10 Series and Sequences

Once again, as with the last chapter, we are going to be looking at a completely different topic. The only material from previous chapters that will be needed here will be the ability to compute limits at infinity (we'll do a fair amount of these), compute the occasional derivative and integral. The integrals will, generally, be fairly simple and needing u substitutions every once in a while although we will see the occasional integral requiring integration by parts or partial fractions. So, basically, the material in this chapter doesn't rely all that much on previous material.

Series is one of those topics that many students don't find all that useful. To be honest, many students will never see series outside of their calculus class. However, series do play an important role in the field of ordinary differential equations and without series large portions of the field of partial differential equations would not be possible.

In other words, series is an important topic even if you won't ever see any of the applications. Most of the applications are beyond the scope of most Calculus courses and tend to occur in classes that many students don't take. So, as you go through this material keep in mind that these do have applications even if we won't really be covering many of them in this class.

The first topic we'll be looking at in this chapter is that of a sequence. We'll define just what we mean by a sequence and look at some basic topics and concepts that we'll need to work with them.

The other topic will be that of (infinite) series. In fact, we will spend the vast majority of this chapter deal with series. We can't, however, fully discuss series without understanding sequences and hence the reason for discussing sequences first. We will define just what an infinite series is and what it means for a series to converge or diverge. The majority of this chapter will then be spent discussing a variety of methods for testing whether or not a series will converge or diverge.

We'll close out the chapter with a discussion of power series and Taylor series as well as a couple of quick applications of series that we can easily discuss without needing any extra knowledge (as is needed for most applications of series).

10.1 Sequences

Let's start off this section with a discussion of just what a sequence is. A sequence is nothing more than a list of numbers written in a specific order. The list may or may not have an infinite number of terms in it although we will be dealing exclusively with infinite sequences in this class. General sequence terms are denoted as follows,

$$\begin{aligned}
&a_1 - \text{first term}\\
&a_2 - \text{second term}\\
&\qquad\vdots\\
&a_n - n^{th} \text{ term}\\
&a_{n+1} - (n+1)^{\text{st}} \text{ term}\\
&\qquad\vdots
\end{aligned}$$

Because we will be dealing with infinite sequences each term in the sequence will be followed by another term as noted above. In the notation above we need to be very careful with the subscripts. The subscript of $n+1$ denotes the next term in the sequence and NOT one plus the n^{th} term! In other words,

$$a_{n+1} \neq a_n + 1$$

so be very careful when writing subscripts to make sure that the "$+1$" doesn't migrate out of the subscript! This is an easy mistake to make when you first start dealing with this kind of thing.

There is a variety of ways of denoting a sequence. Each of the following are equivalent ways of denoting a sequence.

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

In the second and third notations above a_n is usually given by a formula.

A couple of notes are now in order about these notations. First, note the difference between the second and third notations above. If the starting point is not important or is implied in some way by the problem it is often not written down as we did in the third notation. Next, we used a starting point of $n = 1$ in the third notation only so we could write one down. There is absolutely no reason to believe that a sequence will start at $n = 1$. A sequence will start whereever it needs to start.

Let's take a look at a couple of sequences.

Example 1

Write down the first few terms of each of the following sequences.

(a) $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\dfrac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

(c) $\{b_n\}_{n=1}^{\infty}$, where $b_n = n^{th}$ digit of π

Solution

(a) $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$

To get the first few sequence terms here all we need to do is plug in values of n into the formula given and we'll get the sequence terms.

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{\underbrace{2}_{n=1}, \underbrace{\frac{3}{4}}_{n=2}, \underbrace{\frac{4}{9}}_{n=3}, \underbrace{\frac{5}{16}}_{n=4}, \underbrace{\frac{6}{25}}_{n=5}, \ldots\right\}$$

Note the inclusion of the "..." at the end! This is an important piece of notation as it is the only thing that tells us that the sequence continues on and doesn't terminate at the last term.

(b) $\left\{\dfrac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

This one is similar to the first one. The main difference is that this sequence doesn't start at $n = 1$.

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right\}$$

Note that the terms in this sequence alternate in signs. Sequences of this kind are sometimes called alternating sequences.

(c) $\{b_n\}_{n=1}^{\infty}$, where $b_n = n^{th}$ digit of π

This sequence is different from the first two in the sense that it doesn't have a specific formula for each term. However, it does tell us what each term should be. Each term should be the n^{th} digit of π. So we know that $\pi = 3.14159265359\ldots$

The sequence is then,

$$\{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \ldots\}$$

In the first two parts of the previous example note that we were really treating the formulas as functions that can only have integers plugged into them. Or,

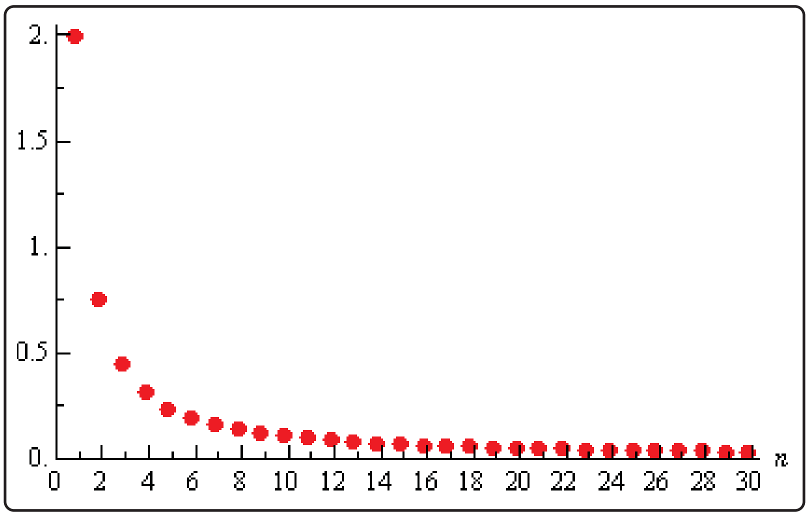
$$f(n) = \frac{n+1}{n^2} \qquad\qquad g(n) = \frac{(-1)^{n+1}}{2^n}$$

This is an important idea in the study of sequences (and series). Treating the sequence terms as function evaluations will allow us to do many things with sequences that we couldn't do otherwise. Before delving further into this idea however we need to get a couple more ideas out of the way.

First, we want to think about "graphing" a sequence. To graph the sequence $\{a_n\}$ we plot the points (n, a_n) as n ranges over all possible values on a graph. For instance, let's graph the sequence $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$. The first few points on the graph are,

$$(1,2)\,,\ \left(2,\frac{3}{4}\right),\ \left(3,\frac{4}{9}\right),\ \left(4,\frac{5}{16}\right),\ \left(5,\frac{6}{25}\right),\ \ldots$$

The graph, for the first 30 terms of the sequence, is then,



This graph leads us to an important idea about sequences. Notice that as n increases the sequence terms in our sequence, in this case, get closer and closer to zero. We then say that zero is the **limit** (or sometimes the **limiting value**) of the sequence and write,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n^2} = 0$$

This notation should look familiar to you. It is the same notation we used when we talked about the limit of a function. In fact, if you recall, we said earlier that we could think of sequences as functions in some way and so this notation shouldn't be too surprising.

Using the ideas that we developed for limits of functions we can write down the following *working definition* for limits of sequences.

Working Definition of Limit

1. We say that
$$\lim_{n\to\infty} a_n = L$$
if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of the a_n's approach L as n approaches infinity.

2. We say that
$$\lim_{n\to\infty} a_n = \infty$$
if we can make a_n as large as we want for all sufficiently large n. Again, in other words, the value of the a_n's get larger and larger without bound as n approaches infinity.

3. We say that
$$\lim_{n\to\infty} a_n = -\infty$$
if we can make a_n as large and negative as we want for all sufficiently large n. Again, in other words, the value of the a_n's are negative and get larger and larger without bound as n approaches infinity.

The working definitions of the various sequence limits are nice in that they help us to visualize what the limit actually is. Just like with limits of functions however, there is also a precise definition for each of these limits. Let's give those before proceeding

Precise Definition of Limit

1. We say that $\lim\limits_{n\to\infty} a_n = L$ if for every number $\varepsilon > 0$ there is an integer N such that
$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N$$

2. We say that $\lim\limits_{n\to\infty} a_n = \infty$ if for every number $M > 0$ there is an integer N such that
$$a_n > M \qquad \text{whenever} \qquad n > N$$

3. We say that $\lim\limits_{n\to\infty} a_n = -\infty$ if for every number $M < 0$ there is an integer N such that
$$a_n < M \qquad \text{whenever} \qquad n > N$$

We won't be using the precise definition often, but it will show up occasionally.

Note that both definitions tell us that in order for a limit to exist and have a finite value all the sequence terms must be getting closer and closer to that finite value as n increases.

Now that we have the definitions of the limit of sequences out of the way we have a bit of terminology that we need to look at. If $\lim\limits_{n\to\infty} a_n$ exists and is finite we say that the sequence is **convergent**. If $\lim\limits_{n\to\infty} a_n$ doesn't exist or is infinite we say the sequence **diverges**. Note that sometimes we will say the sequence **diverges to** ∞ if $\lim\limits_{n\to\infty} a_n = \infty$ and if $\lim\limits_{n\to\infty} a_n = -\infty$ we will sometimes say that the sequence **diverges to** $-\infty$.

Get used to the terms "convergent" and "divergent" as we'll be seeing them quite a bit throughout this chapter.

So just how do we find the limits of sequences? Most limits of most sequences can be found using one of the following theorems.

Theorem 1

Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{n\to\infty} a_n = L$

This theorem is basically telling us that we take the limits of sequences much like we take the limit of functions. In fact, in most cases we'll not even really use this theorem by explicitly writing down a function. We will more often just treat the limit as if it were a limit of a function and take the limit as we always did back in Calculus I when we were taking the limits of functions.

So, now that we know that taking the limit of a sequence is nearly identical to taking the limit of a function we also know that all the properties from the limits of functions will also hold.

Properties

If $\{a_n\}$ and $\{b_n\}$ are both convergent sequences then,

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
2. $\lim\limits_{n\to\infty} c a_n = c \lim\limits_{n\to\infty} a_n$
3. $\lim\limits_{n\to\infty} (a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$
4. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, provided $\lim\limits_{n\to\infty} b_n \neq 0$
5. $\lim\limits_{n\to\infty} a_n^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$ provided $a_n \geq 0$

These properties can be proved using Theorem 1 above and the function limit properties we saw in Calculus I or we can prove them directly using the precise definition of a limit using nearly identical proofs of the function limit properties.

Next, just as we had a Squeeze Theorem for function limits we also have one for sequences and it is pretty much identical to the function limit version.

Squeeze Theorem for Sequences

If $a_n \le c_n \le b_n$ for all $n > N$ for some N and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$ then $\lim\limits_{n\to\infty} c_n = L$.

Note that in this theorem the "for all $n > N$ for some N" is really just telling us that we need to have $a_n \le c_n \le b_n$ for all sufficiently large n, but if it isn't true for the first few n that won't invalidate the theorem.

As we'll see not all sequences can be written as functions that we can actually take the limit of. This will be especially true for sequences that alternate in signs. While we can always write these sequence terms as a function we simply don't know how to take the limit of a function like that. The following theorem will help with some of these sequences.

Theorem 2

If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$.

Note that in order for this theorem to hold the limit MUST be zero and it won't work for a sequence whose limit is not zero. This theorem is easy enough to prove so let's do that.

Proof of Theorem 2

The main thing to this proof is to note that,

$$-|a_n| \leq a_n \leq |a_n|$$

Then note that,

$$\lim_{n\to\infty} (-|a_n|) = -\lim_{n\to\infty} |a_n| = 0$$

We then have $\lim_{n\to\infty} (-|a_n|) = \lim_{n\to\infty} |a_n| = 0$ and so by the Squeeze Theorem we must also have,

$$\lim_{n\to\infty} a_n = 0$$

The next theorem is a useful theorem giving the convergence/divergence and value (for when it's convergent) of a sequence that arises on occasion.

Theorem 3

The sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \leq 1$ and diverges for all other values of r. Also,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Here is a quick (well not so quick, but definitely simple) partial proof of this theorem.

Partial Proof of Theorem 3

We'll do this by a series of cases although the last case will not be completely proven.

Case 1 : $r > 1$

We know from Calculus I that $\lim_{x\to\infty} r^x = \infty$ if $r > 1$ and so by Theorem 1 above we also know that $\lim_{n\to\infty} r^n = \infty$ and so the sequence diverges if $r > 1$.

Case 2 : $r = 1$

In this case we have,

$$\lim_{n\to\infty} r^n = \lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$$

So, the sequence converges for $r = 1$ and in this case its limit is 1.

Case 3 : $0 < r < 1$

We know from Calculus I that $\lim_{x\to\infty} r^x = 0$ if $0 < r < 1$ and so by Theorem 1 above we also know that $\lim\limits_{n\to\infty} r^n = 0$ and so the sequence converges if $0 < r < 1$ and in this case its limit is zero.

Case 4 : $r = 0$

In this case we have,

$$\lim_{n\to\infty} r^n = \lim_{n\to\infty} 0^n = \lim_{n\to\infty} 0 = 0$$

So, the sequence converges for $r = 0$ and in this case its limit is zero.

Case 5 : $-1 < r < 0$

First let's note that if $-1 < r < 0$ then $0 < |r| < 1$ then by Case 3 above we have,

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

Theorem 2 above now tells us that we must also have, $\lim\limits_{n\to\infty} r^n = 0$ and so if $-1 < r < 0$ the sequence converges and has a limit of 0.

Case 6 : $r = -1$

In this case the sequence is,

$$\{r^n\}_{n=0}^{\infty} = \{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1, 1, -1, 1, -1, \ldots\}_{n=0}^{\infty}$$

and hopefully it is clear that $\lim\limits_{n\to\infty} (-1)^n$ doesn't exist. Recall that in order of this limit to exist the terms must be approaching a single value as n increases. In this case however the terms just alternate between 1 and -1 and so the limit does not exist.

So, the sequence diverges for $r = -1$.

Case 7: $r < -1$

In this case we're not going to go through a complete proof. Let's just see what happens if we let $r = -2$ for instance. If we do that the sequence becomes,

$$\{r^n\}_{n=0}^{\infty} = \{(-2)^n\}_{n=0}^{\infty} = \{1, -2, 4, -8, 16, -32, \ldots\}_{n=0}^{\infty}$$

So, if $r = -2$ we get a sequence of terms whose values alternate in sign and get larger and larger and so $\lim\limits_{n\to\infty} (-2)^n$ doesn't exist. It does not settle down to a single value as n increases nor do the terms ALL approach infinity. So, the sequence diverges for $r = -2$.

We could do something similar for any value of r such that $r < -1$ and so the sequence diverges for $r < -1$.

