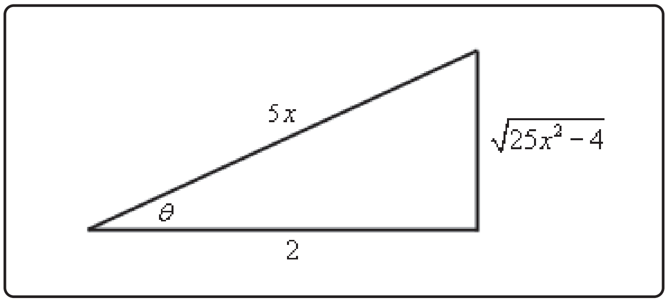
With this substitution we were able to reduce the given integral to an integral involving trig functions and we saw how to do these problems in the previous section. Let's finish the integral.

$$\begin{aligned} \int \frac{\sqrt{25x^2 - 4}}{x}\,dx &= 2\int \sec^2(\theta) - 1\,d\theta \\ &= 2\big(\tan(\theta) - \theta\big) + c \end{aligned}$$

So, we've got an answer for the integral. Unfortunately, the answer isn't given in x's as it should be. So, we need to write our answer in terms of x. We can do this with some right triangle trig. From our original substitution we have,

$$\sec(\theta) = \frac{5x}{2} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

This gives the following right triangle.



From this we can see that,

$$\tan(\theta) = \frac{\sqrt{25x^2-4}}{2}$$

We can deal with the θ in one of any variety of ways. From our substitution we can see that,

$$\theta = \sec^{-1}\left(\frac{5x}{2}\right)$$

While this is a perfectly acceptable method of dealing with the θ we can use any of the possible six inverse trig functions and since sine and cosine are the two trig functions most people are familiar with we will usually use the inverse sine or inverse cosine. In this case we'll use the inverse cosine.

$$\theta = \cos^{-1}\left(\frac{2}{5x}\right)$$

So, with all of this the integral becomes,

$$\begin{aligned}\int \frac{\sqrt{25x^2-4}}{x}\,dx &= 2\left(\frac{\sqrt{25x^2-4}}{2} - \cos^{-1}\left(\frac{2}{5x}\right)\right) + c \\ &= \sqrt{25x^2-4} - 2\cos^{-1}\left(\frac{2}{5x}\right) + c\end{aligned}$$

We now have the answer back in terms of x.

Wow! That was a lot of work. Most of these won't take as long to work however. This first one needed lots of explanation since it was the first one. The remaining examples won't need quite as much explanation and so won't take as long to work.

However, before we move onto more problems let's first address the issue of definite integrals and how the process differs in these cases.

Example 2

Evaluate the following integral.

$$\int_{\frac{2}{5}}^{\frac{4}{5}} \frac{\sqrt{25x^2 - 4}}{x}\, dx$$

Solution

The limits here won't change the substitution so that will remain the same.

$$x = \frac{2}{5}\sec(\theta)$$

Using this substitution the square root still reduces down to,

$$\sqrt{25x^2 - 4} = 2\left|\tan(\theta)\right|$$

However, unlike the previous example we can't just drop the absolute value bars. In this case we've got limits on the integral and so we can use the limits as well as the substitution to determine the range of θ that we're in. Once we've got that we can determine how to drop the absolute value bars.

Here's the limits of θ and note that if you aren't good at solving trig equations in terms of secant you can always convert to cosine as we do below.

$$x = \frac{2}{5} \; : \; \frac{2}{5} = \frac{2}{5}\sec(\theta) = \frac{2}{5}\frac{1}{\cos(\theta)} \quad \rightarrow \quad \cos(\theta) = 1 \quad \Rightarrow \quad \theta = \cos^{-1}(1) = 0$$

$$x = \frac{4}{5} \; : \; \frac{4}{5} = \frac{2}{5}\sec(\theta) = \frac{2}{5}\frac{1}{\cos(\theta)} \quad \rightarrow \quad \cos(\theta) = \frac{1}{2} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Now, we know from solving trig equations, that there are in fact an infinite number of possible answers we could use. In fact, the more "correct" answer for the above work is,

$$\theta = 0 + 2\pi n = 2\pi n \qquad \& \qquad \theta = \frac{\pi}{3} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

So, which ones should we use? The answer is simple. When using a secant trig substitution and converting the limits we always assume that θ is in the range of inverse secant. Or,

$$\text{If } \theta = \sec^{-1}(x) \text{ then } 0 \le \theta < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < \theta \le \pi$$

Note that we have to avoid $\theta = \frac{\pi}{2}$ because secant will not exist at that point. Also note that the range of θ was given in terms of secant even though we actually used inverse cosine to get the answers. This will not be a problem because even though inverse cosine can give $\theta = \frac{\pi}{2}$ we'll never get it in our work above because that would require that we started with the secant being undefined and that will not happen when converting the limits as that would in turn require one of the limits to also be undefined!

So, in finding the new limits we didn't need all possible values of θ we just need the inverse cosine answers we got when we converted the limits. Therefore, if we are in the range $\frac{2}{5} \le x \le \frac{4}{5}$ then θ is in the range of $0 \le \theta \le \frac{\pi}{3}$ and in this range of θ's tangent is positive and so we can just drop the absolute value bars.

Let's do the substitution. Note that the work is identical to the previous example and so most of it is left out. We'll pick up at the final integral and then do the substitution.

$$\begin{aligned} \int_{\frac{2}{5}}^{\frac{4}{5}} \frac{\sqrt{25x^2 - 4}}{x}\, dx &= 2\int_0^{\frac{\pi}{3}} \sec^2(\theta) - 1\, d\theta \\ &= 2\left(\tan(\theta) - \theta\right)\Big|_0^{\pi/3} \\ &= 2\sqrt{3} - \frac{2\pi}{3} \end{aligned}$$

Note that because of the limits we didn't need to resort to a right triangle to complete the problem.

Let's take a look at a different set of limits for this integral.

Example 3

Evaluate the following integral.

$$\int_{-\frac{4}{5}}^{-\frac{2}{5}} \frac{\sqrt{25x^2 - 4}}{x}\, dx$$

Solution

Again, the substitution and square root are the same as the first two examples.

$$x = \frac{2}{5}\sec(\theta) \qquad \sqrt{25x^2 - 4} = 2\left|\tan(\theta)\right|$$

Let's next see the limits θ for this problem.

$$x = -\frac{2}{5} \;:\; -\frac{2}{5} = \frac{2}{5}\sec(\theta) = \frac{2}{5}\frac{1}{\cos(\theta)} \quad \rightarrow \quad \cos(\theta) = -1 \quad \Rightarrow \quad \theta = \cos^{-1}(-1) = \pi$$
$$x = -\frac{4}{5} \;:\; -\frac{4}{5} = \frac{2}{5}\sec(\theta) = \frac{2}{5}\frac{1}{\cos(\theta)} \quad \rightarrow \quad \cos(\theta) = -\frac{1}{2} \quad \Rightarrow \quad \theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

Remember that in converting the limits we use the results from the inverse secant/cosine. So, for this range of x's we have $\frac{2\pi}{3} \le \theta \le \pi$ and in this range of θ tangent is negative and so in this case we can drop the absolute value bars, but will need to add in a minus sign upon doing so. In other words,

$$\sqrt{25x^2 - 4} = -2\tan(\theta)$$

So, the only change this will make in the integration process is to put a minus sign in front of the integral. The integral is then,

$$\begin{aligned}\int_{-\frac{4}{5}}^{-\frac{2}{5}} \frac{\sqrt{25x^2 - 4}}{x}\,dx &= -2\int_{\frac{2\pi}{3}}^{\pi} \sec^2(\theta) - 1\,d\theta \\ &= -2\big(\tan(\theta) - \theta\big)\Big|_{2\pi/3}^{\pi} \\ &= \frac{2\pi}{3} - 2\sqrt{3}\end{aligned}$$

In the last two examples we saw that we have to be very careful with definite integrals. We need to make sure that we determine the limits on θ and whether or not this will mean that we can just drop the absolute value bars or if we need to add in a minus sign when we drop them.

Before moving on to the next example let's get the general form for the secant trig substitution that we used in the previous set of examples and the assumed limits on θ.

Fact

$$\sqrt{b^2x^2 - a^2} \quad \Rightarrow \quad x = \frac{a}{b}\sec(\theta), \qquad 0 \le \theta < \frac{\pi}{2},\; \frac{\pi}{2} < \theta \le \pi$$

Let's work a new and different type of example.

Example 4

Evaluate the following integral.

$$\int \frac{1}{x^4\sqrt{9-x^2}}\,dx$$

Solution

Now, the terms under the root in this problem looks to be (almost) the same as the previous ones so let's try the same type of substitution and see if it will work here as well.

$$x = 3\sec(\theta)$$

Using this substitution, the square root becomes,

$$\sqrt{9-x^2} = \sqrt{9-9\sec^2(\theta)} = 3\sqrt{1-\sec^2(\theta)} = 3\sqrt{-\tan^2(\theta)}$$

So, using this substitution we will end up with a negative quantity (the tangent squared is always positive of course) under the square root and this will be trouble. Using this substitution will give complex values and we don't want that. So, using secant for the substitution won't work.

However, the following substitution (and differential) will work.

$$x = 3\sin(\theta) \qquad\qquad dx = 3\cos(\theta)\,d\theta$$

With this substitution the square root is,

$$\sqrt{9-x^2} = 3\sqrt{1-\sin^2(\theta)} = 3\sqrt{\cos^2(\theta)} = 3\left|\cos(\theta)\right| = 3\cos(\theta)$$

We were able to drop the absolute value bars because we are doing an indefinite integral and so we'll assume that everything is positive.

The integral is now,

$$\begin{aligned}\int \frac{1}{x^4\sqrt{9-x^2}}\,dx &= \int \frac{1}{81\sin^4(\theta)\left(3\cos(\theta)\right)}\,3\cos(\theta)\,d\theta \\ &= \frac{1}{81}\int \frac{1}{\sin^4(\theta)}\,d\theta \\ &= \frac{1}{81}\int \csc^4(\theta)\,d\theta\end{aligned}$$

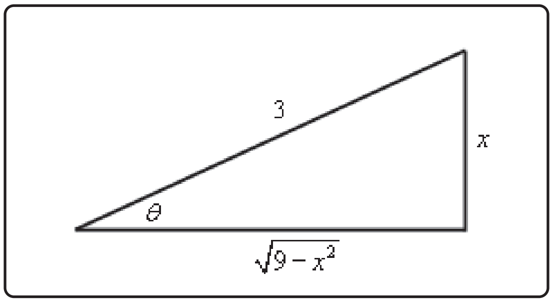
In the previous section we saw how to deal with integrals in which the exponent on the secant was even and since cosecants behave an awful lot like secants we should be able to do something similar with this.

Here is the integral.

$$\begin{aligned}\int \frac{1}{x^4\sqrt{9-x^2}}\,dx &= \frac{1}{81}\int \csc^2(\theta)\,\csc^2(\theta)\,d\theta \\ &= \frac{1}{81}\int \left(\cot^2(\theta)+1\right)\,\csc^2(\theta)\,d\theta \qquad u=\cot(\theta) \\ &= -\frac{1}{81}\int u^2+1\,du \\ &= -\frac{1}{81}\left(\frac{1}{3}\cot^3(\theta)+\cot(\theta)\right)+c\end{aligned}$$

Now we need to go back to x's using a right triangle. Here is the right triangle for this problem and trig functions for this problem.

$$\sin(\theta)=\frac{x}{3} \qquad \cot(\theta)=\frac{\sqrt{9-x^2}}{x}$$



The integral is then,

$$\begin{aligned}\int \frac{1}{x^4\sqrt{9-x^2}}\,dx &= -\frac{1}{81}\left(\frac{1}{3}\left(\frac{\sqrt{9-x^2}}{x}\right)^3+\frac{\sqrt{9-x^2}}{x}\right)+c \\ &= -\frac{\left(9-x^2\right)^{\frac{3}{2}}}{243x^3}-\frac{\sqrt{9-x^2}}{81x}+c\end{aligned}$$

We aren't going to be doing a definite integral example with a sine trig substitution. However, if we had we would need to convert the limits and that would mean eventually needing to evaluate an inverse sine. So, much like with the secant trig substitution, the values of θ that we'll use will be those from the inverse sine or,

$$\text{If } \theta=\sin^{-1}(x) \text{ then } -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$

Here is a summary for the sine trig substitution.

Fact

$$\sqrt{a^2 - b^2x^2} \quad \Rightarrow \quad x = \frac{a}{b}\sin(\theta), \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

There is one final case that we need to look at. The next integral will also contain something that we need to make sure we can deal with.

Example 5

Evaluate the following integral.

$$\int_0^{\frac{1}{6}} \frac{x^5}{(36x^2+1)^{\frac{3}{2}}}\,dx$$

Solution

First, notice that there really is a square root in this problem even though it isn't explicitly written out. To see the root let's rewrite things a little.

$$(36x^2+1)^{\frac{3}{2}} = \left((36x^2+1)^{\frac{1}{2}}\right)^3 = \left(\sqrt{36x^2+1}\right)^3$$

This terms under the root are not in the form we saw in the previous examples. Here we will use the substitution for this root.

$$x = \frac{1}{6}\tan(\theta) \qquad dx = \frac{1}{6}\sec^2(\theta)\,d\theta$$

With this substitution the denominator becomes,

$$\left(\sqrt{36x^2+1}\right)^3 = \left(\sqrt{\tan^2(\theta)+1}\right)^3 = \left(\sqrt{\sec^2(\theta)}\right)^3 = |\sec(\theta)|^3$$

Now, because we have limits we'll need to convert them to θ so we can determine how to drop the absolute value bars.

$$\begin{aligned} x = 0 \ &: \ 0 = \frac{1}{6}\tan(\theta) & \Rightarrow \quad & \theta = \tan^{-1}(0) = 0 \\ x = \frac{1}{6} \ &: \ \frac{1}{6} = \frac{1}{6}\tan(\theta) & \Rightarrow \quad & \theta = \tan^{-1}(1) = \frac{\pi}{4} \end{aligned}$$

As with the previous two cases when converting limits here we will use the results of the inverse tangent or,

$$\text{If } \theta = \tan^{-1}(x) \text{ then } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

So, in this range of θ secant is positive and so we can drop the absolute value bars.

Here is the integral,

$$\int_0^{\frac{1}{6}} \frac{x^5}{(36x^2+1)^{\frac{3}{2}}}\,dx = \int_0^{\frac{\pi}{4}} \frac{\frac{1}{7776}\tan^5(\theta)}{\sec^3(\theta)}\left(\frac{1}{6}\sec^2(\theta)\right)d\theta$$
$$= \frac{1}{46656}\int_0^{\frac{\pi}{4}} \frac{\tan^5(\theta)}{\sec(\theta)}d\theta$$

There are several ways to proceed from this point. Normally with an odd exponent on the tangent we would strip one of them out and convert to secants. However, that would require that we also have a secant in the numerator which we don't have. Therefore, it seems like the best way to do this one would be to convert the integrand to sines and cosines.

$$\int_0^{\frac{1}{6}} \frac{x^5}{(36x^2+1)^{\frac{3}{2}}}\,dx = \frac{1}{46656}\int_0^{\frac{\pi}{4}} \frac{\sin^5(\theta)}{\cos^4(\theta)}d\theta$$
$$= \frac{1}{46656}\int_0^{\frac{\pi}{4}} \frac{\left(1-\cos^2(\theta)\right)^2}{\cos^4(\theta)}\sin(\theta)d\theta$$

We can now use the substitution $u = \cos(\theta)$ and we might as well convert the limits as well.

$$\theta = 0 \qquad\qquad u = \cos(0) = 1$$
$$\theta = \frac{\pi}{4} \qquad\qquad u = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

The integral is then,

$$\int_0^{\frac{1}{6}} \frac{x^5}{(36x^2+1)^{\frac{3}{2}}}\,dx = -\frac{1}{46656}\int_1^{\frac{\sqrt{2}}{2}} u^{-4} - 2u^{-2} + 1\,du$$
$$= -\frac{1}{46656}\left(-\frac{1}{3u^3} + \frac{2}{u} + u\right)\Bigg|_1^{\frac{\sqrt{2}}{2}}$$
$$= \frac{1}{17496} - \frac{11\sqrt{2}}{279936}$$

Here is a summary for this final type of trig substitution.

Fact

$$\sqrt{a^2+b^2x^2} \quad\Rightarrow\quad x = \frac{a}{b}\tan(\theta), \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Before proceeding with some more examples let's discuss just how we knew to use the substitutions that we did in the previous examples.

The main idea was to determine a substitution that would allow us to reduce the two terms under the root that was always in the problem (more on this in a bit) into a single term and in doing so we were also able to easily eliminate the root. To do this we made use of the following formulas.

$$\begin{aligned} 25x^2 - 4 \qquad &\Rightarrow \qquad \sec^2(\theta) - 1 = \tan^2(\theta) \\ 9 - x^2 \qquad &\Rightarrow \qquad 1 - \sin^2(\theta) = \cos^2(\theta) \\ 36x^2 + 1 \qquad &\Rightarrow \qquad \tan^2(\theta) + 1 = \sec^2(\theta) \end{aligned}$$

If we step back a bit we can notice that the terms we reduced look like the trig identities we used to reduce them in a vague way.

For instance, $25x^2 - 4$ is something squared (*i.e.* the $25x^2$) minus a number (*i.e.* the 4) and the left side of formula we used, $\sec^2(\theta) - 1$, also follows this basic form. So, because the two look alike in a very vague way that suggests using a secant substitution for that problem. We can notice similar vague similarities in the other two cases as well.

If we keep this idea in mind we don't need the "formulas" listed after each example to tell us which trig substitution to use and since we have to know the trig identities anyway to do the problems keeping this idea in mind doesn't really add anything to what we need to know for the problems.

Once we've identified the trig function to use in the substitution the coefficient, the $\frac{a}{b}$ in the formulas, is also easy to get. Just remember that in order to use the trig identities the coefficient of the trig function and the number in the identity must be the same, *i.e.* both 4 or 9, so that the trig identity can be used after we factor the common number out. What this means is that we need to "turn" the coefficient of the squared term into the constant number through our substitution.

So, in the first example we needed to "turn" the 25 into a 4 through our substitution. Remembering that we are eventually going to square the substitution that means we need to divide out by a 5 so the 25 will cancel out, upon squaring. Likewise, we'll need to add a 2 to the substitution so the coefficient will "turn" into a 4 upon squaring. In other words, we would need to use the substitution that we did in the problem.

The same idea holds for the other two trig substitutions.

Notice as well that we could have used cosecant in the first case, cosine in the second case and cotangent in the third case. So, why didn't we? Simply because of the differential work. Had we used these trig functions instead we would have picked up a minus sign in the differential that we'd need to keep track of. So, while these could be used they generally aren't to avoid extra minus signs that we need to keep track of.

Next, let's quickly address the fact that a root was in all of these problems. Note that the root is not required in order to use a trig substitution. Instead, the trig substitution gave us a really nice way of eliminating the root from the problem. In this section we will always be having roots in the problems, and in fact our summaries above all assumed roots, roots are not actually required in order use a trig substitution. We will be seeing an example or two of trig substitutions in integrals that do not have roots in the Integrals Involving Quadratics section.

Finally, let's summarize up all the ideas with the trig substitutions we've discussed and again we will be using roots in the summary simply because all the integrals in this section will have roots and those tend to be the most likely places for using trig substitutions but again, are not required in order to use a trig substitution.

Fact

Form	Looks Like	Substitution	Limit Assumptions
$\sqrt{b^2x^2 - a^2}$	$\sec^2(\theta) - 1 = \tan^2\theta$	$x = \frac{a}{b}\sec(\theta)$	$0 \le \theta < \frac{\pi}{2}, \ \frac{\pi}{2} < \theta \le \pi$
$\sqrt{a^2 - b^2x^2}$	$1 - \sin^2(\theta) = \cos^2(\theta)$	$x = \frac{a}{b}\sin(\theta)$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{a^2 + b^2x^2}$	$\tan^2(\theta) + 1 = \sec^2(\theta)$	$x = \frac{a}{b}\tan(\theta)$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Now, we have a couple of final examples to work in this section. Not all trig substitutions will just jump right out at us. Sometimes we need to do a little work on the integrand first to get it into the correct form and that is the point of the remaining examples.