Let's take a look at a couple of examples of limits of sequences.

Example 2

Determine if the following sequences converge or diverge. If the sequence converges determine its limit.

(a) $\left\{\dfrac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

(b) $\left\{\dfrac{\mathbf{e}^{2n}}{n}\right\}_{n=1}^{\infty}$

(c) $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

(d) $\left\{(-1)^n\right\}_{n=0}^{\infty}$

Solution

(a) $\left\{\dfrac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

In this case all we need to do is recall the method that was developed in Calculus I to deal with the limits of rational functions. See the Limits At Infinity, Part I section of the Calculus I notes for a review of this if you need to.

To do a limit in this form all we need to do is factor from the numerator and denominator the largest power of n, cancel and then take the limit.

$$\lim_{n\to\infty}\frac{3n^2-1}{10n+5n^2}=\lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)}=\lim_{n\to\infty}\frac{3-\frac{1}{n^2}}{\frac{10}{n}+5}=\frac{3}{5}$$

So, the sequence converges and its limit is $\frac{3}{5}$.

(b) $\left\{\dfrac{\mathbf{e}^{2n}}{n}\right\}_{n=1}^{\infty}$

We will need to be careful with this one. We will need to use L'Hospital's Rule on this sequence. The problem is that L'Hospital's Rule only works on functions and not on sequences. Normally this would be a problem, but we've got Theorem 1 from above to help us out. Let's define

$$f(x)=\frac{\mathbf{e}^{2x}}{x}$$

and note that,

$$f(n) = \frac{\mathbf{e}^{2n}}{n}$$

Theorem 1 says that all we need to do is take the limit of the function.

$$\lim_{n\to\infty} \frac{\mathbf{e}^{2n}}{n} = \lim_{x\to\infty} \frac{\mathbf{e}^{2x}}{x} = \lim_{x\to\infty} \frac{2\mathbf{e}^{2x}}{1} = \infty$$

So, the sequence in this part diverges (to ∞).

More often than not we just do L'Hospital's Rule on the sequence terms without first converting to x's since the work will be identical regardless of whether we use x or n. However, we really should remember that technically we can't do the derivatives while dealing with sequence terms.

(c) $\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$

We will also need to be careful with this sequence. We might be tempted to just say that the limit of the sequence terms is zero (and we'd be correct). However, technically we can't take the limit of sequences whose terms alternate in sign, because we don't know how to do limits of functions that exhibit that same behavior. Also, we want to be very careful to not rely too much on intuition with these problems. As we will see in the next section, and in later sections, our intuition can lead us astray in these problems if we aren't careful.

So, let's work this one by the book. We will need to use Theorem 2 on this problem. To this we'll first need to compute,

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, since the limit of the sequence terms with absolute value bars on them goes to zero we know by Theorem 2 that,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

which also means that the sequence converges to a value of zero.

(d) $\left\{ (-1)^n \right\}_{n=0}^{\infty}$

For this theorem note that all we need to do is realize that this is the sequence in Theorem 3 above using $r = -1$. So, by Theorem 3 this sequence diverges.

We now need to give a warning about misusing Theorem 2. Theorem 2 only works if the limit is zero. If the limit of the absolute value of the sequence terms is not zero then the theorem will not hold. The last part of the previous example is a good example of this (and in fact this warning is the whole reason that part is there). Notice that

$$\lim_{n\to\infty} |(-1)^n| = \lim_{n\to\infty} 1 = 1$$

and yet, $\lim_{n\to\infty} (-1)^n$ doesn't even exist let alone equal 1. So, be careful using this Theorem 2. You must always remember that it only works if the limit is zero.

Before moving onto the next section we need to give one more theorem that we'll need for a proof down the road.

Theorem 4

For the sequence $\{a_n\}$ if both $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$ then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Proof of Theorem 4

Let $\varepsilon > 0$.

Then since $\lim_{n\to\infty} a_{2n} = L$ there is an $N_1 > 0$ such that if $n > N_1$ we know that,

$$|a_{2n} - L| < \varepsilon$$

Likewise, because $\lim_{n\to\infty} a_{2n+1} = L$ there is an $N_2 > 0$ such that if $n > N_2$ we know that,

$$|a_{2n+1} - L| < \varepsilon$$

Now, let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. Then either $a_n = a_{2k}$ for some $k > N_1$ or $a_n = a_{2k+1}$ for some $k > N_2$ and so in either case we have that,

$$|a_n - L| < \varepsilon$$

Therefore, $\lim_{n\to\infty} a_n = L$ and so $\{a_n\}$ is convergent.

10.2 More on Sequences

In the previous section we introduced the concept of a sequence and talked about limits of sequences and the idea of convergence and divergence for a sequence. In this section we want to take a quick look at some ideas involving sequences.

Let's start off with some terminology and definitions.

Definition

Given any sequence $\{a_n\}$ we have the following.

1. We call the sequence **increasing** if $a_n < a_{n+1}$ for every n.
2. We call the sequence **decreasing** if $a_n > a_{n+1}$ for every n.
3. If $\{a_n\}$ is an increasing sequence or $\{a_n\}$ is a decreasing sequence we call it **monotonic**.
4. If there exists a number m such that $m \leq a_n$ for every n we say the sequence is **bounded below**. The number m is sometimes called a **lower bound** for the sequence.
5. If there exists a number M such that $a_n \leq M$ for every n we say the sequence is **bounded above**. The number M is sometimes called an **upper bound** for the sequence.
6. If the sequence is both bounded below and bounded above we call the sequence **bounded**.

Note that in order for a sequence to be increasing or decreasing it must be increasing/decreasing for every n. In other words, a sequence that increases for three terms and then decreases for the rest of the terms is NOT a decreasing sequence! Also note that a monotonic sequence must always increase or it must always decrease.

Before moving on we should make a quick point about the bounds for a sequence that is bounded above and/or below. We'll make the point about lower bounds, but we could just as easily make it about upper bounds.

A sequence is bounded below if we can find any number m such that $m \leq a_n$ for every n. Note however that if we find one number m to use for a lower bound then any number smaller than m will also be a lower bound. Also, just because we find one lower bound that doesn't mean there won't be a "better" lower bound for the sequence than the one we found. In other words, there are an infinite number of lower bounds for a sequence that is bounded below, some will be better than others. In my class all that I'm after will be a lower bound. I don't necessarily need the best lower bound, just a number that will be a lower bound for the sequence.

Let's take a look at a couple of examples.

Example 1

Determine if the following sequences are monotonic and/or bounded.

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$

(c) $\left\{\dfrac{2}{n^2}\right\}_{n=5}^{\infty}$

Solution

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) because,

$$-n^2 > -(n+1)^2 \qquad \text{for every } n$$

Also, since the sequence terms will be either zero or negative this sequence is bounded above. We can use any positive number or zero as the bound, M, however, it's standard to choose the smallest possible bound if we can and it's a nice number. So, we'll choose $M = 0$ since,

$$-n^2 \le 0 \qquad \text{for every } n$$

This sequence is not bounded below however since we can always get below any potential bound by taking n large enough. Therefore, while the sequence is bounded above it is not bounded.

As a side note we can also note that this sequence diverges (to $-\infty$ if we want to be specific).

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$

The sequence terms in this sequence alternate between 1 and -1 and so the sequence is neither an increasing sequence or a decreasing sequence. Since the sequence is neither an increasing nor decreasing sequence it is not a monotonic sequence.

The sequence is bounded however since it is bounded above by 1 and bounded below by -1.

Again, we can note that this sequence is also divergent.

(c) $\left\{ \frac{2}{n^2} \right\}_{n=5}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) since,

$$\frac{2}{n^2} > \frac{2}{(n+1)^2}$$

The terms in this sequence are all positive and so it is bounded below by zero. Also, since the sequence is a decreasing sequence the first sequence term will be the largest and so we can see that the sequence will also be bounded above by $\frac{2}{25}$. Therefore, this sequence is bounded.

We can also take a quick limit and note that this sequence converges and its limit is zero.

Now, let's work a couple more examples that are designed to make sure that we don't get too used to relying on our intuition with these problems. As we noted in the previous section our intuition can often lead us astray with some of the concepts we'll be looking at in this chapter.

Example 2

Determine if the following sequences are monotonic and/or bounded.

(a) $\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$

(b) $\left\{ \frac{n^3}{n^4 + 10000} \right\}_{n=0}^{\infty}$

Solution

(a) $\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$

We'll start with the bounded part of this example first and then come back and deal with the increasing/decreasing question since that is where students often make mistakes with this type of sequence.

First, n is positive and so the sequence terms are all positive. The sequence is therefore bounded below by zero. Likewise, each sequence term is the quotient of a number divided by a larger number and so is guaranteed to be less than one. The sequence is then bounded above by one. So, this sequence is bounded.

Now let's think about the monotonic question. First, students will often make the mistake of assuming that because the denominator is larger the quotient must be decreasing. This will not always be the case and in this case we would be wrong. This sequence is increasing as we'll see.

To determine the increasing/decreasing nature of this sequence we will need to resort to Calculus I techniques. First consider the following function and its derivative.

$$f(x) = \frac{x}{x+1} \qquad\qquad f'(x) = \frac{1}{(x+1)^2}$$

We can see that the first derivative is always positive and so from Calculus I we know that the function must then be an increasing function. So, how does this help us? Notice that,

$$f(n) = \frac{n}{n+1} = a_n$$

Therefore because $n < n+1$ and $f(x)$ is increasing we can also say that,

$$a_n = \frac{n}{n+1} = f(n) < f(n+1) = \frac{n+1}{n+1+1} = a_{n+1} \qquad \Rightarrow \qquad a_n < a_{n+1}$$

In other words, the sequence must be increasing.

Note that now that we know the sequence is an increasing sequence we can get a better lower bound for the sequence. Since the sequence is increasing the first term in the sequence must be the smallest term and so since we are starting at $n = 1$ we could also use a lower bound of $\frac{1}{2}$ for this sequence. It is important to remember that any number that is always less than or equal to all the sequence terms can be a lower bound. Some are better than others however.

A quick limit will also tell us that this sequence converges with a limit of 1.

Before moving on to the next part there is a natural question that many students will have at this point. Why did we use Calculus to determine the increasing/decreasing nature of the sequence when we could have just plugged in a couple of n's and quickly determined the same thing?

The answer to this question is the next part of this example!

(b) $\left\{\dfrac{n^3}{n^4+10000}\right\}_{n=0}^{\infty}$

This is a messy looking sequence, but it needs to be in order to make the point of this part.

First, notice that, as with the previous part, the sequence terms are all positive and will all be less than one (since the numerator is guaranteed to be less than the denominator) and so the sequence is bounded.

Now, let's move on to the increasing/decreasing question. As with the last problem, many students will look at the exponents in the numerator and denominator and determine based on that that sequence terms must decrease.

This however, isn't a decreasing sequence. Let's take a look at the first few terms to see this.

$$\begin{aligned} a_1 &= \frac{1}{10001} \approx 0.00009999 & a_2 &= \frac{1}{1252} \approx 0.0007987 \\ a_3 &= \frac{27}{10081} \approx 0.005678 & a_4 &= \frac{4}{641} \approx 0.006240 \\ a_5 &= \frac{1}{85} \approx 0.011756 & a_6 &= \frac{27}{1412} \approx 0.019122 \\ a_7 &= \frac{343}{12401} \approx 0.02766 & a_8 &= \frac{32}{881} \approx 0.03632 \\ a_9 &= \frac{729}{16561} \approx 0.04402 & a_{10} &= \frac{1}{20} = 0.05 \end{aligned}$$

The first 10 terms of this sequence are all increasing and so clearly the sequence can't be a decreasing sequence. Recall that a sequence can only be decreasing if ALL the terms are decreasing.

Now, we can't make another common mistake and assume that because the first few terms increase then whole sequence must also increase. If we did that we would also be mistaken as this is also not an increasing sequence.

This sequence is neither decreasing or increasing. The only sure way to see this is to do the Calculus I approach to increasing/decreasing functions.

In this case we'll need the following function and its derivative.

$$f(x) = \frac{x^3}{x^4 + 10000} \qquad f'(x) = \frac{-x^2\left(x^4 - 30000\right)}{\left(x^4 + 10000\right)^2}$$

This function will have the following three critical points,

$$x = 0, \;\; x = \sqrt[4]{30000} \approx 13.1607, \qquad x = -\sqrt[4]{30000} \approx -13.1607$$

Why critical points? Remember these are the only places where the derivative *may* change sign! Our sequence starts at $n = 0$ and so we can ignore the third one since it lies outside the values of n that we're considering. By plugging in some test values of x we can quickly determine that the derivative is positive for $0 < x < \sqrt[4]{30000} \approx 13.16$ and so the function is increasing in this range. Likewise, we can see that the derivative is negative for $x > \sqrt[4]{30000} \approx 13.16$ and so the function will be decreasing in this range.

So, our sequence will be increasing for $0 \le n \le 13$ and decreasing for $n \ge 13$. Therefore, the function is not monotonic.

Finally, note that this sequence will also converge and has a limit of zero.

So, as the last example has shown we need to be careful in making assumptions about sequences. Our intuition will often not be sufficient to get the correct answer and we can NEVER make assumptions about a sequence based on the value of the first few terms. As the last part has shown there are sequences which will increase or decrease for a few terms and then change direction after that.

Note as well that we said “first few terms” here, but it is completely possible for a sequence to decrease for the first 10,000 terms and then start increasing for the remaining terms. In other words, there is no “magical” value of n for which all we have to do is check up to that point and then we’ll know what the whole sequence will do.

The only time that we’ll be able to avoid using Calculus I techniques to determine the increasing/decreasing nature of a sequence is in sequences like part (c) of Example 1. In this case increasing n only changed (in fact increased) the denominator and so we were able to determine the behavior of the sequence based on that.

In Example 2 however, increasing n increased both the denominator and the numerator. In cases like this there is no way to determine which increase will “win out” and cause the sequence terms to increase or decrease and so we need to resort to Calculus I techniques to answer the question.

We’ll close out this section with a nice theorem that we’ll use in some of the proofs later in this chapter.

Theorem

If $\{a_n\}$ is bounded and monotonic then $\{a_n\}$ is convergent.

Be careful to not misuse this theorem. It does not say that if a sequence is not bounded and/or not monotonic that it is divergent. Example 2b is a good case in point. The sequence in that example was not monotonic but it does converge.

Note as well that we can make several variants of this theorem. If $\{a_n\}$ is bounded above and increasing then it converges and likewise if $\{a_n\}$ is bounded below and decreasing then it converges.

10.3 Series - Basics

In this section we will introduce the topic that we will be discussing for the rest of this chapter. That topic is infinite series. So just what is an infinite series? Well, let's start with a sequence $\{a_n\}_{n=1}^{\infty}$ (note the $n = 1$ is for convenience, it can be anything) and define the following,

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
&\vdots \\
s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^{n} a_i
\end{aligned}$$

The s_n are called **partial sums** and notice that they will form a sequence, $\{s_n\}_{n=1}^{\infty}$. Also recall that the Σ is used to represent this summation and called a variety of names. The most common names are : **series notation**, **summation notation**, and **sigma notation**.

You should have seen this notation, at least briefly, back when you saw the definition of a definite integral in Calculus I. If you need a quick refresher on summation notation see the review of summation notation in the Calculus I notes.

Now back to series. We want to take a look at the limit of the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$. To make the notation go a little easier we'll define,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i$$

We will call $\sum\limits_{i=1}^{\infty} a_i$ an **infinite series** and note that the series "starts" at $i = 1$ because that is where our original sequence, $\{a_n\}_{n=1}^{\infty}$, started. Had our original sequence started at 2 then our infinite series would also have started at 2. The infinite series will start at the same value that the sequence of terms (as opposed to the sequence of partial sums) starts.

It is important to note that $\sum\limits_{i=1}^{\infty} a_i$ is really nothing more than a convenient notation for $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} a_i$ so we do not need to keep writing the limit down. We do, however, always need to remind ourselves that we really do have a limit there!

If the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$, is convergent and its limit is finite then we also call the infinite series, $\sum\limits_{i=1}^{\infty} a_i$ **convergent** and if the sequence of partial sums is divergent then the infinite series is also called **divergent**.

Note that sometimes it is convenient to write the infinite series as,

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

We do have to be careful with this however. This implies that an infinite series is just an infinite sum of terms and as we'll see in the next section this is not really true for many series.

In the next section we're going to be discussing in greater detail the value of an infinite series, provided it has one of course, as well as the ideas of convergence and divergence.

This section is going to be devoted mostly to notational issues as well as making sure we can do some basic manipulations with infinite series so we are ready for them when we need to be able to deal with them in later sections.

First, we should note that in most of this chapter we will refer to infinite series as simply series. If we ever need to work with both infinite and finite series we'll be more careful with terminology, but in most sections we'll be dealing exclusively with infinite series and so we'll just call them series.

Now, in $\sum\limits_{i=1}^{\infty} a_i$ the i is called the **index of summation** or just **index** for short and note that the letter we use to represent the index does not matter. So for example the following series are all the same. The only difference is the letter we've used for the index.

$$\sum_{i=0}^{\infty} \frac{3}{i^2+1} = \sum_{k=0}^{\infty} \frac{3}{k^2+1} = \sum_{n=0}^{\infty} \frac{3}{n^2+1} \quad etc.$$

It is important to again note that the index will start at whatever value the sequence of series terms starts at and this can literally be anything. So far we've used $n = 0$ and $n = 1$ but the index could have started anywhere. In fact, we will usually use $\sum a_n$ to represent an infinite series in which the starting point for the index is not important. When we drop the initial value of the index we'll also drop the infinity from the top so don't forget that it is still technically there.

We will be dropping the initial value of the index in quite a few facts and theorems that we'll be seeing throughout this chapter. In these facts/theorems the starting point of the series will not affect the result and so to simplify the notation and to avoid giving the impression that the starting point is important we will drop the index from the notation. Do not forget however, that there is a starting point and that this will be an infinite series.

Note however, that if we do put an initial value of the index on a series in a fact/theorem it is there because it really does need to be there.

Now that some of the notational issues are out of the way we need to start thinking about various ways that we can manipulate series.

We'll start this off with basic arithmetic with infinite series as we'll need to be able to do that on occasion. We have the following properties.

Properties

If $\sum a_n$ and $\sum b_n$ are both convergent series then,

1. $\sum ca_n$, where c is any number, is also convergent and
$$\sum ca_n = c\sum a_n$$

2. $\sum\limits_{n=k}^{\infty} a_n \pm \sum\limits_{n=k}^{\infty} b_n$ is also convergent and,
$$\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n = \sum_{n=k}^{\infty} (a_n \pm b_n)$$

The first property is simply telling us that we can always factor a multiplicative constant out of an infinite series and again recall that if we don't put in an initial value of the index that the series can start at any value. Also recall that in these cases we won't put an infinity at the top either.

The second property says that if we add/subtract series all we really need to do is add/subtract the series terms. Note as well that in order to add/subtract series we need to make sure that both have the same initial value of the index and the new series will also start at this value.

Before we move on to a different topic let's discuss multiplication of series briefly. We'll start both series at $n = 0$ for a later formula and then note that,

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) \neq \sum_{n=0}^{\infty} (a_n b_n)$$

To convince yourself that this isn't true consider the following product of two finite sums.

$$(2 + x)\left(3 - 5x + x^2\right) = 6 - 7x - 3x^2 + x^3$$

Yeah, it was just the multiplication of two polynomials. Each is a finite sum and so it makes the point. In doing the multiplication we didn't just multiply the constant terms, then the x terms, *etc*. Instead we had to distribute the 2 through the second polynomial, then distribute the x through the second polynomial and finally combine like terms.

Multiplying infinite series (even though we said we can't think of an infinite series as an infinite sum) needs to be done in the same manner. With multiplication we're really asking us to do the following,

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = (a_0 + a_1 + a_2 + a_3 + \cdots)(b_0 + b_1 + b_2 + b_3 + \cdots)$$

To do this multiplication we would have to distribute the a_0 through the second term, distribute the a_1 through, *etc* then combine like terms. This is pretty much impossible since both series have an infinite set of terms in them, however the following formula can be used to determine the product of two series.

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n \quad \text{where} \quad c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

We also can't say a lot about the convergence of the product. Even if both of the original series are convergent it is possible for the product to be divergent. The reality is that multiplication of series is a somewhat difficult process and in general is avoided if possible. We will take a brief look at it towards the end of the chapter when we've got more work under our belt and we run across a situation where it might actually be what we want to do. Until then, don't worry about multiplying series.

The next topic that we need to discuss in this section is that of **index shift**. To be honest this is not a topic that we'll see all that often in this course. In fact, we'll use it once in the next section and then not use it again in all likelihood. Despite the fact that we won't use it much in this course doesn't mean however that it isn't used often in other classes where you might run across series. So, we will cover it briefly here so that you can say you've seen it.

The basic idea behind index shifts is to start a series at a different value for whatever the reason (and yes, there are legitimate reasons for doing that).

Consider the following series,

$$\sum_{n=2}^{\infty} \frac{n+5}{2^n}$$

Suppose that for some reason we wanted to start this series at $n = 0$, but we didn't want to change the value of the series. This means that we can't just change the $n = 2$ to $n = 0$ as this would add in two new terms to the series and thus change its value.

Performing an index shift is a fairly simple process to do. We'll start by defining a new index, say i, as follows,

$$i = n - 2$$

Now, when $n = 2$, we will get $i = 0$. Notice as well that if $n = \infty$ then $i = \infty - 2 = \infty$, so only the lower limit will change here. Next, we can solve this for n to get,

$$n = i + 2$$

We can now completely rewrite the series in terms of the index i instead of the index n simply by plugging in our equation for n in terms of i.

$$\sum_{n=2}^{\infty} \frac{n+5}{2^n} = \sum_{i=0}^{\infty} \frac{(i+2)+5}{2^{i+2}} = \sum_{i=0}^{\infty} \frac{i+7}{2^{i+2}}$$

To finish the problem out we'll recall that the letter we used for the index doesn't matter and so we'll change the final i back into an n to get,

$$\sum_{n=2}^{\infty} \frac{n+5}{2^n} = \sum_{n=0}^{\infty} \frac{n+7}{2^{n+2}}$$

To convince yourselves that these really are the same summation let's write out the first couple of terms for each of them,

$$\sum_{n=2}^{\infty} \frac{n+5}{2^n} = \frac{7}{2^2} + \frac{8}{2^3} + \frac{9}{2^4} + \frac{10}{2^5} + \cdots$$
$$\sum_{n=0}^{\infty} \frac{n+7}{2^{n+2}} = \frac{7}{2^2} + \frac{8}{2^3} + \frac{9}{2^4} + \frac{10}{2^5} + \cdots$$

So, sure enough the two series do have exactly the same terms.

There is actually an easier way to do an index shift. The method given above is the technically correct way of doing an index shift. However, notice in the above example we decreased the initial value of the index by 2 and all the n's in the series terms increased by 2 as well.

This will always work in this manner. If we decrease the initial value of the index by a set amount then all the other n's in the series term will increase by the same amount. Likewise, if we increase the initial value of the index by a set amount, then all the n's in the series term will decrease by the same amount.

Let's do a couple of examples using this shorthand method for doing index shifts.

Example 1

Perform the following index shifts.

(a) Write $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$ as a series that starts at $n = 0$.

(b) Write $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{1-3^{n+1}}$ as a series that starts at $n = 3$.

Solution

(a) Write $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$ as a series that starts at $n = 0$.

In this case we need to decrease the initial value by 1 and so the n's (okay the single n) in the term must increase by 1 as well.

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{(n+1)-1} = \sum_{n=0}^{\infty} ar^{n}$$

(b) Write $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{1-3^{n+1}}$ as a series that starts at $n=3$.

For this problem we want to increase the initial value by 2 and so all the n's in the series term must decrease by 2.

$$\sum_{n=1}^{\infty} \frac{n^2}{1-3^{n+1}} = \sum_{n=3}^{\infty} \frac{(n-2)^2}{1-3^{(n-2)+1}} = \sum_{n=3}^{\infty} \frac{(n-2)^2}{1-3^{n-1}}$$

The final topic in this section is again a topic that we'll not be seeing all that often in this class, although we will be seeing it more often than the index shifts. This final topic is really more about alternate ways to write series when the situation requires it.

Let's start with the following series and note that the $n=1$ starting point is only for convenience since we need to start the series somewhere.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

Notice that if we ignore the first term the remaining terms will also be a series that will start at $n=2$ instead of $n=1$ So, we can rewrite the original series as follows,

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$$

In this example we say that we've stripped out the first term.

We could have stripped out more terms if we wanted to. In the following series we've stripped out the first two terms and the first four terms respectively.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \sum_{n=3}^{\infty} a_n$$
$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \sum_{n=5}^{\infty} a_n$$

Being able to strip out terms will, on occasion, simplify our work or allow us to reuse a prior result so it's an important idea to remember.

Notice that in the second example above we could have also denoted the four terms that we stripped out as a finite series as follows,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \sum_{n=5}^{\infty} a_n = \sum_{n=1}^{4} a_n + \sum_{n=5}^{\infty} a_n$$

This is a convenient notation when we are stripping out a large number of terms or if we need to strip out an undetermined number of terms. In general, we can write a series as follows,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

We'll leave this section with an important warning about terminology. Don't get sequences and series confused! A sequence is a list of numbers written in a specific order while an infinite series is a limit of a sequence of finite series and hence, if it exists will be a single value.

So, once again, a sequence is a list of numbers while a series is a single number, provided it makes sense to even compute the series. Students will often confuse the two and try to use facts pertaining to one on the other. However, since they are different beasts this just won't work. There will be problems where we are using both sequences and series so we'll always have to remember that they are different.

10.4 Convergence & Divergence of Series

In the previous section we spent some time getting familiar with series and we briefly defined convergence and divergence. Before worrying about convergence and divergence of a series we wanted to make sure that we've started to get comfortable with the notation involved in series and some of the various manipulations of series that we will, on occasion, need to be able to do.

As noted in the previous section most of what we were doing there won't be done much in this chapter. So, it is now time to start talking about the convergence and divergence of a series as this will be a topic that we'll be dealing with to one extent or another in almost all of the remaining sections of this chapter.

So, let's recap just what an infinite series is and what it means for a series to be convergent or divergent. We'll start with a sequence $\{a_n\}_{n=1}^{\infty}$ and again note that we're starting the sequence at $n = 1$ only for the sake of convenience and it can, in fact, be anything.

Next, we define the partial sums of the series as,

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
&\vdots \\
s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^{n} a_i
\end{aligned}$$

and these form a new sequence, $\{s_n\}_{n=1}^{\infty}$.

An infinite series, or just series here since almost every series that we'll be looking at will be an infinite series, is then the limit of the partial sums. Or,

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n$$

It is important to remember that $\sum_{i=1}^{\infty} a_i$ is really nothing more than a convenient notation for $\lim\limits_{n\to\infty} \sum_{i=1}^{n} a_i$ so we do not need to keep writing the limit down. We do, however, always need to remind ourselves that we really do have a limit there!

If the sequence of partial sums is a convergent sequence (*i.e.* its limit exists and is finite) then the series is also called **convergent** and in this case if $\lim\limits_{n\to\infty} s_n = s$ then, $\sum_{i=1}^{\infty} a_i = s$. Likewise, if the sequence of partial sums is a divergent sequence (*i.e.* its limit doesn't exist or is plus or minus infinity) then the series is also called **divergent**.

Let's take a look at some series and see if we can determine if they are convergent or divergent and see if we can determine the value of any convergent series we find.

Example 1

Determine if the following series is convergent or divergent. If it converges determine the value of the series.

$$\sum_{n=1}^{\infty} n$$

Solution

To determine if the series is convergent we first need to get our hands on a formula for the general term in the sequence of partial sums.

$$s_n = \sum_{i=1}^{n} i$$

This is a known series and its value can be shown to be,

$$s_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Don't worry if you didn't know this formula (we'd be surprised if anyone knew it...) as you won't be required to know it in my course.

So, to determine if the series is convergent we will first need to see if the sequence of partial sums,

$$\left\{ \frac{n(n+1)}{2} \right\}_{n=1}^{\infty}$$

is convergent or divergent. That's not terribly difficult in this case. The limit of the sequence terms is,

$$\lim_{n \to \infty} \frac{n(n+1)}{2} = \infty$$

Therefore, the sequence of partial sums diverges to ∞ and so the series also diverges.

So, as we saw in this example we had to know a fairly obscure formula in order to determine the convergence of this series. In general finding a formula for the general term in the sequence of partial sums is a very difficult process. In fact after the next section we'll not be doing much with the partial sums of series due to the extreme difficulty faced in finding the general formula. This also means that we'll not be doing much work with the value of series since in order to get the value we'll also need to know the general formula for the partial sums.

We will continue with a few more examples however, since this is technically how we determine convergence and the value of a series. Also, the remaining examples we'll be looking at in this section will lead us to a very important fact about the convergence of series.

So, let's take a look at a couple more examples.

Example 2

Determine if the following series converges or diverges. If it converges determine the value of the series.

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

Solution

This is actually one of the few series in which we are able to determine a formula for the general term in the sequence of partial fractions. However, in this section we are more interested in the general idea of convergence and divergence and so we'll put off discussing the process for finding the formula until the next section.