Example 6

Evaluate the following integral.

$$\int \frac{x}{\sqrt{2x^2 - 4x - 7}}\,dx$$

Solution

In this case the quantity under the root doesn't obviously fit into any of the cases we looked at above and in fact isn't in the any of the forms we saw in the previous examples. Note however that if we complete the square on the quadratic we can make it look somewhat like the above integrals.

Remember that completing the square requires a coefficient of one in front of the x^2. Once we have that we take half the coefficient of the x, square it, and then add and subtract it to the quantity. Here is the completing the square for this problem.

$$2\left(x^2 - 2x - \frac{7}{2}\right) = 2\left(x^2 - 2x + 1 - 1 - \frac{7}{2}\right) = 2\left((x-1)^2 - \frac{9}{2}\right) = 2(x-1)^2 - 9$$

So, the root becomes,

$$\sqrt{2x^2-4x-7}=\sqrt{2(x-1)^2-9}$$

Now, this looks (very) vaguely like $\sec^2(\theta)-1$ (*i.e.* something squared minus a number) except we've got something more complicated in the squared term. That is okay we'll still be able to do a secant substitution and it will work in pretty much the same way.

$$x-1=\frac{3}{\sqrt{2}}\sec(\theta)\qquad x=1+\frac{3}{\sqrt{2}}\sec(\theta)\qquad dx=\frac{3}{\sqrt{2}}\sec(\theta)\tan(\theta)\,d\theta$$

Using this substitution the root reduces to,

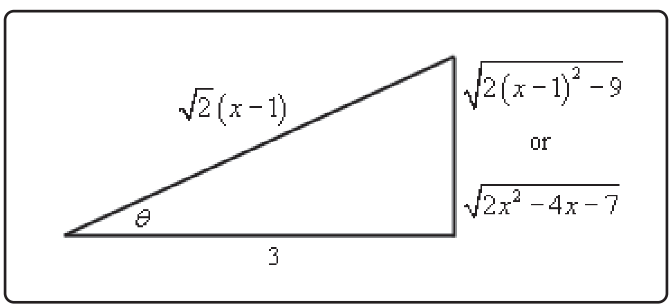
$$\sqrt{2x^2-4x-7}=\sqrt{2(x-1)^2-9}=\sqrt{9\sec^2(\theta)-9}=3\sqrt{\tan^2(\theta)}=3\left|\tan(\theta)\right|=3\tan(\theta)$$

Note we could drop the absolute value bars since we are doing an indefinite integral. Here is the integral.

$$\begin{aligned}\int\frac{x}{\sqrt{2x^2-4x-7}}\,dx&=\int\frac{1+\frac{3}{\sqrt{2}}\sec(\theta)}{3\tan(\theta)}\left(\frac{3}{\sqrt{2}}\sec(\theta)\tan(\theta)\right)d\theta\\&=\int\frac{1}{\sqrt{2}}\sec(\theta)+\frac{3}{2}\sec^2(\theta)\,d\theta\\&=\frac{1}{\sqrt{2}}\ln\left|\sec(\theta)+\tan(\theta)\right|+\frac{3}{2}\tan(\theta)+c\end{aligned}$$

And here is the right triangle for this problem.

$$\sec(\theta)=\frac{\sqrt{2}\,(x-1)}{3}\qquad\qquad\tan(\theta)=\frac{\sqrt{2x^2-4x-7}}{3}$$



The integral is then,

$$\int\frac{x}{\sqrt{2x^2-4x-7}}\,dx=\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}\,(x-1)}{3}+\frac{\sqrt{2x^2-4x-7}}{3}\right|+\frac{\sqrt{2x^2-4x-7}}{2}+c$$

Example 7

Evaluate the following integral.

$$\int \mathbf{e}^{4x}\sqrt{1+\mathbf{e}^{2x}}\,dx$$

Solution

This doesn't look to be anything like the other problems in this section. However it is. To see this we first need to notice that,

$$\mathbf{e}^{2x} = (\mathbf{e}^{x})^2$$

Upon noticing this we can use the following standard Calculus I substitution.

$$u = \mathbf{e}^{x} \qquad\qquad du = \mathbf{e}^{x}\,dx$$

We do need to be a little careful with the differential work however. We don't have just an $\mathbf{e}^{x}$ out in front of the root. Instead we have an $\mathbf{e}^{4x}$. So, we'll need to strip one of those out for the differential and then use the substitution on the rest. Here is the substitution work.

$$\begin{aligned}\int \mathbf{e}^{4x}\sqrt{1+\mathbf{e}^{2x}}\,dx &= \int \mathbf{e}^{3x}\mathbf{e}^{x}\sqrt{1+\mathbf{e}^{2x}}\,dx\\ &= \int (\mathbf{e}^{x})^3\sqrt{1+(\mathbf{e}^{x})^2}\,\mathbf{e}^{x}dx = \int u^3\sqrt{1+u^2}\,du\end{aligned}$$

This is now a fairly obvious trig substitution (hopefully). The quantity under the root looks almost exactly like $1+\tan^2(\theta)$ and so we can use a tangent substitution. Here is that work.

$$u = \tan(\theta) \qquad du = \sec^2(\theta)\,d\theta \qquad\qquad \sqrt{1+u^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2(\theta)} = |\sec(\theta)|$$

Because we are doing an indefinite integral we can assume secant is positive and drop the absolute value bars. Applying this substitution to the integral gives,

$$\int \mathbf{e}^{4x}\sqrt{1+\mathbf{e}^{2x}}\,dx = \int \tan^3(\theta)\,(\sec(\theta))\,\left(\sec^2(\theta)\right)d\theta = \int \tan^3(\theta)\sec^3(\theta)d\theta$$

We'll finish this integral off in a bit. Before we get to that there is a "quicker" (although not super obvious) way of doing the substitutions above. Let's cover that first then we'll come back and finish working the integral.

We can notice that the u in the Calculus I substitution and the trig substitution are the same u and so we can combine them into the following substitution.

$$\mathbf{e}^{x} = \tan(\theta)$$

We can then compute the differential. Just remember that all we do is differentiate both sides and then tack on dx or $d\theta$ onto the appropriate side. Doing this gives,

$$\mathbf{e}^x\,dx = \sec^2(\theta)\,d\theta$$

With this substitution the square root becomes,

$$\sqrt{1+\mathbf{e}^{2x}} = \sqrt{1+(\mathbf{e}^x)^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = |\sec(\theta)| = \sec(\theta)$$

Again, we can drop the absolute value bars because we are doing an indefinite integral. The integral then becomes,

$$\begin{aligned}\int \mathbf{e}^{4x}\sqrt{1+\mathbf{e}^{2x}}\,dx &= \int \mathbf{e}^{3x}\mathbf{e}^{x}\sqrt{1+\mathbf{e}^{2x}}\,dx\\ &= \int (\mathbf{e}^x)^3\sqrt{1+\mathbf{e}^{2x}}\,(\mathbf{e}^x)\,dx\\ &= \int \tan^3(\theta)\,(\sec(\theta))\left(\sec^2(\theta)\right)\,d\theta = \int \tan^3(\theta)\sec^3(\theta)d\theta\end{aligned}$$

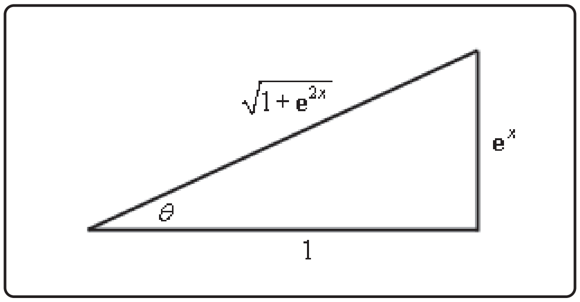
So, the same integral with less work. However, it does require that you be able to combine the two substitutions in to a single substitution. How you do this type of problem is up to you but if you don't feel comfortable with the single substitution (and there's nothing wrong if you don't!) then just do the two individual substitutions. The single substitution method was given only to show you that it can be done so that those that are really comfortable with both kinds of substitutions can do the work a little quicker.

Now, let's finish the integral work.

$$\begin{aligned}\int \mathbf{e}^{4x}\sqrt{1+\mathbf{e}^{2x}}\,dx &= \int \tan^3(\theta)\sec^3(\theta)d\theta\\ &= \int \left(\sec^2(\theta)-1\right)\sec^2(\theta)\,\sec(\theta)\tan(\theta)\,d\theta \qquad\qquad v=\sec(\theta)\\ &= \int v^4 - v^2\,dv\\ &= \frac{1}{5}\sec^5(\theta) - \frac{1}{3}\sec^3(\theta)+c\end{aligned}$$

Here is the right triangle for this integral.

$$\tan(\theta) = \frac{\mathbf{e}^x}{1} \qquad\qquad \sec(\theta) = \frac{\sqrt{1+\mathbf{e}^{2x}}}{1} = \sqrt{1+\mathbf{e}^{2x}}$$



The integral is then,

$$\int \mathbf{e}^{4x}\sqrt{1+\mathbf{e}^{2x}}\,dx = \frac{1}{5}\left(1+\mathbf{e}^{2x}\right)^{\frac{5}{2}} - \frac{1}{3}\left(1+\mathbf{e}^{2x}\right)^{\frac{3}{2}} + c$$

This was a messy problem, but we will be seeing some of this type of integral in later sections on occasion so we needed to make sure you'd seen at least one like it.

So, as we've seen in the final two examples in this section some integrals that look nothing like the first few examples can in fact be turned into a trig substitution problem with a little work.

7.4 Partial Fractions

In this section we are going to take a look at integrals of rational expressions of polynomials and once again let's start this section out with an integral that we can already do so we can contrast it with the integrals that we'll be doing in this section.

$$\begin{aligned}\int \frac{2x-1}{x^2-x-6}\,dx &= \int \frac{1}{u}\,du \qquad \text{using } u = x^2-x-6 \text{ and } du = (2x-1)\,dx\\ &= \ln\left|x^2-x-6\right| + c\end{aligned}$$

So, if the numerator is the derivative of the denominator (or a constant multiple of the derivative of the denominator) doing this kind of integral is fairly simple. However, often the numerator isn't the derivative of the denominator (or a constant multiple). For example, consider the following integral.