The general formula for the partial sums is,

$$s_n = \sum_{i=2}^{n} \frac{1}{i^2 - 1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$$

and in this case we have,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)} \right) = \frac{3}{4}$$

The sequence of partial sums converges and so the series converges also and its value is,

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}$$

Example 3

Determine if the following series converges or diverges. If it converges determine the value of the series.

$$\sum_{n=0}^{\infty} (-1)^n$$

Solution

In this case we really don't need a general formula for the partial sums to determine the convergence of this series. Let's just write down the first few partial sums.

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 - 1 = 0 \\ s_2 &= 1 - 1 + 1 = 1 \\ s_3 &= 1 - 1 + 1 - 1 = 0 \\ &etc. \end{aligned}$$

So, it looks like the sequence of partial sums is,

$$\{s_n\}_{n=0}^{\infty} = \{1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots\}$$

and this sequence diverges since $\lim\limits_{n \to \infty} s_n$ doesn't exist. Therefore, the series also diverges.

Example 4

Determine if the following series converges or diverges. If it converges determine the value of the series.

$$\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}$$

Solution

Here is the general formula for the partial sums for this series.

$$s_n = \sum_{i=1}^{n} \frac{1}{3^{i-1}} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$$

Again, do not worry about knowing this formula. This is not something that you'll ever be asked to know in my class.

In this case the limit of the sequence of partial sums is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{3}{2}\left(1 - \frac{1}{3^n}\right) = \frac{3}{2}$$

The sequence of partial sums is convergent and so the series will also be convergent. The value of the series is,

$$\sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = \frac{3}{2}$$

As we already noted, do not get excited about determining the general formula for the sequence of partial sums. There is only going to be one type of series where you will need to determine this formula and the process in that case isn't too bad. In fact, you already know how to do most of the work in the process as you'll see in the next section.

So, we've determined the convergence of four series now. Two of the series converged and two diverged. Let's go back and examine the series terms for each of these. For each of the series let's take the limit as n goes to infinity of the series terms (not the partial sums!!).

$$\lim_{n\to\infty} n = \infty \qquad \text{this series diverged}$$

$$\lim_{n\to\infty} \frac{1}{n^2-1} = 0 \qquad \text{this series converged}$$

$$\lim_{n\to\infty} (-1)^n \text{ doesn't exist} \qquad \text{this series diverged}$$

$$\lim_{n\to\infty} \frac{1}{3^{n-1}} = 0 \qquad \text{this series converged}$$

Notice that for the two series that converged the series term itself was zero in the limit. This will always be true for convergent series and leads to the following theorem.

Theorem

If $\sum a_n$ converges then $\lim\limits_{n\to\infty} a_n = 0$.

Proof

First let's suppose that the series starts at $n = 1$. If it doesn't then we can modify things as appropriate below. Then the partial sums are,

$$s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_{n-1} \qquad s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_{n-1} + a_n$$

Next, we can use these two partial sums to write,

$$a_n = s_n - s_{n-1}$$

Now because we know that $\sum a_n$ is convergent we also know that the sequence $\{s_n\}_{n=1}^{\infty}$ is also convergent and that $\lim\limits_{n\to\infty} s_n = s$ for some finite value s. However, since $n-1\to\infty$ as $n\to\infty$ we also have $\lim\limits_{n\to\infty} s_{n-1} = s$.

We now have,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

Be careful to not misuse this theorem! This theorem gives us a requirement for convergence but not a guarantee of convergence. In other words, the converse is NOT true. If $\lim\limits_{n\to\infty} a_n = 0$ the series may actually diverge! Consider the following two series.

$$\sum_{n=1}^{\infty}\frac{1}{n} \qquad\qquad \sum_{n=1}^{\infty}\frac{1}{n^2}$$

In both cases the series terms are zero in the limit as n goes to infinity, yet only the second series converges. The first series diverges. It will be a couple of sections before we can prove this, so at this point please believe this and know that you'll be able to prove the convergence of these two series in a couple of sections.

Again, as noted above, all this theorem does is give us a requirement for a series to converge. In order for a series to converge the series terms must go to zero in the limit. If the series terms do not go to zero in the limit then there is no way the series can converge since this would violate the theorem.

This leads us to the first of many tests for the convergence/divergence of a series that we'll be seeing in this chapter.

Divergence Test

If $\lim\limits_{n\to\infty} a_n \neq 0$ then $\sum a_n$ will diverge.

Again, do NOT misuse this test. This test only says that a series is guaranteed to diverge if the series terms don't go to zero in the limit. If the series terms do happen to go to zero the series may or may not converge! Again, recall the following two series,

$$\sum_{n=1}^{\infty}\frac{1}{n} \qquad \text{diverges}$$
$$\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad \text{converges}$$

One of the more common mistakes that students make when they first get into series is to assume that if $\lim\limits_{n\to\infty} a_n = 0$ then $\sum a_n$ will converge. There is just no way to guarantee this so be careful!

Let's take a quick look at an example of how this test can be used.

Example 5

Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}$$

Solution

With almost every series we'll be looking at in this chapter the first thing that we should do is take a look at the series terms and see if they go to zero or not. If it's clear that the terms don't go to zero use the Divergence Test and be done with the problem.

That's what we'll do here.

$$\lim_{n\to\infty} \frac{4n^2 - n^3}{10 + 2n^3} = -\frac{1}{2} \neq 0$$

The limit of the series terms isn't zero and so by the Divergence Test the series diverges.

The divergence test is the first test of many tests that we will be looking at over the course of the next several sections. You will need to keep track of all these tests, the conditions under which they can be used and their conclusions all in one place so you can quickly refer back to them as you need to.

Next we should briefly revisit arithmetic of series and convergence/divergence. As we saw in the previous section if $\sum a_n$ and $\sum b_n$ are both convergent series then so are $\sum c a_n$ and $\sum\limits_{n=k}^{\infty} (a_n \pm b_n)$. Furthermore, these series will have the following sums or values.

$$\sum c a_n = c \sum a_n \qquad\qquad \sum_{n=k}^{\infty} (a_n \pm b_n) = \sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n$$

We'll see an example of this in the next section after we get a few more examples under our belt. At this point just remember that a sum of convergent series is convergent and multiplying a convergent series by a number will not change its convergence.

We need to be a little careful with these facts when it comes to divergent series. In the first case if $\sum a_n$ is divergent then $\sum c a_n$ will also be divergent (provided c isn't zero of course) since multiplying a series that is infinite in value or doesn't have a value by a finite value (*i.e.* c) won't change the fact that the series has an infinite or no value. However, it is possible to have both $\sum a_n$ and $\sum b_n$ be divergent series and yet have $\sum\limits_{n=k}^{\infty}(a_n \pm b_n)$ be a convergent series.

Now, since the main topic of this section is the convergence of a series we should mention a stronger type of convergence. A series $\sum a_n$ is said to **converge absolutely** if $\sum |a_n|$ also converges. Absolute convergence is *stronger* than convergence in the sense that a series that is absolutely convergent will also be convergent, but a series that is convergent may or may not be absolutely convergent.

In fact if $\sum a_n$ converges and $\sum |a_n|$ diverges the series $\sum a_n$ is called **conditionally convergent**.

At this point we don't really have the tools at hand to properly investigate this topic in detail nor do we have the tools in hand to determine if a series is absolutely convergent or not. So we'll not say anything more about this subject for a while. When we finally have the tools in hand to discuss this topic in more detail we will revisit it. Until then don't worry about it. The idea is mentioned here only because we were already discussing convergence in this section and it ties into the last topic that we want to discuss in this section.

In the previous section after we'd introduced the idea of an infinite series we commented on the fact that we shouldn't think of an infinite series as an infinite sum despite the fact that the notation we use for infinite series seems to imply that it is an infinite sum. It's now time to briefly discuss this.

First, we need to introduce the idea of a **rearrangement**. A rearrangement of a series is exactly what it might sound like, it is the same series with the terms rearranged into a different order.

For example, consider the following infinite series.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \cdots$$

A rearrangement of this series is,

$$\sum_{n=1}^{\infty} a_n = a_2 + a_1 + a_3 + a_{14} + a_5 + a_9 + a_4 + \cdots$$

The issue we need to discuss here is that for some series each of these arrangements of terms can have different values despite the fact that they are using exactly the same terms.

Here is an example of this. It can be shown that,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2 \tag{10.1}$$

Since this series converges we know that if we multiply it by a constant c its value will also be multiplied by c. So, let's multiply this by $\frac{1}{2}$ to get,

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots = \frac{1}{2}\ln 2 \tag{10.2}$$

Now, let's add in a zero between each term as follows.

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \frac{1}{10} + 0 - \frac{1}{12} + 0 + \cdots = \frac{1}{2}\ln 2 \tag{10.3}$$

Note that this won't change the value of the series because the partial sums for this series will be the partial sums for the Equation 10.2 except that each term will be repeated. Repeating terms in a series will not affect its limit however and so both Equation 10.2 and Equation 10.3 will be the same.

We know that if two series converge we can add them by adding term by term and so add Equation 10.1 and Equation 10.3 to get,

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2 \tag{10.4}$$

Now, notice that the terms of Equation 10.4 are simply the terms of Equation 10.1 rearranged so that each negative term comes after two positive terms. The values however are definitely different despite the fact that the terms are the same.

Note as well that this is not one of those "tricks" that you see occasionally where you get a contradictory result because of a hard to spot math/logic error. This is a very real result and we've not made any logic mistakes/errors.

Here is a nice set of facts that govern this idea of when a rearrangement will lead to a different value of a series.

Facts

Given the series $\sum a_n$,

1. If $\sum a_n$ is absolutely convergent and its value is s then any rearrangement of $\sum a_n$ will also have a value of s.
2. If $\sum a_n$ is conditionally convergent and r is any real number then there is a rearrangement of $\sum a_n$ whose value will be r.

Again, we do not have the tools in hand yet to determine if a series is absolutely convergent and so don't worry about this at this point. This is here just to make sure that you understand that we have to be very careful in thinking of an infinite series as an infinite sum. There are times when we can (*i.e.* the series is absolutely convergent) and there are times when we can't (*i.e.* the series is conditionally convergent).

As a final note, the fact above tells us that the series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

must be conditionally convergent since two rearrangements gave two separate values of this series. Eventually it will be very simple to show that this series is conditionally convergent.

10.5 Special Series

In this section we are going to take a brief look at three special series. Actually, special may not be the correct term. All three have been named which makes them special in some way, however the main reason that we're going to look at two of them in this section is that they are the only types of series that we'll be looking at for which we will be able to get actual values for the series. The third type is divergent and so won't have a value to worry about.

In general, determining the value of a series is very difficult and outside of these two kinds of series that we'll look at in this section we will not be determining the value of series in this chapter.

So, let's get started.

Geometric Series

A geometric series is any series that can be written in the form,

$$\sum_{n=1}^{\infty} ar^{n-1}$$

or, with an index shift the geometric series will often be written as,

$$\sum_{n=0}^{\infty} ar^{n}$$

These are identical series and will have identical values, provided they converge of course.

If we start with the first form it can be shown that the partial sums are,

$$s_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

The series will converge provided the partial sums form a convergent sequence, so let's take the limit of the partial sums.

$$\begin{aligned} \lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right) \\ &= \lim_{n\to\infty}\frac{a}{1-r} - \lim_{n\to\infty}\frac{ar^n}{1-r} \\ &= \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n \end{aligned}$$

Now, from Theorem 3 from the Sequences section we know that the limit above will exist and be finite provided $-1 < r \le 1$. However, note that we can't let $r = 1$ since this will give division by zero. Therefore, this will exist and be finite provided $-1 < r < 1$ and in this case the limit is zero and so we get,

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}$$

Therefore, a geometric series will converge if $-1 < r < 1$, which is usually written $|r| < 1$, its value is,

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{n} = \frac{a}{1-r}$$

Note that in using this formula we'll need to make sure that we are in the correct form. In other words, if the series starts at $n = 0$ then the exponent on the r must be n. Likewise, if the series starts at $n = 1$ then the exponent on the r must be $n - 1$.

Example 1

Determine if the following series converge or diverge. If they converge give the value of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$

This series doesn't really look like a geometric series. However, notice that both parts of the series term are numbers raised to a power. This means that it can be put into the form of a geometric series. We will just need to decide which form is the correct form. Since the series starts at $n = 1$ we will want the exponents on the numbers to be $n - 1$.

It will be fairly easy to get this into the correct form. Let's first rewrite things slightly. One of the n's in the exponent has a negative in front of it and that can't be there in the geometric form. So, let's first get rid of that.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 9^{-(n-2)}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}}$$

Now let's get the correct exponent on each of the numbers. This can be done using simple exponent properties.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}} = \sum_{n=1}^{\infty} \frac{4^{n-1}4^{2}}{9^{n-1}9^{-1}}$$

Now, rewrite the term a little.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 16\,(9)\,\frac{4^{n-1}}{9^{n-1}} = \sum_{n=1}^{\infty} 144\left(\frac{4}{9}\right)^{n-1}$$

So, this is a geometric series with $a = 144$ and $r = \frac{4}{9} < 1$. Therefore, since $|r| < 1$ we know the series will converge and its value will be,

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \frac{144}{1-\frac{4}{9}} = \frac{9}{5}\,(144) = \frac{1296}{5}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

Again, this doesn't look like a geometric series, but it can be put into the correct form. In this case the series starts at $n = 0$ so we'll need the exponents to be n on the terms. Note that this means we're going to need to rewrite the exponent on the numerator a little

$$\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}} = \sum_{n=0}^{\infty} \frac{\left((-4)^3\right)^n}{5^n 5^{-1}} = \sum_{n=0}^{\infty} 5\frac{(-64)^n}{5^n} = \sum_{n=0}^{\infty} 5\left(\frac{-64}{5}\right)^n$$

So, we've got it into the correct form and we can see that $a = 5$ and $r = -\frac{64}{5}$. Also note that $|r| \geq 1$ and so this series diverges.

Back in the Series - The Basics section we talked about stripping out terms from a series, but didn't really provide any examples of how this idea could be used in practice. We can now do some examples.

Example 2

Use the results from the previous example to determine the value of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} 9^{-n+2}4^{n+1}$

(b) $\displaystyle\sum_{n=3}^{\infty} 9^{-n+2}4^{n+1}$

Solution

(a) $\sum_{n=0}^{\infty} 9^{-n+2}4^{n+1}$

In this case we could just acknowledge that this is a geometric series that starts at $n = 0$ and so we could put it into the correct form and be done with it. However, this does provide us with a nice example of how to use the idea of stripping out terms to our advantage.