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

In this case the numerator is definitely not the derivative of the denominator nor is it a constant multiple of the derivative of the denominator. Therefore, the simple substitution that we used above won't work. However, if we notice that the integrand can be broken up as follows,

$$\frac{3x+11}{x^2-x-6} = \frac{4}{x-3} - \frac{1}{x+2}$$

then the integral is actually quite simple.

$$\begin{aligned}\int \frac{3x+11}{x^2-x-6}\,dx &= \int \frac{4}{x-3} - \frac{1}{x+2}dx\\ &= 4\ln|x-3| - \ln|x+2| + c\end{aligned}$$

This process of taking a rational expression and decomposing it into simpler rational expressions that we can add or subtract to get the original rational expression is called **partial fraction decomposition**. Many integrals involving rational expressions can be done if we first do partial fractions on the integrand.

So, let's do a quick review of partial fractions. We'll start with a rational expression in the form,

$$f(x) = \frac{P(x)}{Q(x)}$$

where both $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is smaller than the degree of $Q(x)$. Recall that the degree of a polynomial is the largest exponent in the polynomial. Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember.

So, once we've determined that partial fractions can be done we factor the denominator as completely as possible. Then for each factor in the denominator we can use the following table to determine the term(s) we pick up in the partial fraction decomposition.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k},\ k=1,2,3,\ldots$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k},\ k=1,2,3,\ldots$

Notice that the first and third cases are really special cases of the second and fourth cases respectively.

There are several methods for determining the coefficients for each term and we will go over each of those in the following examples.

Let's start the examples by doing the integral above.

Example 1

Evaluate the following integral.

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

Solution

The first step is to factor the denominator as much as possible and get the form of the partial fraction decomposition. Doing this gives,

$$\frac{3x+11}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}$$

The next step is to actually add the right side back up.

$$\frac{3x+11}{(x-3)(x+2)}=\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

Now, we need to choose A and B so that the numerators of these two are equal for every x. To do this we'll need to set the numerators equal.

$$3x+11=A(x+2)+B(x-3)$$

Note that in most problems we will go straight from the general form of the decomposition to this step and not bother with actually adding the terms back up. The only point to adding the terms is to get the numerator and we can get that without actually writing down the results of the addition.

At this point we have one of two ways to proceed. One way will always work but is often more work. The other, while it won't always work, is often quicker when it does work. In this case both will work and so we'll use the quicker way for this example. We'll take a look at the other method in a later example.

What we're going to do here is to notice that the numerators must be equal for *any x* that we would choose to use. In particular the numerators must be equal for $x = -2$ and $x = 3$. So, let's plug these in and see what we get.

$$\begin{array}{llll} x = -2: & 5 = A(0) + B(-5) & \Rightarrow & B = -1 \\ x = 3\ \ : & 20 = A(5) + B(0) & \Rightarrow & A = 4 \end{array}$$

So, by carefully picking the x's we got the unknown constants to quickly drop out. Note that these are the values we claimed they would be above.

At this point there really isn't a whole lot to do other than the integral.

$$\begin{aligned} \int \frac{3x+11}{x^2-x-6}\,dx &= \int \frac{4}{x-3} - \frac{1}{x+2}dx \\ &= \int \frac{4}{x-3}\,dx - \int \frac{1}{x+2}dx \\ &= 4\ln|x-3| - \ln|x+2| + c \end{aligned}$$

Recall that to do this integral we first split it up into two integrals and then used the substitutions,

$$u = x - 3 \qquad\qquad v = x + 2$$

on the integrals to get the final answer.

Before moving onto the next example a couple of quick notes are in order here. First, many of the integrals in partial fractions problems come down to the type of integral seen above. Make sure that you can do those integrals.

There is also another integral that often shows up in these kinds of problems so we may as well give the formula for it here since we are already on the subject.

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

It will be an example or two before we use this so don't forget about it.

Now, let's work some more examples.

Example 2

Evaluate the following integral.

$$\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx$$

Solution

We won't be putting as much detail into this solution as we did in the previous example. The first thing is to factor the denominator and get the form of the partial fraction decomposition.

$$\frac{x^2+4}{x\,(x+2)\,(3x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{3x-2}$$

The next step is to set numerators equal. If you need to actually add the right side together to get the numerator for that side then you should do so, however, it will definitely make the problem quicker if you can do the addition in your head to get,

$$x^2+4 = A\,(x+2)\,(3x-2) + Bx\,(3x-2) + Cx\,(x+2)$$

As with the previous example it looks like we can just pick a few values of x and find the constants so let's do that.

$$\begin{array}{llll} x=0\quad: & 4 = A\,(2)\,(-2) & \Rightarrow & A=-1 \\ x=-2: & 8 = B\,(-2)\,(-8) & \Rightarrow & B=\dfrac{1}{2} \\ x=\dfrac{2}{3}\ : & \dfrac{40}{9} = C\left(\dfrac{2}{3}\right)\left(\dfrac{8}{3}\right) & \Rightarrow & C=\dfrac{40}{16}=\dfrac{5}{2} \end{array}$$

Note that unlike the first example most of the coefficients here are fractions. That is not unusual so don't get excited about it when it happens.

Now, let's do the integral.

$$\begin{aligned} \int \frac{x^2+4}{3x^3+4x^2-4x}\,dx &= \int -\frac{1}{x} + \frac{\frac{1}{2}}{x+2} + \frac{\frac{5}{2}}{3x-2}\,dx \\ &= -\ln|x| + \frac{1}{2}\ln|x+2| + \frac{5}{6}\ln|3x-2| + c \end{aligned}$$

Again, as noted above, integrals that generate natural logarithms are very common in these problems so make sure you can do them. Also, you were able to correctly do the last integral right? The coefficient of $\frac{5}{6}$ is correct. Make sure that you do the substitution required for the term properly.

Example 3

Evaluate the following integral.

$$\int \frac{x^2 - 29x + 5}{(x-4)^2 (x^2+3)}\, dx$$

Solution

This time the denominator is already factored so let's just jump right to the partial fraction decomposition.

$$\frac{x^2 - 29x + 5}{(x-4)^2 (x^2+3)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+3}$$

Setting numerators gives,

$$x^2 - 29x + 5 = A(x-4)\left(x^2+3\right) + B\left(x^2+3\right) + (Cx+D)(x-4)^2$$

In this case we aren't going to be able to just pick values of x that will give us all the constants. Therefore, we will need to work this the second (and often longer) way. The first step is to multiply out the right side and collect all the like terms together. Doing this gives,

$$x^2 - 29x + 5 = (A+C)\,x^3 + (-4A + B - 8C + D)\,x^2 + (3A + 16C - 8D)\,x - 12A + 3B + 16D$$

Now we need to choose A, B, C, and D so that these two are equal. In other words, we will need to set the coefficients of like powers of x equal. This will give a system of equations that can be solved.

$$\left.\begin{array}{lr} x^3: & A + C = 0 \\ x^2: & -4A + B - 8C + D = 1 \\ x^1: & 3A + 16C - 8D = -29 \\ x^0: & -12A + 3B + 16D = 5 \end{array}\right\} \quad \Rightarrow \quad A = 1,\ B = -5,\ C = -1,\ D = 2$$

Note that we used x^0 to represent the constants. Also note that these systems can often be quite large and have a fair amount of work involved in solving them. The best way to deal with these is to use some form of computer aided solving techniques.

Now, let's take a look at the integral.

$$\begin{aligned} \int \frac{x^2 - 29x + 5}{(x-4)^2 (x^2+3)}\, dx &= \int \frac{1}{x-4} - \frac{5}{(x-4)^2} + \frac{-x+2}{x^2+3}\, dx \\ &= \int \frac{1}{x-4} - \frac{5}{(x-4)^2} - \frac{x}{x^2+3} + \frac{2}{x^2+3}\, dx \\ &= \ln|x-4| + \frac{5}{x-4} - \frac{1}{2}\ln\left|x^2+3\right| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c \end{aligned}$$

In order to take care of the third term we needed to split it up into two separate terms. Once we've done this we can do all the integrals in the problem. The first two use the substitution $u = x - 4$, the third uses the substitution $v = x^2 + 3$ and the fourth term uses the formula given above for inverse tangents.

Example 4

Evaluate the following integral.

$$\int \frac{x^3 + 10x^2 + 3x + 36}{(x-1)\left(x^2+4\right)^2}\,dx$$

Solution

Let's first get the general form of the partial fraction decomposition.

$$\frac{x^3 + 10x^2 + 3x + 36}{(x-1)\left(x^2+4\right)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{\left(x^2+4\right)^2}$$

Now, set numerators equal, expand the right side and collect like terms.

$$\begin{aligned} x^3 + 10x^2 + 3x + 36 &= A\left(x^2+4\right)^2 + (Bx+C)(x-1)\left(x^2+4\right) + (Dx+E)(x-1) \\ &= (A+B)\,x^4 + (C-B)\,x^3 + (8A+4B-C+D)\,x^2 + \\ &\qquad (-4B+4C-D+E)\,x + 16A - 4C - E \end{aligned}$$

Setting coefficient equal gives the following system.

$$\left.\begin{array}{lr} x^4: & A+B=0 \\ x^3: & C-B=1 \\ x^2: & 8A+4B-C+D=10 \\ x^1: & -4B+4C-D+E=3 \\ x^0: & 16A-4C-E=36 \end{array}\right\} \quad \Rightarrow \quad A=2,\ B=-2,\ C=-1,\ D=1,\ E=0$$

Don't get excited if some of the coefficients end up being zero. It happens on occasion.

Here's the integral.

$$\begin{aligned} \int \frac{x^3 + 10x^2 + 3x + 36}{(x-1)\left(x^2+4\right)^2}\,dx &= \int \frac{2}{x-1} + \frac{-2x-1}{x^2+4} + \frac{x}{\left(x^2+4\right)^2}\,dx \\ &= \int \frac{2}{x-1} - \frac{2x}{x^2+4} - \frac{1}{x^2+4} + \frac{x}{\left(x^2+4\right)^2}\,dx \\ &= 2\ln|x-1| - \ln\left|x^2+4\right| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\frac{1}{x^2+4} + c \end{aligned}$$

To this point we've only looked at rational expressions where the degree of the numerator was strictly less that the degree of the denominator. Of course, not all rational expressions will fit into this form and so we need to take a look at a couple of examples where this isn't the case.

Example 5

Evaluate the following integral.

$$\int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2}\,dx$$

Solution

So, in this case the degree of the numerator is 4 and the degree of the denominator is 3. Therefore, partial fractions can't be done on this rational expression.

To fix this up we'll need to do long division on this to get it into a form that we can deal with. Here is the work for that.

$$\begin{array}{r@{}l} & \quad\; x - 2 \\ x^3 - 3x^2 \big) & \overline{x^4 - 5x^3 + 6x^2 - 18} \\ & \underline{-\left(x^4 - 3x^3\right)\qquad\qquad} \\ & \qquad -2x^3 + 6x^2 - 18 \\ & \quad \underline{-\left(-2x^3 + 6x^2\right)\quad} \\ & \qquad\qquad\qquad -18 \end{array}$$

So, from the long division we see that,

$$\frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2} = x - 2 - \frac{18}{x^3 - 3x^2}$$

and the integral becomes,

$$\begin{aligned} \int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2}\,dx &= \int x - 2 - \frac{18}{x^3 - 3x^2}\,dx \\ &= \int x - 2\,dx - \int \frac{18}{x^3 - 3x^2}\,dx \end{aligned}$$

The first integral we can do easily enough and the second integral is now in a form that allows us to do partial fractions. So, let's get the general form of the partial fractions for the second integrand.

$$\frac{18}{x^2\left(x - 3\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 3}$$

Setting numerators equal gives us,

$$18 = Ax\left(x - 3\right) + B\left(x - 3\right) + Cx^2$$

Now, there is a variation of the method we used in the first couple of examples that will work here. There are a couple of values of x that will allow us to quickly get two of the three constants, but there is no value of x that will just hand us the third.

What we'll do in this example is pick x's to get the two constants that we can easily get and then we'll just pick another value of x that will be easy to work with (*i.e.* it won't give large/messy numbers anywhere) and then we'll use the fact that we also know the other two constants to find the third.

$$\begin{array}{llll} x=0: & 18 = B(-3) & \Rightarrow & B=-6 \\ x=3: & 18 = C(9) & \Rightarrow & C=2 \\ x=1: & 18 = A(-2)+B(-2)+C = -2A+14 & \Rightarrow & A=-2 \end{array}$$

The integral is then,

$$\begin{aligned}\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,dx &= \int x-2\,dx - \int -\frac{2}{x} - \frac{6}{x^2} + \frac{2}{x-3}\,dx \\ &= \frac{1}{2}x^2 - 2x + 2\ln|x| - \frac{6}{x} - 2\ln|x-3| + c\end{aligned}$$

In the previous example there were actually two different ways of dealing with the x^2 in the denominator. One is to treat it as a quadratic which would give the following term in the decomposition

$$\frac{Ax+B}{x^2}$$

and the other is to treat it as a linear term in the following way,

$$x^2 = (x-0)^2$$

which gives the following two terms in the decomposition,

$$\frac{A}{x} + \frac{B}{x^2}$$

We used the second way of thinking about it in our example. Notice however that the two will give identical partial fraction decompositions. So, why talk about this? Simple. This will work for x^2, but what about x^3 or x^4? In these cases, we really will need to use the second way of thinking about these kinds of terms.

$$x^3 \quad\Rightarrow\quad \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3} \qquad\qquad x^4 \quad\Rightarrow\quad \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x^4}$$

Let's take a look at one more example.

Example 6

Evaluate the following integral.

$$\int \frac{x^2}{x^2-1}\,dx$$

Solution

In this case the numerator and denominator have the same degree. As with the last example we'll need to do long division to get this into the correct form. We'll leave the details of that to you to check.

$$\int \frac{x^2}{x^2-1}\,dx = \int 1 + \frac{1}{x^2-1}\,dx = \int dx + \int \frac{1}{x^2-1}\,dx$$

So, we'll need to partial fraction the second integral. Here's the decomposition.

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Setting numerator equal gives,

$$1 = A(x+1) + B(x-1)$$

Picking value of x gives us the following coefficients.

$$\begin{array}{llll} x=-1: & 1 = B(-2) & \Rightarrow & B = -\frac{1}{2} \\ x = 1 \;\; : & 1 = A(2) & \Rightarrow & A = \frac{1}{2} \end{array}$$

The integral is then,

$$\begin{aligned} \int \frac{x^2}{x^2-1}\,dx &= \int dx + \int \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\,dx \\ &= x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c \end{aligned}$$

7.5 Integrals Involving Roots

In this section we're going to look at an integration technique that can be useful for *some* integrals with roots in them. We've already seen some integrals with roots in them. Some can be done quickly with a simple Calculus I substitution and some can be done with trig substitutions.

However, not all integrals with roots will allow us to use one of these methods. Let's look at a couple of examples to see another technique that can be used on occasion to help with these integrals.

Example 1

Evaluate the following integral.

$$\int \frac{x+2}{\sqrt[3]{x-3}}\,dx$$

Solution

Sometimes when faced with an integral that contains a root we can use the following substitution to simplify the integral into a form that can be easily worked with.

$$u = \sqrt[3]{x-3}$$

So, instead of letting u be the stuff under the radical as we often did in Calculus I we let u be the whole radical. Now, there will be a little more work here since we will also need to know what x is so we can substitute in for that in the numerator and so we can compute the differential, dx. This is easy enough to get however. Just solve the substitution for x as follows,

$$x = u^3 + 3 \qquad\qquad dx = 3u^2\,du$$

Using this substitution the integral is now,

$$\begin{aligned}\int \frac{(u^3+3)+2}{u}\,3u^2du &= \int 3u^4 + 15u\,du \\ &= \frac{3}{5}u^5 + \frac{15}{2}u^2 + c \\ &= \frac{3}{5}(x-3)^{\frac{5}{3}} + \frac{15}{2}(x-3)^{\frac{2}{3}} + c\end{aligned}$$

So, sometimes, when an integral contains the root $\sqrt[n]{g(x)}$ the substitution,

$$u = \sqrt[n]{g(x)}$$

can be used to simplify the integral into a form that we can deal with.

Let's take a look at another example real quick.

Example 2

Evaluate the following integral.

$$\int \frac{2}{x - 3\sqrt{x+10}}\,dx$$

Solution

We'll do the same thing we did in the previous example. Here's the substitution and the extra work we'll need to do to get x in terms of u.

$$u = \sqrt{x+10} \qquad x = u^2 - 10 \qquad dx = 2u\,du$$

With this substitution the integral is,

$$\int \frac{2}{x - 3\sqrt{x+10}}\,dx = \int \frac{2}{u^2 - 10 - 3u}\,(2u)\,du = \int \frac{4u}{u^2 - 3u - 10}\,du$$

This integral can now be done with partial fractions.

$$\frac{4u}{(u-5)(u+2)} = \frac{A}{u-5} + \frac{B}{u+2}$$

Setting numerators equal gives,

$$4u = A\,(u+2) + B\,(u-5)$$

Picking value of u gives the coefficients.

$$\begin{array}{llll} u = -2 & -8 = B\,(-7) & \Rightarrow & B = \frac{8}{7} \\ u = 5 & 20 = A\,(7) & \Rightarrow & A = \frac{20}{7} \end{array}$$

The integral is then,

$$\begin{aligned} \int \frac{2}{x - 3\sqrt{x+10}}\,dx &= \int \frac{\frac{20}{7}}{u-5} + \frac{\frac{8}{7}}{u+2}\,du \\ &= \frac{20}{7}\ln|u-5| + \frac{8}{7}\ln|u+2| + c \\ &= \frac{20}{7}\ln\left|\sqrt{x+10} - 5\right| + \frac{8}{7}\ln\left|\sqrt{x+10} + 2\right| + c \end{aligned}$$

So, we've seen a nice method to eliminate roots from the integral and put it into a form that we can deal with. Note however, that this won't always work and sometimes the new integral will be just as difficult to do.

7.6 Integrals Involving Quadratics

To this point we've seen quite a few integrals that involve quadratics. A couple of examples are,

$$\int \frac{x}{x^2 \pm a}\,dx = \frac{1}{2}\ln\left|x^2 \pm a\right| + c \qquad \int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

We also saw that integrals involving $\sqrt{b^2x^2 - a^2}$, $\sqrt{a^2 - b^2x^2}$ and $\sqrt{a^2 + b^2x^2}$ could be done with a trig substitution.

Notice however that all of these integrals were missing an x term. They all consist of only a quadratic term and a constant.

Some integrals involving general quadratics are easy enough to do. For instance, the following integral can be done with a quick substitution.

$$\begin{aligned}\int \frac{2x+3}{4x^2+12x-1}\,dx &= \frac{1}{4}\int \frac{1}{u}\,du \qquad u = 4x^2+12x-1 \qquad du = 4\,(2x+3)\,dx\\ &= \frac{1}{4}\ln\left|4x^2+12x-1\right| + c\end{aligned}$$

Some integrals with quadratics can be done with partial fractions. For instance,

$$\int \frac{10x-6}{3x^2+16x+5}\,dx = \int \frac{4}{x+5} - \frac{2}{3x+1}\,dx = 4\ln|x+5| - \frac{2}{3}\ln|3x+1| + c$$

Unfortunately, these methods won't work on a lot of integrals. A simple substitution will only work if the numerator is a constant multiple of the derivative of the denominator and partial fractions will only work if the denominator can be factored.

The topic of this section is how to deal with integrals involving quadratics when the techniques that we've looked at to this point simply won't work.

Back in the Trig Substitution section we saw how to deal with square roots that had a general quadratic in them. Let's take a quick look at another one like that since the idea involved in doing that kind of integral is exactly what we are going to need for the other integrals in this section.