Let's notice that if we strip out the first term from this series we arrive at,

$$\sum_{n=0}^{\infty} 9^{-n+2}4^{n+1} = 9^2 4^1 + \sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = 324 + \sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$$

From the previous example we know the value of the new series that arises here and so the value of the series in this example is,

$$\sum_{n=0}^{\infty} 9^{-n+2}4^{n+1} = 324 + \frac{1296}{5} = \frac{2916}{5}$$

(b) $\sum_{n=3}^{\infty} 9^{-n+2}4^{n+1}$

In this case we can't strip out terms from the given series to arrive at the series used in the previous example. However, we can start with the series used in the previous example and strip terms out of it to get the series in this example. So, let's do that. We will strip out the first two terms from the series we looked at in the previous example.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = 9^1 4^2 + 9^0 4^3 + \sum_{n=3}^{\infty} 9^{-n+2}4^{n+1} = 208 + \sum_{n=3}^{\infty} 9^{-n+2}4^{n+1}$$

We can now use the value of the series from the previous example to get the value of this series.

$$\sum_{n=3}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} - 208 = \frac{1296}{5} - 208 = \frac{256}{5}$$

Notice that we didn't discuss the convergence of either of the series in the above example. Here's why. Consider the following series written in two separate ways (*i.e.* we stripped out a couple of terms from it).

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \sum_{n=3}^{\infty} a_n$$

Let's suppose that we know $\sum_{n=3}^{\infty} a_n$ is a convergent series. This means that it's got a finite value and adding three finite terms onto this will not change that fact. So the value of $\sum_{n=0}^{\infty} a_n$ is also finite and so is convergent.

Likewise, suppose that $\sum_{n=0}^{\infty} a_n$ is convergent. In this case if we subtract three finite values from this value we will remain finite and arrive at the value of $\sum_{n=3}^{\infty} a_n$. This is now a finite value and so this series will also be convergent.

In other words, if we have two series and they differ only by the presence, or absence, of a finite number of finite terms they will either both be convergent or they will both be divergent. The difference of a few terms one way or the other will not change the convergence of a series. This is an important idea and we will use it several times in the following sections to simplify some of the tests that we'll be looking at.

Telescoping Series

It's now time to look at the second of the three series in this section. In this portion we are going to look at a series that is called a telescoping series. The name in this case comes from what happens with the partial sums and is best shown in an example.

Example 3

Determine if the following series converges or diverges. If it converges find its value.

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + 3n + 2}$$

Solution

We first need the partial sums for this series.

$$s_n = \sum_{i=0}^{n} \frac{1}{i^2 + 3i + 2}$$

Now, let's notice that we can use partial fractions on the series term to get,

$$\frac{1}{i^2 + 3i + 2} = \frac{1}{(i+2)(i+1)} = \frac{1}{i+1} - \frac{1}{i+2}$$

We'll leave the details of the partial fractions to you. By now you should be fairly adept at this since we spent a fair amount of time doing partial fractions back in the Integration Techniques chapter. If you need a refresher you should go back and review that section.

So, what does this do for us? Well, let's start writing out the terms of the general partial sum for this series using the partial fraction form.

$$\begin{aligned} s_n &= \sum_{i=0}^{n} \left(\frac{1}{i+1} - \frac{1}{i+2} \right) \\ &= \left(\frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \right) + \cdots + \left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}} \right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2} \right) \\ &= 1 - \frac{1}{n+2} \end{aligned}$$

Notice that every term except the first and last term canceled out. This is the origin of the name **telescoping series**.

This also means that we can determine the convergence of this series by taking the limit of the partial sums.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+2} \right) = 1$$

The sequence of partial sums is convergent and so the series is convergent and has a value of

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = 1$$

In telescoping series be careful to not assume that successive terms will be the ones that cancel. Consider the following example.

Example 4

Determine if the following series converges or diverges. If it converges find its value.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$$

Solution

As with the last example we'll leave the partial fractions details to you to verify. The partial sums are,

$$\begin{aligned} s_n &= \sum_{i=1}^{n}\left(\frac{\frac{1}{2}}{i+1}-\frac{\frac{1}{2}}{i+3}\right)=\frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{i+1}-\frac{1}{i+3}\right) \\ &= \frac{1}{2}\left[\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\cdots+\left(\cancel{\frac{1}{n}}-\frac{1}{n+2}\right)+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+3}\right)\right] \\ &= \frac{1}{2}\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right] \end{aligned}$$

In this case instead of successive terms canceling a term will cancel with a term that is farther down the list. The end result this time is two initial and two final terms are left. Notice as well that in order to help with the work a little we factored the $\frac{1}{2}$ out of the series.

The limit of the partial sums is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{1}{2}\left(\frac{5}{6}-\frac{1}{n+2}-\frac{1}{n+3}\right)=\frac{5}{12}$$

So, this series is convergent (because the partial sums form a convergent sequence) and its value is,

$$\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3}=\frac{5}{12}$$

Note that it's not always obvious if a series is telescoping or not until you try to get the partial sums and then see if they are in fact telescoping. There is no test that will tell us that we've got a telescoping series right off the bat. Also note that just because you can do partial fractions on a series term does not mean that the series will be a telescoping series. The following series, for example, is not a telescoping series despite the fact that we can partial fraction the series terms.

$$\sum_{n=1}^{\infty}\frac{3+2n}{n^2+3n+2}=\sum_{n=1}^{\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)$$

In order for a series to be a telescoping series we must get terms to cancel and all of these terms are positive and so none will cancel.

Next, we need to go back and address an issue that was first raised in the previous section. In that section we stated that the sum or difference of convergent series was also convergent and that the presence of a multiplicative constant would not affect the convergence of a series. Now that we have a few more series in hand let's work a quick example showing that.

Example 5

Determine the value of the following series.

$$\sum_{n=1}^{\infty}\left(\frac{4}{n^2+4n+3}-9^{-n+2}4^{n+1}\right)$$

Solution

To get the value of this series all we need to do is rewrite it and then use the previous results.

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{4}{n^2+4n+3}-9^{-n+2}4^{n+1}\right) &= \sum_{n=1}^{\infty}\frac{4}{n^2+4n+3}-\sum_{n=1}^{\infty}9^{-n+2}4^{n+1}\\ &= 4\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3}-\sum_{n=1}^{\infty}9^{-n+2}4^{n+1}\\ &= 4\left(\frac{5}{12}\right)-\frac{1296}{5}\\ &= -\frac{3863}{15}\end{aligned}$$

We didn't discuss the convergence of this series because it was the sum of two convergent series and that guaranteed that the original series would also be convergent.

Harmonic Series

This is the third and final series that we're going to look at in this section. Here is the harmonic series.

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

You can read a little bit about why it is called a harmonic series (has to do with music) at the Wikipedia page for the harmonic series.

The harmonic series is divergent and we'll need to wait until the next section to show that. This series is here because it's got a name and so we wanted to put it here with the other two named series that we looked at in this section. We're also going to use the harmonic series to illustrate a couple of ideas about divergent series that we've already discussed for convergent series. We'll do that with the following example.

Example 6

Show that each of the following series are divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n}$

(b) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n}$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n}$

To see that this series is divergent all we need to do is use the fact that we can factor a constant out of a series as follows,

$$\sum_{n=1}^{\infty} \frac{5}{n} = 5\sum_{n=1}^{\infty} \frac{1}{n}$$

Now, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and so five times this will still not be a finite number and so the series has to be divergent. In other words, if we multiply a divergent series by a constant it will still be divergent.

(b) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n}$

In this case we'll start with the harmonic series and strip out the first three terms.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \sum_{n=4}^{\infty} \frac{1}{n} \qquad \Rightarrow \qquad \sum_{n=4}^{\infty} \frac{1}{n} = \left(\sum_{n=1}^{\infty} \frac{1}{n}\right) - \frac{11}{6}$$

In this case we are subtracting a finite number from a divergent series. This subtraction will not change the divergence of the series. We will either have infinity minus a finite number, which is still infinity, or a series with no value minus a finite number, which will still have no value.

Therefore, this series is divergent.

Just like with convergent series, adding/subtracting a finite number from a divergent series is not going to change the divergence of the series.

So, some general rules about the convergence/divergence of a series are now in order. Multiplying a series by a constant will not change the convergence/divergence of the series and adding or subtracting a constant from a series will not change the convergence/divergence of the series. These are nice ideas to keep in mind.

10.6 Integral Test

The last topic that we discussed in the previous section was the harmonic series. In that discussion we stated that the harmonic series was a divergent series. It is now time to prove that statement. This proof will also get us started on the way to our next test for convergence that we'll be looking at.

So, we will be trying to prove that the harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

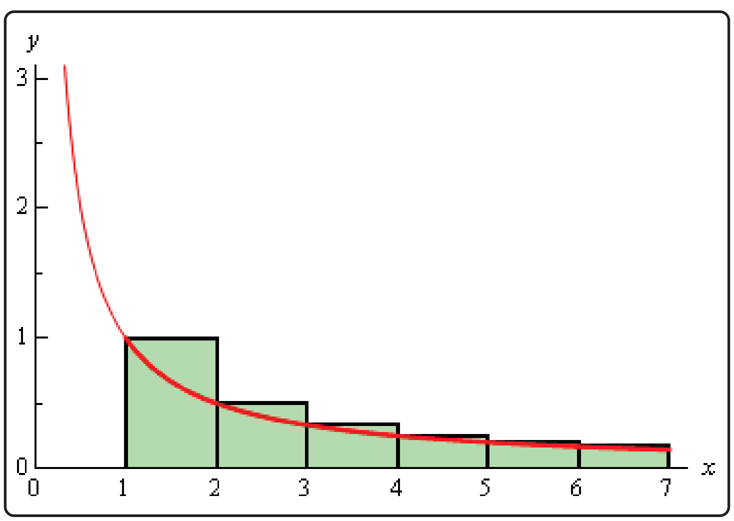
We'll start this off by looking at an apparently unrelated problem. Let's start off by asking what the area under $f(x) = \frac{1}{x}$ on the interval $[1, \infty)$. From the section on Improper Integrals we know that this is,

$$\int_1^{\infty} \frac{1}{x}\, dx = \infty$$

and so we called this integral divergent (yes, that's the same term we're using here with series....).

So, just how does that help us to prove that the harmonic series diverges? Well, recall that we can always estimate the area by breaking up the interval into segments and then sketching in rectangles and using the sum of the area all of the rectangles as an estimate of the actual area. Let's do that for this problem as well and see what we get.

We will break up the interval into subintervals of width 1 and we'll take the function value at the left endpoint as the height of the rectangle. The image below shows the first few rectangles for this area.



So, the area under the curve is approximately,

$$\begin{aligned} A &\approx \left(\frac{1}{1}\right)(1) + \left(\frac{1}{2}\right)(1) + \left(\frac{1}{3}\right)(1) + \left(\frac{1}{4}\right)(1) + \left(\frac{1}{5}\right)(1) + \cdots \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \end{aligned}$$

Now note a couple of things about this approximation. First, each of the rectangles overestimates the actual area and secondly the formula for the area is exactly the harmonic series!

Putting these two facts together gives the following,

$$A \approx \sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\,dx = \infty$$

Notice that this tells us that we must have,

$$\sum_{n=1}^{\infty} \frac{1}{n} > \infty \qquad \Rightarrow \qquad \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Since we can't really be larger than infinity the harmonic series must also be infinite in value. In other words, the harmonic series is in fact divergent.

So, we've managed to relate a series to an improper integral that we could compute and it turns out that the improper integral and the series have exactly the same convergence.

Let's see if this will also be true for a series that converges. When discussing the Divergence Test we made the claim that

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

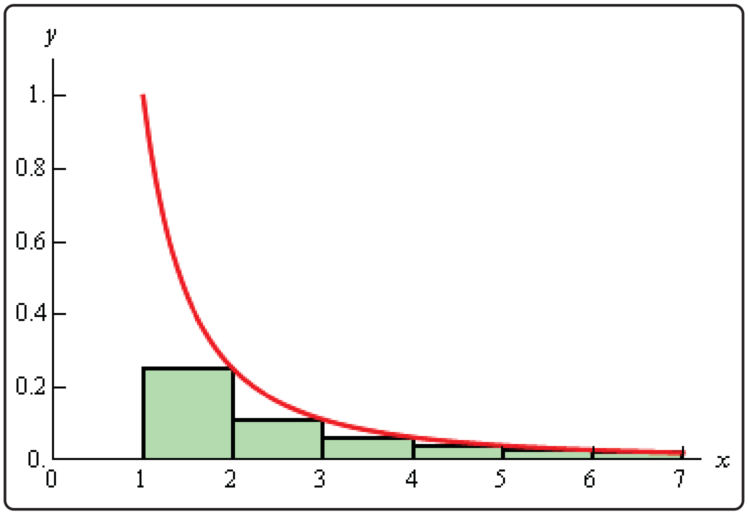
converges. Let's see if we can do something similar to the above process to prove this.

We will try to relate this to the area under $f(x) = \frac{1}{x^2}$ is on the interval $[1, \infty)$. Again, from the Improper Integral section we know that,

$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

and so this integral converges.

We will once again try to estimate the area under this curve. We will do this in an almost identical manner as the previous part with the exception that instead of using the left end points for the height of our rectangles we will use the right end points. Here is a sketch of this case,



In this case the area estimation is,

$$\begin{aligned} A &\approx \left(\frac{1}{2^2}\right)(1) + \left(\frac{1}{3^2}\right)(1) + \left(\frac{1}{4^2}\right)(1) + \left(\frac{1}{5^2}\right)(1) + \cdots \\ &= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \end{aligned}$$

This time, unlike the first case, the area will be an underestimation of the actual area and the estimation is not quite the series that we are working with. Notice however that the only difference is that we're missing the first term. This means we can do the following,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \underbrace{\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots}_{\text{Area Estimation}} < 1 + \int_1^{\infty} \frac{1}{x^2}\, dx = 1 + 1 = 2$$

Or, putting all this together we see that,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$$

With the harmonic series this was all that we needed to say that the series was divergent. With this series however, this isn't quite enough. For instance, $-\infty < 2$, and if the series did have a value of $-\infty$ then it would be divergent (when we want convergent). So, let's do a little more work.

First, let's notice that all the series terms are positive (that's important) and that the partial sums are,

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$$

Because the terms are all positive we know that the partial sums must be an increasing sequence. In other words,

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2} < \sum_{i=1}^{n+1} \frac{1}{i^2} = s_{n+1}$$

In s_{n+1} we are adding a single positive term onto s_n and so must get larger. Therefore, the partial sums form an increasing (and hence monotonic) sequence.

Also note that, since the terms are all positive, we can say,

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < 2 \qquad \Rightarrow \qquad s_n < 2$$

and so the sequence of partial sums is a bounded sequence.

In the second section on Sequences we gave a theorem that stated that a bounded and monotonic sequence was guaranteed to be convergent. This means that the sequence of partial sums is a convergent sequence. So, who cares right? Well recall that this means that the series must then also be convergent!