Example 1

Evaluate the following integral.

$$\int \sqrt{x^2+4x+5}\,dx$$

Solution

Recall from the Trig Substitution section that in order to do a trig substitution here we first needed to complete the square on the quadratic. This gives,

$$x^2 + 4x + 5 = x^2 + 4x + 4 - 4 + 5 = (x+2)^2 + 1$$

After completing the square the integral becomes,

$$\int \sqrt{x^2 + 4x + 5}\, dx = \int \sqrt{(x+2)^2 + 1}\, dx$$

Upon doing this we can identify the trig substitution that we need. Here it is,

$$x + 2 = \tan(\theta) \qquad x = \tan(\theta) - 2 \qquad dx = \sec^2(\theta)\, d\theta$$

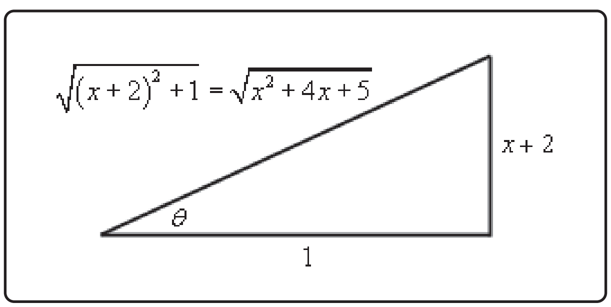
$$\sqrt{(x+2)^2 + 1} = \sqrt{\tan^2(\theta) + 1} = \sqrt{\sec^2(\theta)} = |\sec(\theta)| = \sec(\theta)$$

Recall that since we are doing an indefinite integral we can drop the absolute value bars. Using this substitution the integral becomes,

$$\begin{aligned}\int \sqrt{x^2 + 4x + 5}\, dx &= \int \sec^3(\theta)\, d\theta \\ &= \frac{1}{2}\Big(\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\Big) + c\end{aligned}$$

We can finish the integral out with the following right triangle.

$$\tan(\theta) = \frac{x+2}{1} \qquad \sec(\theta) = \frac{\sqrt{x^2 + 4x + 5}}{1} = \sqrt{x^2 + 4x + 5}$$



$$\int \sqrt{x^2 + 4x + 5}\, dx = \frac{1}{2}\left((x+2)\sqrt{x^2 + 4x + 5} + \ln\left|x + 2 + \sqrt{x^2 + 4x + 5}\right|\right) + c$$

So, by completing the square we were able to take an integral that had a general quadratic in it and convert it into a form that allowed us to use a known integration technique.

Let's do a quick review of completing the square before proceeding. Here is the general completing the square formula that we'll use.

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

This will always take a general quadratic and write it in terms of a squared term and a constant term.

Recall as well that in order to do this we must have a coefficient of one in front of the x^2. If not, we'll need to factor out the coefficient before completing the square. In other words,

$$ax^2 + bx + c = a \underbrace{\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)}_{\text{complete the square on this!}}$$

Now, let's see how completing the square can be used to do integrals that we aren't able to do at this point.

Example 2

Evaluate the following integral.

$$\int \frac{1}{2x^2 - 3x + 2}\,dx$$

Solution

Okay, this doesn't factor so partial fractions just won't work on this. Likewise, since the numerator is just "1" we can't use the substitution $u = 2x^2 - 3x + 2$. So, let's see what happens if we complete the square on the denominator.

$$\begin{aligned} 2x^2 - 3x + 2 &= 2\left(x^2 - \frac{3}{2}x + 1\right) \\ &= 2\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16} + 1\right) \\ &= 2\left(\left(x - \frac{3}{4}\right)^2 + \frac{7}{16}\right) \end{aligned}$$

With this the integral is,

$$\int \frac{1}{2x^2 - 3x + 2}\,dx = \frac{1}{2}\int \frac{1}{\left(x - \frac{3}{4}\right)^2 + \frac{7}{16}}\,dx$$

Now this may not seem like all that great of a change. However, notice that we can now use the following substitution.

$$u = x - \frac{3}{4} \qquad\qquad du = dx$$

and the integral is now,

$$\int \frac{1}{2x^2 - 3x + 2}\,dx = \frac{1}{2}\int \frac{1}{u^2 + \frac{7}{16}}\,du$$

We can now see that this is an inverse tangent! So, using the formula from the start of the section we get,

$$\begin{aligned}\int \frac{1}{2x^2 - 3x + 2}\,dx &= \frac{1}{2}\left(\frac{4}{\sqrt{7}}\right)\tan^{-1}\left(\frac{4u}{\sqrt{7}}\right) + c \\ &= \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{4x-3}{\sqrt{7}}\right) + c\end{aligned}$$

Example 3

Evaluate the following integral.

$$\int \frac{3x - 1}{x^2 + 10x + 28}\,dx$$

Solution

This example is a little different from the previous one. In this case we do have an x in the numerator however the numerator still isn't a multiple of the derivative of the denominator and so a simple Calculus I substitution won't work.

So, let's again complete the square on the denominator and see what we get,

$$x^2 + 10x + 28 = x^2 + 10x + 25 - 25 + 28 = (x+5)^2 + 3$$

Upon completing the square the integral becomes,

$$\int \frac{3x - 1}{x^2 + 10x + 28}\,dx = \int \frac{3x - 1}{(x+5)^2 + 3}\,dx$$

At this point we can use the same type of substitution that we did in the previous example. The only real difference is that we'll need to make sure that we plug the substitution back into the numerator as well.

$$u = x + 5 \qquad x = u - 5 \qquad dx = du$$

$$\begin{aligned}\int \frac{3x-1}{x^2+10x+28}\,dx &= \int \frac{3\left(u-5\right)-1}{u^2+3}\,du \\ &= \int \frac{3u}{u^2+3} - \frac{16}{u^2+3}\,du \\ &= \frac{3}{2}\ln\left|u^2+3\right| - \frac{16}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+c \\ &= \frac{3}{2}\ln\left|(x+5)^2+3\right| - \frac{16}{\sqrt{3}}\tan^{-1}\left(\frac{x+5}{\sqrt{3}}\right)+c\end{aligned}$$

So, in general when dealing with an integral in the form,

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx \tag{7.5}$$

Here we are going to assume that the denominator doesn't factor and the numerator isn't a constant multiple of the derivative of the denominator. In these cases, we complete the square on the denominator and then do a substitution that will yield an inverse tangent and/or a logarithm depending on the exact form of the numerator.

Let's now take a look at a couple of integrals that are in the same general form as Equation 7.5 except the denominator will also be raised to a power. In other words, let's look at integrals in the form,

$$\int \frac{Ax+B}{\left(ax^2+bx+c\right)^n}\,dx \tag{7.6}$$

Example 4

Evaluate the following integral.

$$\int \frac{x}{\left(x^2-6x+11\right)^3}\,dx$$

Solution

For the most part this integral will work the same as the previous two with one exception that will occur down the road. So, let's start by completing the square on the quadratic in the denominator.

$$x^2-6x+11 = x^2-6x+9-9+11 = \left(x-3\right)^2+2$$

The integral is then,

$$\int \frac{x}{\left(x^2 - 6x + 11\right)^3}\,dx = \int \frac{x}{\left[(x-3)^2 + 2\right]^3}\,dx$$

Now, we will use the same substitution that we've used to this point in the previous two examples.

$$u = x - 3 \qquad x = u + 3 \qquad dx = du$$

$$\begin{aligned}\int \frac{x}{\left(x^2 - 6x + 11\right)^3}\,dx &= \int \frac{u+3}{\left(u^2+2\right)^3}\,du \\ &= \int \frac{u}{\left(u^2+2\right)^3}\,du + \int \frac{3}{\left(u^2+2\right)^3}\,du\end{aligned}$$

Now, here is where the differences start cropping up. The first integral can be done with the substitution $v = u^2 + 2$ and isn't too difficult. The second integral however, can't be done with the substitution used on the first integral and it isn't an inverse tangent.

It turns out that a trig substitution will work nicely on the second integral and it will be the same as we did when we had square roots in the problem.

$$u = \sqrt{2}\tan(\theta) \qquad du = \sqrt{2}\sec^2(\theta)\,d\theta$$

With these two substitutions the integrals become,

$$\begin{aligned}\int \frac{x}{\left(x^2 - 6x + 11\right)^3}\,dx &= \frac{1}{2}\int \frac{1}{v^3}\,dv + \int \frac{3}{\left(2\tan^2(\theta) + 2\right)^3}\left(\sqrt{2}\sec^2(\theta)\right)d\theta \\ &= -\frac{1}{4}\frac{1}{v^2} + \int \frac{3\sqrt{2}\sec^2(\theta)}{8\left(\tan^2(\theta)+1\right)^3}\,d\theta \\ &= -\frac{1}{4}\frac{1}{\left(u^2+2\right)^2} + \frac{3\sqrt{2}}{8}\int \frac{\sec^2(\theta)}{\left(\sec^2(\theta)\right)^3}\,d\theta \\ &= -\frac{1}{4}\frac{1}{\left((x-3)^2+2\right)^2} + \frac{3\sqrt{2}}{8}\int \frac{1}{\sec^4(\theta)}\,d\theta \\ &= -\frac{1}{4}\frac{1}{\left((x-3)^2+2\right)^2} + \frac{3\sqrt{2}}{8}\int \cos^4(\theta)\,d\theta\end{aligned}$$

Okay, at this point we've got two options for the remaining integral. We can either use the ideas we learned in the section about integrals involving trig integrals or we could use the following formula.

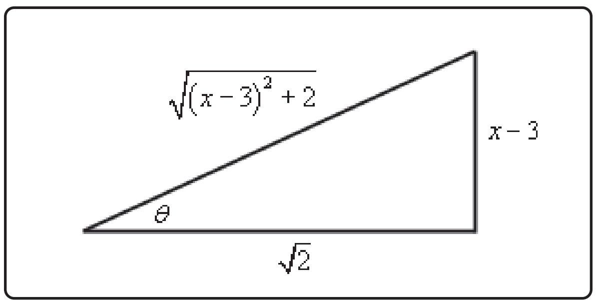
$$\int \cos^m(\theta)\,d\theta = \frac{1}{m}\sin(\theta)\cos^{m-1}(\theta) + \frac{m-1}{m}\int \cos^{m-2}(\theta)\,d\theta$$

Let's use this formula to do the integral.

$$\begin{aligned}\int \cos^4(\theta)\,d\theta &= \frac{1}{4}\sin(\theta)\cos^3(\theta) + \frac{3}{4}\int \cos^2(\theta)\,d\theta \\ &= \frac{1}{4}\sin(\theta)\cos^3(\theta) + \frac{3}{4}\left(\frac{1}{2}\sin(\theta)\cos(\theta) + \frac{1}{2}\int \cos^0(\theta)\,d\theta\right) \qquad \cos^0(\theta) = 1! \\ &= \frac{1}{4}\sin(\theta)\cos^3(\theta) + \frac{3}{8}\sin(\theta)\cos(\theta) + \frac{3}{8}\theta\end{aligned}$$

Next, let's use the following right triangle to get this back to x's.

$$\tan(\theta) = \frac{u}{\sqrt{2}} = \frac{x-3}{\sqrt{2}} \qquad \sin(\theta) = \frac{x-3}{\sqrt{(x-3)^2+2}} \qquad \cos(\theta) = \frac{\sqrt{2}}{\sqrt{(x-3)^2+2}}$$



The cosine integral is then,

$$\begin{aligned}\int \cos^4(\theta)\,d\theta &= \frac{1}{4}\frac{2\sqrt{2}\,(x-3)}{\left((x-3)^2+2\right)^2} + \frac{3}{8}\frac{\sqrt{2}\,(x-3)}{(x-3)^2+2} + \frac{3}{8}\tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) \\ &= \frac{\sqrt{2}}{2}\frac{x-3}{\left((x-3)^2+2\right)^2} + \frac{3\sqrt{2}}{8}\frac{x-3}{(x-3)^2+2} + \frac{3}{8}\tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right)\end{aligned}$$

All told then the original integral is,

$$\begin{aligned}\int \frac{x}{(x^2-6x+11)^3}\,dx &= -\frac{1}{4}\frac{1}{\left((x-3)^2+2\right)^2} + \\ &\quad \frac{3\sqrt{2}}{8}\left(\frac{\sqrt{2}}{2}\frac{x-3}{\left((x-3)^2+2\right)^2} + \frac{3\sqrt{2}}{8}\frac{x-3}{(x-3)^2+2} + \frac{3}{8}\tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right)\right) \\ &= \frac{1}{8}\frac{3x-11}{\left((x-3)^2+2\right)^2} + \frac{9}{32}\frac{x-3}{(x-3)^2+2} + \frac{9\sqrt{2}}{64}\tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) + c\end{aligned}$$

It's a long and messy answer, but there it is.

Example 5

Evaluate the following integral.

$$\int \frac{x-3}{\left(4-2x-x^2\right)^2}\,dx$$

Solution

As with the other problems we'll first complete the square on the denominator.

$$4-2x-x^2 = -\left(x^2+2x-4\right) = -\left(x^2+2x+1-1-4\right) = -\left((x+1)^2-5\right) = 5-(x+1)^2$$

The integral is,

$$\int \frac{x-3}{\left(4-2x-x^2\right)^2}\,dx = \int \frac{x-3}{\left[5-(x+1)^2\right]^2}\,dx$$

Now, let's do the substitution.

$$u = x+1 \qquad x = u-1 \qquad dx = du$$

and the integral is now,

$$\begin{aligned}\int \frac{x-3}{\left(4-2x-x^2\right)^2}\,dx &= \int \frac{u-4}{\left(5-u^2\right)^2}\,du \\ &= \int \frac{u}{\left(5-u^2\right)^2}\,du - \int \frac{4}{\left(5-u^2\right)^2}\,du\end{aligned}$$

In the first integral we'll use the substitution

$$v = 5-u^2$$

and in the second integral we'll use the following trig substitution

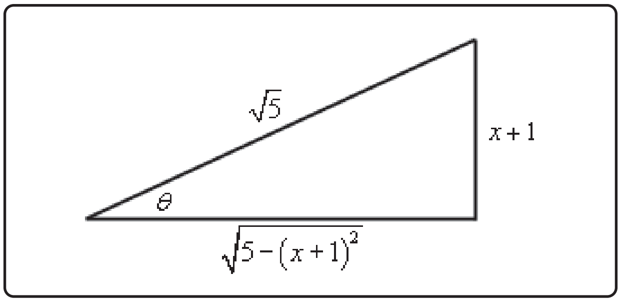
$$u = \sqrt{5}\sin(\theta) \qquad du = \sqrt{5}\cos(\theta)\,d\theta$$

Using these substitutions the integral becomes,

$$\begin{aligned}\int \frac{x-3}{\left(4-2x-x^2\right)^2}\,dx &= -\frac{1}{2}\int \frac{1}{v^2}\,dv - \int \frac{4}{\left(5-5\sin^2(\theta)\right)^2}\left(\sqrt{5}\cos(\theta)\right)\,d\theta \\ &= \frac{1}{2}\frac{1}{v} - \frac{4\sqrt{5}}{25}\int \frac{\cos(\theta)}{\left(1-\sin^2(\theta)\right)^2}\,d\theta \\ &= \frac{1}{2}\frac{1}{v} - \frac{4\sqrt{5}}{25}\int \frac{\cos(\theta)}{\cos^4(\theta)}\,d\theta \\ &= \frac{1}{2}\frac{1}{v} - \frac{4\sqrt{5}}{25}\int \sec^3(\theta)\,d\theta \\ &= \frac{1}{2}\frac{1}{v} - \frac{2\sqrt{5}}{25}\left(\sec(\theta)\tan(\theta) + \ln\left|\sec(\theta)+\tan(\theta)\right|\right) + c\end{aligned}$$

We'll need the following right triangle to finish this integral out.

$$\sin(\theta) = \frac{u}{\sqrt{5}} = \frac{x+1}{\sqrt{5}} \qquad \sec(\theta) = \frac{\sqrt{5}}{\sqrt{5-(x+1)^2}} \qquad \tan(\theta) = \frac{x+1}{\sqrt{5-(x+1)^2}}$$



So, going back to x's the integral becomes,

$$\begin{aligned}\int \frac{x-3}{(4-2x-x^2)^2}\,dx &= \frac{1}{2}\frac{1}{5-u^2} - \\ &\qquad \frac{2\sqrt{5}}{25}\left(\frac{\sqrt{5}\,(x+1)}{5-(x+1)^2} + \ln\left|\frac{\sqrt{5}}{\sqrt{5-(x+1)^2}} + \frac{x+1}{\sqrt{5-(x+1)^2}}\right|\right) + c \\ &= \frac{1}{10}\frac{1-4x}{5-(x+1)^2} - \frac{2\sqrt{5}}{25}\ln\left|\frac{x+1+\sqrt{5}}{\sqrt{5-(x+1)^2}}\right| + c\end{aligned}$$

Often the following formula is needed when using the trig substitution that we used in the previous example.

$$\int \sec^m(\theta)\,d\theta = \frac{1}{m-1}\tan(\theta)\sec^{m-2}(\theta) + \frac{m-2}{m-1}\int \sec^{m-2}(\theta)\,d\theta$$

Note that we'll only need the two trig substitutions (sine and tangent) that we used here. The third trig substitution (secant) that we used will not be needed here. Any quadratic that could use a secant substitution can be turned into a sine substitution simply by factoring a minus sign out of the quadratic. Note that we can do that for these types of problems because we don't have a root and so the minus sign can be completely factored out of the integrand while we couldn't do that with the roots we had in the problems back in the Trig Substitution section.

7.7 Integration Strategy

We've now seen a fair number of different integration techniques and so we should probably pause at this point and talk a little bit about a strategy to use for determining the correct technique to use when faced with an integral.

There are a couple of points that need to be made about this strategy. First, it isn't a hard and fast set of rules for determining the method that should be used. It is really nothing more than a general set of guidelines that will help us to identify techniques that may work. Some integrals can be done in more than one way and so depending on the path you take through the strategy you may end up with a different technique than somebody else who also went through this strategy.

Second, while the strategy is presented as a way to identify the technique that could be used on an integral also keep in mind that, for many integrals, it can also automatically exclude certain techniques as well. When going through the strategy keep two lists in mind. The first list is integration techniques that simply won't work and the second list is techniques that look like they might work. After going through the strategy and the second list has only one entry then that is the technique to use. If, on the other hand, there is more than one possible technique to use we will then have to decide on which is liable to be the best for us to use. Unfortunately, there is no way to teach which technique is the best as that usually depends upon the person and which technique they find to be the easiest.

Third, don't forget that many integrals can be evaluated in multiple ways and so more than one technique may be used on it. This has already been mentioned in each of the previous points but is important enough to warrant a separate mention. Sometimes one technique will be significantly easier than the others and so don't just stop at the first technique that appears to work. Always identify all possible techniques and then go back and determine which you feel will be the easiest for you to use.

Next, it's entirely possible that you will need to use more than one method to completely do an integral. For instance, a substitution may lead to using integration by parts or partial fractions integral.

Finally, in my class I will accept any valid integration technique as a solution. As already noted there is often more than one way to do an integral and just because I find one technique to be the easiest doesn't mean that you will as well. So, in my class, there is no one right way of doing an integral. You may use any integration technique that I've taught you in this class or you learned in Calculus I to evaluate integrals in this class. In other words, always take the approach that you find to be the easiest.

Note that this final point is more geared towards my class and it's completely possible that your instructor may not agree with this and so be careful in applying this point if you aren't in my class.

Okay, let's get on with the strategy.

Integration Strategy

1. **Simplify the integrand, if possible.** This step is very important in the integration process. Many integrals can be taken from impossible or very difficult to very easy with a little simplification or manipulation. Don't forget basic trig and algebraic identities as these can often be used to simplify the integral. We used this idea when we were looking at integrals involving trig functions. For example, consider the following integral.

$$\int \cos^2(x)\,dx$$

This integral can't be done as is, however simply by recalling the identity,

$$\cos^2(x) = \frac{1}{2}\left(1+\cos(2x)\right)$$

the integral becomes very easy to do.

Note that this example also shows that simplification does not necessarily mean that we'll write the integrand in a "simpler" form. It only means that we'll write the integrand into a form that we can deal with and this is often longer and/or "messier" than the original integral.

2. **See if a "simple", *i.e.* a u-substitution will work.** Look to see if a simple substitution can be used instead of the often more complicated methods from Calculus II. For example, consider both of the following integrals.

$$\int \frac{x}{x^2-1}\,dx \qquad\qquad \int x\sqrt{x^2-1}\,dx$$

The first integral can be done with partial fractions and the second could be done with a trig substitution.