So, once again we were able to relate a series to an improper integral (that we could compute) and the series and the integral had the same convergence.

We went through a fair amount of work in both of these examples to determine the convergence of the two series. Luckily for us we don't need to do all this work every time. The ideas in these two examples can be summarized in the following test.

Integral Test

Suppose that $f(x)$ is a continuous, positive and decreasing function on the interval $[k, \infty)$ and that $f(n) = a_n$ then,

1. If $\int_k^{\infty} f(x)\, dx$ is convergent so is $\sum_{n=k}^{\infty} a_n$.
2. If $\int_k^{\infty} f(x)\, dx$ is divergent so is $\sum_{n=k}^{\infty} a_n$.

A formal proof of this test can be found at the end of this section.

There are a couple of things to note about the integral test. First, the lower limit on the improper integral must be the same value that starts the series.

Second, the function does not actually need to be decreasing and positive everywhere in the interval. All that's really required is that eventually the function is decreasing and positive. In other words, it is okay if the function (and hence series terms) increases or is negative for a while, but eventually the function (series terms) must decrease and be positive for all terms. To see why this is true let's suppose that the series terms increase and or are negative in the range $k \le n \le N$ and then decrease and are positive for $n \ge N+1$. In this case the series can be written as,

$$\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

Now, the first series is nothing more than a finite sum (no matter how large N is) of finite terms and so will be finite. So, the original series will be convergent/divergent only if the second infinite series on the right is convergent/divergent and the test can be done on the second series as it satisfies the conditions of the test.

A similar argument can be made using the improper integral as well.

The requirement in the test that the function/series be decreasing and positive everywhere in the range is required for the proof. In practice however, we only need to make sure that the function/series is eventually a decreasing and positive function/series. Also note that when computing the integral in the test we don't actually need to strip out the increasing/negative portion since the presence of a small range on which the function is increasing/negative will not change the integral from convergent to divergent or from divergent to convergent.

There is one more very important point that must be made about this test. This test does NOT give the value of a series. It will only give the convergence/divergence of the series. That's it. No value. We can use the above series as a perfect example of this. All that the test gave us was that,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$$

So, we got an upper bound on the value of the series, but not an actual value for the series. In fact, from this point on we will not be asking for the value of a series we will only be asking whether a series converges or diverges. In a later section we look at estimating values of series, but even in that section still won't actually be getting values of series.

Just for the sake of completeness the value of this series is known.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.644934... < 2$$

Let's work a couple of examples.

Example 1

Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

Solution

In this case the function we'll use is,

$$f(x) = \frac{1}{x \ln(x)}$$

This function is clearly positive and if we make x larger the denominator will get larger and so the function is also decreasing. Therefore, all we need to do is determine the convergence of the following integral.

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln(x)}\, dx &= \lim_{t\to\infty} \int_2^{t} \frac{1}{x \ln(x)}\, dx \qquad\qquad u = \ln(x) \\ &= \lim_{t\to\infty} \left(\ln\left(\ln(x)\right)\right)\Big|_2^t \\ &= \lim_{t\to\infty} \left(\ln\left(\ln(t)\right) - \ln\left(\ln 2\right)\right) \\ &= \infty \end{aligned}$$

The integral is divergent and so the series is also divergent by the Integral Test.

Example 2

Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} n \mathbf{e}^{-n^2}$$

Solution

The function that we'll use in this example is,

$$f(x) = x\mathbf{e}^{-x^2}$$

This function is always positive on the interval that we're looking at. Now we need to check that the function is decreasing. It is not clear that this function will always be decreasing on the interval given. We can use our Calculus I knowledge to help us however. The derivative of this function is,

$$f'(x) = \mathbf{e}^{-x^2}\left(1 - 2x^2\right)$$

This function has two critical points (which will tell us where the derivative changes sign) at $x = \pm\frac{1}{\sqrt{2}}$. Since we are starting at $n = 0$ we can ignore the negative critical point. Picking a couple of test points we can see that the function is increasing on the interval $\left[0, \frac{1}{\sqrt{2}}\right]$ and it is decreasing on $\left[\frac{1}{\sqrt{2}}, \infty\right)$. Therefore, eventually the function will be decreasing and that's all that's required for us to use the Integral Test.

$$\begin{aligned} \int_0^\infty x\mathbf{e}^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t x\mathbf{e}^{-x^2}\,dx \qquad\qquad u = -x^2 \\ &= \lim_{t\to\infty}\left(-\frac{1}{2}\mathbf{e}^{-x^2}\right)\Bigg|_0^t \\ &= \lim_{t\to\infty}\left(\frac{1}{2} - \frac{1}{2}\mathbf{e}^{-t^2}\right) = \frac{1}{2} \end{aligned}$$

The integral is convergent and so the series must also be convergent by the Integral Test.

We can use the Integral Test to get the following fact/test for some series.

Fact (The p -series Test)

If $k > 0$ then $\displaystyle\sum_{n=k}^{\infty}\frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Sometimes the series in this fact are called p-series and so this fact is sometimes called the p-series test. This fact follows directly from the Integral Test and a similar fact we saw in the Improper Integral section. This fact says that the integral,

$$\int_k^\infty \frac{1}{x^p}\,dx$$

converges if $p > 1$ and diverges if $p \le 1$.

Using the p-series test makes it very easy to determine the convergence of some series.

Example 3

Determine if the following series are convergent or divergent.

(a) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^7}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Solution

(a) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^7}$

In this case $p = 7 > 1$ and so by this fact the series is convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

For this series $p = \frac{1}{2} \le 1$ and so the series is divergent by the fact.

The last thing that we'll do in this section is give a quick proof of the Integral Test. We've essentially done the proof already at the beginning of the section when we were introducing the Integral Test, but let's go through it formally for a general function.

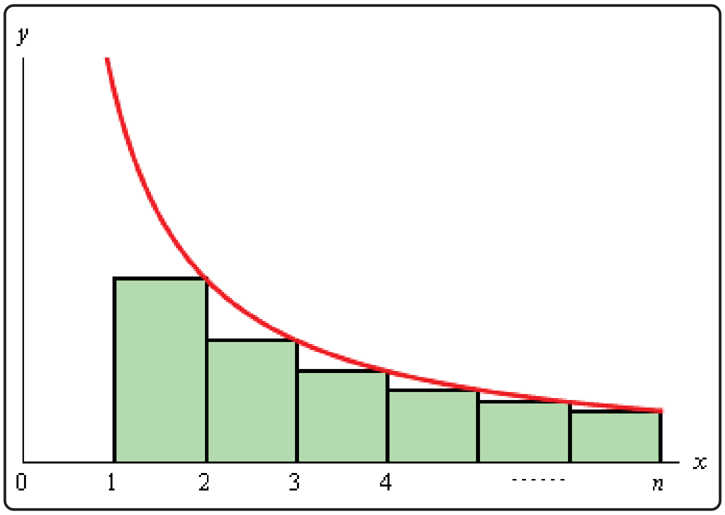
Proof of Integral Test

First, for the sake of the proof we'll be working with the series $\sum\limits_{n=1}^{\infty} a_n$. The original test statement was for a series that started at a general $n = k$ and while the proof can be done for that it will be easier if we assume that the series starts at $n = 1$.

Another way of dealing with the $n = k$ is we could do an index shift and start the series at $n = 1$ and then do the Integral Test. Either way proving the test for $n = 1$ will be sufficient.

Also note that while we allowed for the first few terms of the series to increase and/or be negative in working problems this proof does require that all the terms be decreasing and positive.

Let's start off and estimate the area under the curve on the interval $[1, n]$ and we'll underestimate the area by taking rectangles of width one and whose height is the right endpoint. This gives the following figure.



Now, note that,

$$f(2) = a_2 \qquad f(3) = a_3 \qquad \cdots \qquad f(n) = a_n$$

The approximate area is then,

$$A \approx (1)\,f(2) + (1)\,f(3) + \cdots + (1)\,f(n) = a_2 + a_3 + \cdots a_n$$

and we know that this underestimates the actual area so,

$$\sum_{i=2}^{n} a_i = a_2 + a_3 + \cdots a_n < \int_1^n f(x)\,dx$$

Now, let's suppose that $\int_1^\infty f(x)\,dx$ is convergent and so $\int_1^\infty f(x)\,dx$ must have a finite value. Also, because $f(x)$ is positive we know that,

$$\int_1^n f(x)\,dx < \int_1^\infty f(x)\,dx$$

This in turn means that,

$$\sum_{i=2}^{n} a_i < \int_1^n f(x)\,dx < \int_1^\infty f(x)\,dx$$

Our series starts at $n = 1$ so this isn't quite what we need. However, that's easy enough to deal with.

$$\sum_{i=1}^{n} a_i = a_1 + \sum_{i=2}^{n} a_i < a_1 + \int_1^\infty f(x)\,dx = M$$

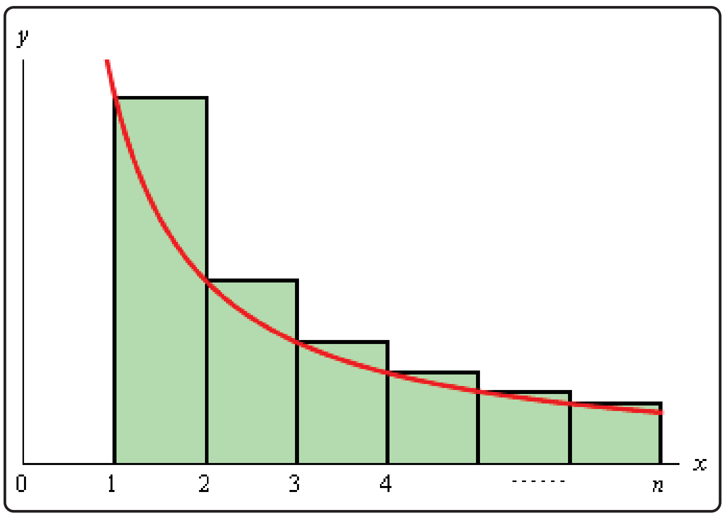
So, just what has this told us? Well we now know that the sequence of partial sums, $s_n = \sum\limits_{i=1}^{n} a_i$ are bounded above by M.

Next, because the terms are positive we also know that,

$$s_n \le s_n + a_{n+1} = \sum_{i=1}^{n} a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = s_{n+1} \qquad \Rightarrow \qquad s_n \le s_{n+1}$$

and so the sequence $\{s_n\}_{n=1}^{\infty}$ is also an increasing sequence. So, we now know that the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ converges and hence our series $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

So, the first part of the test is proven. The second part is somewhat easier. This time let's overestimate the area under the curve by using the left endpoints of interval for the height of the rectangles as shown below.



In this case the area is approximately,

$$A \approx (1)\, f(1) + (1)\, f(2) + \cdots + (1)\, f(n-1) = a_1 + a_2 + \cdots a_{n-1}$$

Since we know this overestimates the area we also then know that,

$$s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + \cdots a_{n-1} > \int_1^{n-1} f(x)\, dx$$

Now, suppose that $\int_1^{\infty} f(x)\, dx$ is divergent. In this case this means that $\int_1^{n} f(x)\, dx \to \infty$ as $n \to \infty$ because $f(x) \ge 0$. However, because $n-1 \to \infty$ as $n \to \infty$ we also know that $\int_1^{n-1} f(x)\, dx \to \infty$.

Therefore, since $s_{n-1} > \int_1^{n-1} f(x)\, dx$ we know that as $n \to \infty$ we must have $s_{n-1} \to \infty$. This in turn tells us that $s_n \to \infty$ as $n \to \infty$.

So, we now know that the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$, is a divergent sequence and so $\sum\limits_{n=1}^{\infty} a_n$ is a divergent series.

It is important to note before leaving this section that in order to use the Integral Test the series terms MUST eventually be decreasing and positive. If they are not then the test doesn't work. Also remember that the test only determines the convergence of a series and does NOT give the value of the series.

10.7 Comparison Test/Limit Comparison Test

In the previous section we saw how to relate a series to an improper integral to determine the convergence of a series. While the integral test is a nice test, it does force us to do improper integrals which aren't always easy and, in some cases, may be impossible to determine the convergence of.

For instance, consider the following series.

$$\sum_{n=0}^{\infty} \frac{1}{3^n + n}$$

In order to use the Integral Test we would have to integrate

$$\int_0^{\infty} \frac{1}{3^x + x}\, dx$$

and we're not even sure if it's possible to do this integral. Nicely enough for us there is another test that we can use on this series that will be much easier to use.

First, let's note that the series terms are positive. As with the Integral Test that will be important in this section. Next let's note that we must have $x > 0$ since we are integrating on the interval $0 \le x < \infty$. Likewise, regardless of the value of x we will always have $3^x > 0$. So, if we drop the x from the denominator the denominator will get smaller and hence the whole fraction will get larger. So,

$$\frac{1}{3^n + n} < \frac{1}{3^n}$$

Now,

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

is a geometric series and we know that since $|r| = \left|\frac{1}{3}\right| < 1$ the series will converge and its value will be,

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

Now, if we go back to our original series and write down the partial sums we get,

$$s_n = \sum_{i=0}^{n} \frac{1}{3^i + i}$$

Since all the terms are positive adding a new term will only make the number larger and so the sequence of partial sums must be an increasing sequence.

$$s_n = \sum_{i=0}^{n} \frac{1}{3^i + i} < \sum_{i=0}^{n+1} \frac{1}{3^i + i} = s_{n+1}$$

Then since,

$$\frac{1}{3^n+n} < \frac{1}{3^n}$$

and because the terms in these two sequences are positive we can also say that,

$$s_n = \sum_{i=0}^{n} \frac{1}{3^i+i} < \sum_{i=0}^{n} \frac{1}{3^i} < \sum_{i=0}^{\infty} \frac{1}{3^n} = \frac{3}{2} \qquad \Rightarrow \qquad s_n < \frac{3}{2}$$

Therefore, the sequence of partial sums is also a bounded sequence. Then from the second section on sequences we know that a monotonic and bounded sequence is also convergent.

So, the sequence of partial sums of our series is a convergent sequence. This means that the series itself,

$$\sum_{n=0}^{\infty} \frac{1}{3^n+n}$$

is also convergent.

So, what did we do here? We found a series whose terms were always larger than the original series terms and this new series was also convergent. Then since the original series terms were positive (very important) this meant that the original series was also convergent.

To show that a series (with only positive terms) was divergent we could go through a similar argument and find a new divergent series whose terms are always smaller than the original series. In this case the original series would have to take a value larger than the new series. However, since the new series is divergent its value will be infinite. This means that the original series must also be infinite and hence divergent.

We can summarize all this in the following test.

Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \ge 0$ for all n and $a_n \le b_n$ for all n. Then,

1. If $\sum b_n$ is convergent then so is $\sum a_n$.
2. If $\sum a_n$ is divergent then so is $\sum b_n$.

In other words, we have two series of positive terms and the terms of one of the series is always larger than the terms of the other series. Then if the larger series is convergent the smaller series must also be convergent. Likewise, if the smaller series is divergent then the larger series must also be divergent. Note as well that in order to apply this test we need both series to start at the same place.

A formal proof of this test is at the end of this section.

Do not misuse this test. Just because the smaller of the two series converges does not say anything about the larger series. The larger series may still diverge. Likewise, just because we know that the larger of two series diverges we can't say that the smaller series will also diverge! Be very careful in using this test

Recall that we had a similar test for improper integrals back when we were looking at integration techniques. So, if you could use the comparison test for improper integrals you can use the comparison test for series as they are pretty much the same idea.

Note as well that the requirement that $a_n, b_n \geq 0$ and $a_n \leq b_n$ really only need to be true eventually. In other words, if a couple of the first terms are negative or $a_n \not\leq b_n$ for a couple of the first few terms we're okay. As long as we eventually reach a point where $a_n, b_n \geq 0$ and $a_n \leq b_n$ for all sufficiently large n the test will work.

To see why this is true let's suppose that the series start at $n = k$ and that the conditions of the test are only true for for $n \geq N + 1$ and for $k \leq n \leq N$ at least one of the conditions is not true. If we then look at $\sum a_n$ (the same thing could be done for $\sum b_n$) we get,

$$\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

The first series is nothing more than a finite sum (no matter how large N is) of finite terms and so will be finite. So, the original series will be convergent/divergent only if the second infinite series on the right is convergent/divergent and the test can be done on the second series as it satisfies the conditions of the test.

Let's take a look at some examples.

Example 1

Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2(n)}$$

Solution

Since the cosine term in the denominator doesn't get too large we can assume that the series terms will behave like,

$$\frac{n}{n^2} = \frac{1}{n}$$

which, as a series, will diverge. So, from this we can guess that the series will probably diverge and so we'll need to find a smaller series that will also diverge.

Recall that from the comparison test with improper integrals that we determined that we can make a fraction smaller by either making the numerator smaller or the denominator larger.

In this case the two terms in the denominator are both positive. So, if we drop the cosine term we will in fact be making the denominator larger since we will no longer be subtracting off a positive quantity. Therefore,

$$\frac{n}{n^2 - \cos^2(n)} > \frac{n}{n^2} = \frac{1}{n}$$

Then, since

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (it's harmonic or the p-series test) by the Comparison Test our original series must also diverge.

Example 2

Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\mathbf{e}^{-n}}{n + \cos^2(n)}$$

Solution

This example looks somewhat similar to the first one but we are going to have to be careful with it as there are some significant differences.

First, as with the first example the cosine term in the denominator will not get very large and so it won't affect the behavior of the terms in any meaningful way. Therefore, the temptation at this point is to focus in on the n in the denominator and think that because it is just an n the series will diverge.

That would be correct if we didn't have much going on in the numerator. In this example, however, we also have an exponential in the numerator that is going to zero very fast. In fact, it is going to zero so fast that it will, in all likelihood, force the series to converge.

So, let's guess that this series will converge and we'll need to find a larger series that will also converge.

First, because we are adding two positive numbers in the denominator we can drop the cosine term from the denominator. This will, in turn, make the denominator smaller and so the term will get larger or,

$$\frac{\mathbf{e}^{-n}}{n + \cos^2(n)} \le \frac{\mathbf{e}^{-n}}{n}$$

Next, we know that $n \geq 1$ and so if we replace the n in the denominator with its smallest possible value (i.e. 1) the term will again get larger. Doing this gives,

$$\frac{\mathbf{e}^{-n}}{n + \cos^2(n)} \leq \frac{\mathbf{e}^{-n}}{n} \leq \frac{\mathbf{e}^{-n}}{1} = \mathbf{e}^{-n}$$

We can't do much more, in a way that is useful anyway, to make this larger so let's see if we can determine if,

$$\sum_{n=1}^{\infty} \mathbf{e}^{-n}$$

converges or diverges.

We can notice that $f(x) = \mathbf{e}^{-x}$ is always positive and it is also decreasing (you can verify that correct?) and so we can use the Integral Test on this series. Doing this gives,

$$\int_1^{\infty} \mathbf{e}^{-x}\,dx = \lim_{t\to\infty} \int_1^{t} \mathbf{e}^{-x}\,dx = \lim_{t\to\infty} \left(-\mathbf{e}^{-x}\right)\Big|_1^t = \lim_{t\to\infty} \left(-\mathbf{e}^{-t} + \mathbf{e}^{-1}\right) = \mathbf{e}^{-1}$$

Okay, we now know that the integral is convergent and so the series $\sum\limits_{n=1}^{\infty} \mathbf{e}^{-n}$ must also be convergent.

Therefore, because $\sum\limits_{n=1}^{\infty} \mathbf{e}^{-n}$ is larger than the original series we know that the original series must also converge.

With each of the previous examples we saw that we can't always just focus in on the denominator when making a guess about the convergence of a series. Sometimes there is something going on in the numerator that will change the convergence of a series from what the denominator tells us should be happening.

We also saw in the previous example that, unlike most of the examples of the comparison test that we've done (or will do) both in this section and in the Comparison Test for Improper Integrals, that it won't always be the denominator that is driving the convergence or divergence. Sometimes it is the numerator that will determine if something will converge or diverge so do not get too locked into only looking at the denominator.

One of the more common mistakes is to just focus in on the denominator and make a guess based just on that. If we'd done that with both of the previous examples we would have guessed wrong so be careful.

Let's work another example of the comparison test before we move on to a different topic.

Example 3

Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$$

Solution

In this case the "$+2$" and the "$+5$" don't really add anything to the series and so the series terms should behave pretty much like

$$\frac{n^2}{n^4} = \frac{1}{n^2}$$

which will converge as a series. Therefore, we can guess that the original series will converge and we will need to find a larger series which also converges.

This means that we'll either have to make the numerator larger or the denominator smaller. We can make the denominator smaller by dropping the "$+5$". Doing this gives,

$$\frac{n^2+2}{n^4+5} < \frac{n^2+2}{n^4}$$

At this point, notice that we can't drop the "$+2$" from the numerator since this would make the term smaller and that's not what we want. However, this is actually the furthest that we need to go. Let's take a look at the following series.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^2+2}{n^4} &= \sum_{n=1}^{\infty} \frac{n^2}{n^4} + \sum_{n=1}^{\infty} \frac{2}{n^4} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{2}{n^4}\end{aligned}$$

As shown, we can write the series as a sum of two series and both of these series are convergent by the p-series test. Therefore, since each of these series are convergent we know that the sum,

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n^4}$$

is also a convergent series. Recall that the sum of two convergent series will also be convergent.

Now, since the terms of this series are larger than the terms of the original series we know that the original series must also be convergent by the Comparison Test.

The comparison test is a nice test that allows us to do problems that either we couldn't have done with the integral test or at the best would have been very difficult to do with the integral test. That doesn't mean that it doesn't have problems of its own.

Consider the following series.

$$\sum_{n=0}^{\infty} \frac{1}{3^n - n}$$

This is not much different from the first series that we looked at. The original series converged because the 3^n gets very large very fast and will be significantly larger than the n. Therefore, the n doesn't really affect the convergence of the series in that case. The fact that we are now subtracting the n off instead of adding the n on really shouldn't change the convergence. We can say this because the 3^n gets very large very fast and the fact that we're subtracting n off won't really change the size of this term for all sufficiently large values of n.

So, we would expect this series to converge. However, the comparison test won't work with this series. To use the comparison test on this series we would need to find a larger series that we could easily determine the convergence of. In this case we can't do what we did with the original series. If we drop the n we will make the denominator larger (since the n was subtracted off) and so the fraction will get smaller and just like when we looked at the comparison test for improper integrals knowing that the smaller of two series converges does not mean that the larger of the two will also converge.

So, we will need something else to do help us determine the convergence of this series. The following variant of the comparison test will allow us to determine the convergence of this series.

Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n \geq 0, b_n > 0$ for all n. Define,

$$c = \lim_{n \to \infty} \frac{a_n}{b_n}$$

If c is positive (*i.e.* $c > 0$) and is finite (*i.e.* $c < \infty$) then either both series converge or both series diverge.

The proof of this test is at the end of this section.

Note that it doesn't really matter which series term is in the numerator for this test, we could just have easily defined c as,

$$c = \lim_{n \to \infty} \frac{b_n}{a_n}$$

and we would get the same results. To see why this is, consider the following two definitions.

$$c = \lim_{n \to \infty} \frac{a_n}{b_n} \qquad \bar{c} = \lim_{n \to \infty} \frac{b_n}{a_n}$$

Start with the first definition and rewrite it as follows, then take the limit.

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\frac{b_n}{a_n}} = \frac{1}{\lim\limits_{n\to\infty} \frac{b_n}{a_n}} = \frac{1}{\overline{c}}$$

In other words, if c is positive and finite then so is $\overline{c}$ and if $\overline{c}$ is positive and finite then so is c. Likewise if $\overline{c} = 0$ then $c = \infty$ and if $\overline{c} = \infty$ then $c = 0$. Both definitions will give the same results from the test so don't worry about which series terms should be in the numerator and which should be in the denominator. Choose this to make the limit easy to compute.

Also, this really is a comparison test in some ways. If c is positive and finite this is saying that both of the series terms will behave in generally the same fashion and so we can expect the series themselves to also behave in a similar fashion. If $c = 0$ or $c = \infty$ we can't say this and so the test fails to give any information.

The limit in this test will often be written as,

$$c = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n}$$

since often both terms will be fractions and this will make the limit easier to deal with.

Let's see how this test works.

Example 4

Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{1}{3^n - n}$$

Solution

To use the limit comparison test we need to find a second series that we can determine the convergence of easily and has what we assume is the same convergence as the given series. On top of that we will need to choose the new series in such a way as to give us an easy limit to compute for c.

We've already guessed that this series converges and since it's vaguely geometric let's use

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

as the second series. We know that this series converges and there is a chance that since both series have the 3^n in it the limit won't be too bad.

Here's the limit.

$$c = \lim_{n\to\infty} \frac{1}{3^n} \frac{3^n - n}{1}$$
$$= \lim_{n\to\infty} 1 - \frac{n}{3^n}$$

Now, we'll need to use L'Hospital's Rule on the second term in order to actually evaluate this limit.

$$c = 1 - \lim_{n\to\infty} \frac{1}{3^n \ln(3)}$$
$$= 1$$

So, c is positive and finite so by the Comparison Test both series must converge since

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

converges.

Example 5

Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{4n^2 + n}{\sqrt[3]{n^7 + n^3}}$$

Solution

Fractions involving only polynomials or polynomials under radicals will behave in the same way as the largest power of n will behave in the limit. So, the terms in this series should behave as,

$$\frac{n^2}{\sqrt[3]{n^7}} = \frac{n^2}{n^{\frac{7}{3}}} = \frac{1}{n^{\frac{1}{3}}}$$

and as a series this will diverge by the p-series test. In fact, this would make a nice choice for our second series in the limit comparison test so let's use it.

$$\lim_{n\to\infty}\frac{4n^2+n}{\sqrt[3]{n^7+n^3}}\frac{n^{\frac{1}{3}}}{1}=\lim_{n\to\infty}\frac{4n^{\frac{7}{3}}+n^{\frac{4}{3}}}{\sqrt[3]{n^7\left(1+\frac{1}{n^4}\right)}}$$

$$=\lim_{n\to\infty}\frac{n^{\frac{7}{3}}\left(4+\frac{1}{n}\right)}{n^{\frac{7}{3}}\sqrt[3]{1+\frac{1}{n^4}}}$$

$$=\frac{4}{\sqrt[3]{1}}=4=c$$

So, c is positive and finite and so both limits will diverge since

$$\sum_{n=2}^{\infty}\frac{1}{n^{\frac{1}{3}}}$$

diverges.

Finally, to see why we need c to be positive and finite (*i.e.* $c\neq 0$ and $c\neq\infty$) consider the following two series.

$$\sum_{n=1}^{\infty}\frac{1}{n}\qquad\qquad\sum_{n=1}^{\infty}\frac{1}{n^2}$$

The first diverges and the second converges.

Now compute each of the following limits.

$$\lim_{n\to\infty}\frac{1}{n}\cdot\frac{n^2}{1}=\lim_{n\to\infty}n=\infty\qquad\qquad\lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{n}{1}=\lim_{n\to\infty}\frac{1}{n}=0$$

In the first case the limit from the limit comparison test yields $c=\infty$ and in the second case the limit yields $c=0$. Clearly, both series do not have the same convergence.

Note however, that just because we get $c=0$ or $c=\infty$ doesn't mean that the series will have the opposite convergence. To see this consider the series,

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\qquad\qquad\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Both of these series converge and here are the two possible limits that the limit comparison test uses.

$$\lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n^2}{1}=\lim_{n\to\infty}\frac{1}{n}=0\qquad\qquad\lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{n^3}{1}=\lim_{n\to\infty}n=\infty$$

So, even though both series had the same convergence we got both $c=0$ and $c=\infty$.

The point of all of this is to remind us that if we get $c=0$ or $c=\infty$ from the limit comparison test we will know that we have chosen the second series incorrectly and we'll need to find a different choice in order to get any information about the convergence of the series.

We'll close out this section with proofs of the two tests.

Proof of Comparison Test

The test statement did not specify where each series should start. We only need to require that they start at the same place so to help with the proof we'll assume that the series start at $n = 1$. If the series don't start at $n = 1$ the proof can be redone in exactly the same manner or you could use an index shift to start the series at $n = 1$ and then this proof will apply.

We'll start off with the partial sums of each series.

$$s_n = \sum_{i=1}^{n} a_i \qquad\qquad t_n = \sum_{i=1}^{n} b_i$$

Let's notice a couple of nice facts about these two partial sums. First, because $a_n, b_n \geq 0$ we know that,

$$s_n \leq s_n + a_{n+1} = \sum_{i=1}^{n} a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = s_{n+1} \qquad \Rightarrow \qquad s_n \leq s_{n+1}$$
$$t_n \leq t_n + b_{n+1} = \sum_{i=1}^{n} b_i + b_{n+1} = \sum_{i=1}^{n+1} b_i = t_{n+1} \qquad \Rightarrow \qquad t_n \leq t_{n+1}$$

So, both partial sums form increasing sequences.

Also, because $a_n \leq b_n$ for all n we know that we must have $s_n \leq t_n$ for all n.

With these preliminary facts out of the way we can proceed with the proof of the test itself.