However, both could also be evaluated using the substitution $u = x^2 - 1$ and the work involved in the substitution would be significantly less than the work involved in either partial fractions or trig substitution.

So, always look for quick, simple substitutions before moving on to the more complicated Calculus II techniques.

3. **Identify the type of integral.** Note that any integral may fall into more than one of these types. Because of this fact it's usually best to go all the way through the list and identify all possible types since one may be easier than the other and it's entirely possible that the easier type is listed lower in the list.

   (a) Is the integrand a rational expression (*i.e* is the integrand a polynomial divided by a polynomial)? If so, then partial fractions may work on the integral.

(b) Is the integrand a polynomial times a trig function, exponential, or logarithm? If so, then integration by parts may work.

(c) Is the integrand a product of sines and cosines, secant and tangents, or cosecants and cotangents? If so, then the topics from the second section may work.

Likewise, don't forget that some quotients involving these functions can also be done using these techniques.

(d) Does the integrand involve $\sqrt{b^2x^2 + a^2}$, $\sqrt{b^2x^2 - a^2}$, or $\sqrt{a^2 - b^2x^2}$? If so, then a trig substitution might work nicely.

(e) Does the integrand have roots other than those listed above in it? If so, then the substitution $u = \sqrt[n]{g(x)}$ might work.

(f) Does the integrand have a quadratic in it? If so, then completing the square on the quadratic might put it into a form that we can deal with.

4. **Can we relate the integral to an integral we already know how to do?** In other words, can we use a substitution or manipulation to write the integrand into a form that does fit into the forms we've looked at previously in this chapter. typical example here is the following integral.

$$\int \cos(x)\sqrt{1 + \sin^2(x)}\,dx$$

This integral doesn't obviously fit into any of the forms we looked at in this chapter. However, with the substitution $u = \sin(x)$ we can reduce the integral to the form,

$$\int \sqrt{1 + u^2}\,du$$

which is a trig substitution problem.

5. **Do we need to use multiple techniques?** In this step we need to ask ourselves if it is possible that we'll need to use multiple techniques. The example in the previous part is a good example. Using a substitution didn't allow us to actually do the integral. All it did was put the integral and put it into a form that we could use a different technique on. Don't ever get locked into the idea that an integral will only require one step to completely evaluate it. Many will require more than one step.

6. **Try again.** If everything that you've tried to this point doesn't work then go back through the process and try again. This time try a technique that you didn't use the first time around.

As noted above this strategy is not a hard and fast set of rules. It is only intended to guide you through the process of best determining how to do any given integral. Note as well that the only place Calculus II actually arises is in the third step. Steps 1, 2 and 4 involve nothing more than manipulation of the integrand either through direct manipulation of the integrand or by using a substitution. The last two steps are simply ideas to think about in going through this strategy.

Many students go through this process and concentrate almost exclusively on Step 3 (after all this is Calculus II, so it's easy to see why they might do that....) to the exclusion of the other steps. One very large consequence of that exclusion is that often a simple manipulation or substitution is overlooked that could make the integral very easy to do.

Before moving on to the next section we should work a couple of quick problems illustrating a couple of not so obvious simplifications/manipulations and a not so obvious substitution.

Example 1

Evaluate the following integral.

$$\int \frac{\tan(x)}{\sec^4(x)}\,dx$$

Solution

This integral almost falls into the form given in **3c**. It is a quotient of tangent and secant and we know that sometimes we can use the same methods for products of tangents and secants on quotients.

The process from that section tells us that if we have even powers of secant to strip two of them off and convert the rest to tangents. That won't work here. We can split two secants off, but they would be in the denominator and they won't do us any good there. Remember that the point of splitting them off is so they would be there for the substitution $u = \tan(x)$. That requires them to be in the numerator. So, that won't work and so we'll have to find another solution method.

There are in fact two solution methods to this integral depending on how you want to go about it. We'll take a look at both.

<u>**Solution 1**</u>
In this solution method we could just convert everything to sines and cosines and see if that

gives us an integral we can deal with.

$$\begin{aligned}\int \frac{\tan(x)}{\sec^4(x)}\,dx &= \int \frac{\sin(x)}{\cos(x)}\cos^4(x)\,dx \\ &= \int \sin(x)\cos^3(x)\,dx \qquad\qquad u=\cos(x) \\ &= -\int u^3\,du \\ &= -\frac{1}{4}\cos^4(x)+c\end{aligned}$$

Note that just converting to sines and cosines won't always work and if it does it won't always work this nicely. Often there will be a lot more work that would need to be done to complete the integral.

Solution 2

This solution method goes back to dealing with secants and tangents. Let's notice that if we had a secant in the numerator we could just use $u=\sec(x)$ as a substitution and it would be a fairly quick and simple substitution to use. We don't have a secant in the numerator. However, we could very easily get a secant in the numerator simply by multiplying the numerator and denominator by secant.

$$\begin{aligned}\int \frac{\tan(x)}{\sec^4(x)}\,dx &= \int \frac{\tan(x)\sec(x)}{\sec^5(x)}\,dx \qquad\qquad u=\sec(x) \\ &= \int \frac{1}{u^5}\,du \\ &= -\frac{1}{4}\frac{1}{\sec^4(x)}+c \\ &= -\frac{1}{4}\cos^4(x)+c\end{aligned}$$

In the previous example we saw two "simplifications" that allowed us to do the integral. The first was using identities to rewrite the integral into terms we could deal with and the second involved multiplying the numerator and the denominator by something to again put the integral into terms we could deal with.

Using identities to rewrite an integral is an important "simplification" and we should not forget about it. Integrals can often be greatly simplified or at least put into a form that can be dealt with by using an identity.

The second "simplification" is not used as often, but does show up on occasion so again, it's best to not forget about it. In fact, let's take another look at an example in which multiplying the numerator and denominator by something will allow us to do an integral.

Example 2

Evaluate the following integral.

$$\int \frac{1}{1+\sin(x)}\,dx$$

Solution

This is an integral in which if we just concentrate on the third step we won't get anywhere. This integral doesn't appear to be any of the kinds of integrals that we worked in this chapter.

We can do the integral however, if we do the following,

$$\begin{aligned}\int \frac{1}{1+\sin(x)}\,dx &= \int \frac{1}{1+\sin(x)}\frac{1-\sin(x)}{1-\sin(x)}\,dx\\ &= \int \frac{1-\sin(x)}{1-\sin^2(x)}\,dx\end{aligned}$$

This does not appear to have done anything for us. However, if we now remember the first "simplification" we looked at above we will notice that we can use an identity to rewrite the denominator. Once we do that we can further reduce the integral into something we can deal with.

$$\begin{aligned}\int \frac{1}{1+\sin(x)}\,dx &= \int \frac{1-\sin(x)}{\cos^2(x)}\,dx\\ &= \int \frac{1}{\cos^2(x)} - \frac{\sin(x)}{\cos(x)}\frac{1}{\cos(x)}\,dx\\ &= \int \sec^2(x) - \tan(x)\sec(x)\,dx\\ &= \tan(x) - \sec(x) + c\end{aligned}$$

So, we've seen once again that multiplying the numerator and denominator by something can put the integral into a form that we can integrate. Notice as well that this example also showed that "simplifications" do not necessarily put an integral into a simpler form. They only put the integral into a form that is easier to integrate.

Let's now take a quick look at an example of a substitution that is not so obvious.

Example 3

Evaluate the following integral.

$$\int \cos\left(\sqrt{x}\right)\, dx$$

Solution

We introduced this example saying that the substitution was not so obvious. However, this is really an integral that falls into the form given by 3e in our strategy above. However, many people miss that form and so don't think about it. So, let's try the following substitution.

$$u = \sqrt{x} \qquad x = u^2 \qquad dx = 2u\, du$$

With this substitution the integral becomes,

$$\int \cos\left(\sqrt{x}\right)\, dx = 2\int u \cos(u)\, du$$

This is now an integration by parts integral. Remember that often we will need to use more than one technique to completely do the integral. This is a fairly simple integration by parts problem so we'll leave the remainder of the details to you to check.

$$\int \cos\left(\sqrt{x}\right)\, dx = 2\left(\cos\left(\sqrt{x}\right) + \sqrt{x}\sin\left(\sqrt{x}\right)\right) + c$$

Before leaving this section we should also point out that there are integrals out there in the world that just can't be done in terms of functions that we know. Some examples of these are.

$$\int \mathbf{e}^{-x^2}\, dx \qquad \int \cos\left(x^2\right)\, dx \qquad \int \frac{\sin(x)}{x}\, dx \qquad \int \cos\left(\mathbf{e}^x\right)\, dx$$

That doesn't mean that these integrals can't be done at some level. If you go to a computer algebra system such as Maple or Mathematica and have it do these integrals it will return the following.

$$\int \mathbf{e}^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)$$

$$\int \cos\left(x^2\right)\, dx = \sqrt{\frac{\pi}{2}}\operatorname{FresnelC}\left(x\sqrt{\frac{2}{\pi}}\right)$$

$$\int \frac{\sin(x)}{x}\, dx = \operatorname{Si}(x)$$

$$\int \cos\left(\mathbf{e}^x\right)\, dx = \operatorname{Ci}\left(\mathbf{e}^x\right)$$

So, it appears that these integrals can in fact be done. However, this is a little misleading. Here are the definitions of each of the functions given above.

Error Function

$$\text{erf}\left(x\right) = \frac{2}{\sqrt{\pi}}\int_0^x \mathbf{e}^{-t^2}\,dt$$

The Sine Integral

$$\text{Si}\left(x\right) = \int_0^x \frac{\sin(t)}{t}\,dt$$

The Fresnel Cosine Integral

$$\text{FresnelC}\left(x\right) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)\,dt$$

The Cosine Integral

$$\text{Ci}\left(x\right) = \gamma + \ln\left(x\right) + \int_0^x \frac{\cos(t) - 1}{t}\,dt$$

Where γ is the Euler-Mascheroni constant.

Note that the first three are simply defined in terms of themselves and so when we say we can integrate them all we are really doing is renaming the integral. The fourth one is a little different and yet it is still defined in terms of an integral that can't be done in practice.

It will be possible to integrate every integral given in this class, but it is important to note that there are integrals that just can't be done. We should also note that after we look at Series we will be able to write down series representations of each of the integrals above.