Let's start out by assuming that $\sum\limits_{n=1}^{\infty} b_n$ is a convergent series. Since $b_n \geq 0$ we know that,

$$t_n = \sum_{i=1}^{n} b_i \leq \sum_{i=1}^{\infty} b_i$$

However, we also have established that $s_n \leq t_n$ for all n and so for all n we also have,

$$s_n \leq \sum_{i=1}^{\infty} b_i$$

Finally, since $\sum\limits_{n=1}^{\infty} b_n$ is a convergent series it must have a finite value and so the partial sums, s_n are bounded above. Therefore, from the second section on sequences we know that a monotonic and bounded sequence is also convergent and so $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence and so $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

Next, let's assume that $\sum_{n=1}^{\infty} a_n$ is divergent. Because $a_n \geq 0$ we then know that we must have $s_n \to \infty$ as $n \to \infty$. However, we also know that for all n we have $s_n \leq t_n$ and therefore we also know that $t_n \to \infty$ as $n \to \infty$.

So, $\{t_n\}_{n=1}^{\infty}$ is a divergent sequence and so $\sum_{n=1}^{\infty} b_n$ is divergent.

Proof of Limit Comparison Test

Because $0 < c < \infty$ we can find two positive and finite numbers, m and M, such that $m < c < M$. Now, because $c = \lim_{n\to\infty} \frac{a_n}{b_n}$ we know that for large enough n the quotient $\frac{a_n}{b_n}$ must be close to c and so there must be a positive integer N such that if $n > N$ we also have,

$$m < \frac{a_n}{b_n} < M$$

Multiplying through by b_n gives,

$$mb_n < a_n < Mb_n$$

provided $n > N$.

Now, if $\sum b_n$ diverges then so does $\sum mb_n$ and so since $mb_n < a_n$ for all sufficiently large n by the Comparison Test $\sum a_n$ also diverges.

Likewise, if $\sum b_n$ converges then so does $\sum Mb_n$ and since $a_n < Mb_n$ for all sufficiently large n by the Comparison Test $\sum a_n$ also converges.

10.8 Alternating Series Test

The last two tests that we looked at for series convergence have required that all the terms in the series be positive. Of course there are many series out there that have negative terms in them and so we now need to start looking at tests for these kinds of series.

The test that we are going to look into in this section will be a test for alternating series. An **alternating series** is any series, $\sum a_n$, for which the series terms can be written in one of the following two forms.

$$a_n = (-1)^n b_n \qquad b_n \geq 0$$
$$a_n = (-1)^{n+1} b_n \qquad b_n \geq 0$$

There are many other ways to deal with the alternating sign, but they can all be written as one of the two forms above. For instance,

$$(-1)^{n+2} = (-1)^n(-1)^2 = (-1)^n$$
$$(-1)^{n-1} = (-1)^{n+1}(-1)^{-2} = (-1)^{n+1}$$

There are of course many others, but they all follow the same basic pattern of reducing to one of the first two forms given. If you should happen to run into a different form than the first two, don't worry about converting it to one of those forms, just be aware that it can be and so the test from this section can be used.

Alternating Series Test

Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. Then if,

1. $\lim\limits_{n \to \infty} b_n = 0$ and,
2. $\{b_n\}$ is a decreasing sequence

the series $\sum a_n$ is convergent.

A proof of this test is at the end of the section.

There are a couple of things to note about this test. First, unlike the Integral Test and the Comparison/Limit Comparison Test, this test will only tell us when a series converges and not if a series will diverge.

Secondly, in the second condition all that we need to require is that the series terms, b_n will be eventually decreasing. It is possible for the first few terms of a series to increase and still have the test be valid. All that is required is that eventually we will have $b_n \geq b_{n+1}$ for all n after some point.

To see why this is consider the following series,

$$\sum_{n=1}^{\infty} (-1)^n b_n$$

Let's suppose that for $1 \le n \le N$ $\{b_n\}$ is not decreasing and that for $n \ge N+1$ $\{b_n\}$ is decreasing. The series can then be written as,

$$\sum_{n=1}^{\infty} (-1)^n b_n = \sum_{n=1}^{N} (-1)^n b_n + \sum_{n=N+1}^{\infty} (-1)^n b_n$$

The first series is a finite sum (no matter how large N is) of finite terms and so we can compute its value and it will be finite. The convergence of the series will depend solely on the convergence of the second (infinite) series. If the second series has a finite value then the sum of two finite values is also finite and so the original series will converge to a finite value. On the other hand, if the second series is divergent either because its value is infinite or it doesn't have a value then adding a finite number onto this will not change that fact and so the original series will be divergent.

The point of all this is that we don't need to require that the series terms be decreasing for all n. We only need to require that the series terms will eventually be decreasing since we can always strip out the first few terms that aren't actually decreasing and look only at the terms that are actually decreasing.

Note that, in practice, we don't actually strip out the terms that aren't decreasing. All we do is check that eventually the series terms are decreasing and then apply the test.

Let's work a couple of examples.

Example 1

Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Solution

First, identify the b_n for the test.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \qquad\qquad b_n = \frac{1}{n}$$

Now, all that we need to do is run through the two conditions in the test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$$

Both conditions are met and so by the Alternating Series Test the series must converge.

The series from the previous example is sometimes called the **Alternating Harmonic Series**. Also, the $(-1)^{n+1}$ could be $(-1)^n$ or any other form of alternating sign and we'd still call it an Alternating Harmonic Series.

In the previous example it was easy to see that the series terms decreased since increasing n only increased the denominator for the term and hence made the term smaller. In general however, we will need to resort to Calculus I techniques to prove the series terms decrease. We'll see an example of this in a bit.

Example 2

Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$$

Solution

First, identify the b_n for the test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5} = \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+5} \qquad \Rightarrow \qquad b_n = \frac{n^2}{n^2+5}$$

Let's check the conditions.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^2+5} = 1 \neq 0$$

So, the first condition isn't met and so there is no reason to check the second. Since this condition isn't met we'll need to use another test to check convergence. In these cases where the first condition isn't met it is usually best to use the divergence test.

So, the divergence test requires us to compute the following limit.

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5}$$

This limit can be somewhat tricky to evaluate. For a second let's consider the following,

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} = \left(\lim_{n\to\infty} (-1)^n\right)\left(\lim_{n\to\infty} \frac{n^2}{n^2+5}\right)$$

Splitting this limit like this can't be done because this operation requires that both limits exist and while the second one does the first clearly does not. However, it does show us how we can at least convince ourselves that the overall limit does not exist (even if it won't be a direct proof of that fact).

So, let's start with,

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} = \lim_{n\to\infty} \left[(-1)^n \frac{n^2}{n^2+5}\right]$$

Now, the second part of this clearly is going to 1 as $n \to \infty$ while the first part just alternates between 1 and -1. So, as $n \to \infty$ the terms are alternating between positive and negative values that are getting closer and closer to 1 and -1 respectively.

In order for limits to exist we know that the terms need to settle down to a single number and since these clearly don't this limit doesn't exist and so by the Divergence Test this series diverges.

Example 3

Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$$

Solution

Notice that in this case the exponent on the "-1" isn't n or $n+1$. That won't change how the test works however so we won't worry about that. In this case we have,

$$b_n = \frac{\sqrt{n}}{n+4}$$

so let's check the conditions.

The first is easy enough to check.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = 0$$

The second condition requires some work however. It is not immediately clear that these terms will decrease. Increasing n to $n+1$ will increase both the numerator and the denominator. Increasing the numerator says the term should also increase while increasing the denominator says that the term should decrease. Since it's not clear which of these will win out we will need to resort to Calculus I techniques to show that the terms decrease.

Let's start with the following function and its derivative.

$$f(x) = \frac{\sqrt{x}}{x+4} \qquad f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2}$$

Now, there are two critical points for this function, $x = 0$, and $x = 4$. Note that $x = -4$ is not a critical point because the function is not defined at $x = -4$. The first is outside the bound of our series so we won't need to worry about that one. Using the test points,

$$f'(1) = \frac{3}{50} \qquad f'(5) = -\frac{\sqrt{5}}{810}$$

and so we can see that the function in increasing on $0 \le x \le 4$ and decreasing on $x \ge 4$. Therefore, since $f(n) = b_n$ we know as well that the b_n are also increasing on $0 \le n \le 4$ and decreasing on $n \ge 4$.

The b_n are then eventually decreasing and so the second condition is met.

Both conditions are met and so by the Alternating Series Test the series must be converging.

As the previous example has shown, we sometimes need to do a fair amount of work to show that the terms are decreasing. Do not just make the assumption that the terms will be decreasing and let it go at that.

Let's do one more example just to make a point.

Example 4

Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

Solution

The point of this problem is really just to acknowledge that it is in fact an alternating series. To see this we need to acknowledge that,

$$\cos(n\pi) = (-1)^n$$

If you aren't sure of this you can easily convince yourself that this is correct by plugging in a few values of n and checking.

So the series is really,

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad \Rightarrow \qquad b_n = \frac{1}{\sqrt{n}}$$

Checking the two condition gives,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$b_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = b_{n+1}$$

The two conditions of the test are met and so by the Alternating Series Test the series is convergent.

It should be pointed out that the rewrite we did in previous example only works because n is an integer and because of the presence of the π. Without the π we couldn't do this and if n wasn't guaranteed to be an integer we couldn't do this.

Let's close this section out with a proof of the Alternating Series Test.

Proof of Alternating Series Test

Without loss of generality we can assume that the series starts at $n = 1$. If not we could modify the proof below to meet the new starting place or we could do an index shift to get the series to start at $n = 1$.

Also note that the assumption here is that we have $a_n = (-1)^{n+1} b_n$. To get the proof for $a_n = (-1)^n b_n$ we only need to make minor modifications of the proof and so will not give that proof.

Finally, in the examples all we really needed was for the b_n to be positive and decreasing eventually but for this proof to work we really do need them to be positive and decreasing for all n.

First, notice that because the terms of the sequence are decreasing for any two successive terms we can say,

$$b_n - b_{n+1} \geq 0$$

Now, let's take a look at the even partial sums.

$$\begin{aligned}
s_2 &= b_1 - b_2 \geq 0 \\
s_4 &= b_1 - b_2 + b_3 - b_4 = s_2 + b_3 - b_4 \geq s_2 && \text{because } b_3 - b_4 \geq 0 \\
s_6 &= s_4 + b_5 - b_6 \geq s_4 && \text{because } b_5 - b_6 \geq 0 \\
&\vdots \\
s_{2n} &= s_{2n-2} + b_{2n-1} - b_{2n} \geq s_{2n-2} && \text{because } b_{2n-1} - b_{2n} \geq 0
\end{aligned}$$

So, $\{s_{2n}\}$ is an increasing sequence.

Next, we can also write the general term as,

$$\begin{aligned}
s_{2n} &= b_1 - b_2 + b_3 - b_4 + b_5 + \cdots - b_{2n-2} + b_{2n-1} - b_{2n} \\
&= b_1 - (b_2 - b_3) - (b_4 - b_5) + \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}
\end{aligned}$$

Each of the quantities in parenthesis are positive and by assumption we know that b_{2n} is also positive. So, this tells us that $s_{2n} \leq b_1$ for all n.

We now know that $\{s_{2n}\}$ is an increasing sequence that is bounded above and so we know that it must also converge. So, let's assume that its limit is s or,

$$\lim_{n\to\infty} s_{2n} = s$$

Next, we can quickly determine the limit of the sequence of odd partial sums, $\{s_{2n+1}\}$, as follows,

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} (s_{2n} + b_{2n+1}) = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} = s + 0 = s$$

So, we now know that both $\{s_{2n}\}$ and $\{s_{2n+1}\}$ are convergent sequences and they both have the same limit and so we also know that $\{s_n\}$ is a convergent sequence with a limit of s. This in turn tells us that $\sum a_n$ is convergent.

10.9 Absolute Convergence

When we first talked about series convergence we briefly mentioned a stronger type of convergence but didn't do anything with it because we didn't have any tools at our disposal that we could use to work problems involving it. We now have some of those tools so it's now time to talk about absolute convergence in detail.

First, let's go back over the definition of absolute convergence.

Definition

A series $\sum a_n$ is called **absolutely convergent** if $\sum |a_n|$ is convergent. If $\sum a_n$ is convergent and $\sum |a_n|$ is divergent we call the series **conditionally convergent**.

We also have the following fact about absolute convergence.

Fact

If $\sum a_n$ is absolutely convergent then it is also convergent.

Proof

First notice that $|a_n|$ is either a_n or it is $-a_n$ depending on its sign. This means that we can then say,

$$0 \le a_n + |a_n| \le 2|a_n|$$

Now, since we are assuming that $\sum |a_n|$ is convergent then $\sum 2|a_n|$ is also convergent since we can just factor the 2 out of the series and 2 times a finite value will still be finite. This however allows us to use the Comparison Test to say that $\sum (a_n + |a_n|)$ is also a convergent series.

Finally, we can write,

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

and so $\sum a_n$ is the difference of two convergent series and so is also convergent.

This fact is one of the ways in which absolute convergence is a "stronger" type of convergence. Series that are absolutely convergent are guaranteed to be convergent. However, series that are convergent may or may not be absolutely convergent.

Let's take a quick look at a couple of examples of absolute convergence.

Example 1

Determine if each of the following series are absolute convergent, conditionally convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

This is the alternating harmonic series and we saw in the last section that it is a convergent series so we don't need to check that here. So, let's see if it is an absolutely convergent series. To do this we'll need to check the convergence of.

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

This is the harmonic series and we know from the integral test section that it is divergent.

Therefore, this series is not absolutely convergent. It is however conditionally convergent since the series itself does converge.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$

In this case let's just check absolute convergence first since if it's absolutely convergent we won't need to bother checking convergence as we will get that for free.

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+2}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This series is convergent by the p-series test and so the series is absolute convergent. Note that this does say as well that it's a convergent series.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$

In this part we need to be a little careful. First, this is NOT an alternating series and so we can't use any tools from that section.

What we'll do here is check for absolute convergence first again since that will also give convergence. This means that we need to check the convergence of the following series.

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^3} \right| = \sum_{n=1}^{\infty} \frac{\left|\sin(n)\right|}{n^3}$$

To do this we'll need to note that

$$-1 \le \sin(n) \le 1 \qquad \Rightarrow \qquad \left|\sin(n)\right| \le 1$$

and so we have,

$$\frac{\left|\sin(n)\right|}{n^3} \le \frac{1}{n^3}$$

Now we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges by the p-series test and so by the Comparison Test we also know that

$$\sum_{n=1}^{\infty} \frac{\left|\sin(n)\right|}{n^3}$$

converges.

Therefore, the original series is absolutely convergent (and hence convergent).

Let's close this section off by recapping a topic we saw earlier. When we first discussed the convergence of series in detail we noted that we can't think of series as an infinite sum because some series can have different sums if we rearrange their terms. In fact, we gave two rearrangements of an Alternating Harmonic series that gave two different values. We closed that section off with the following fact,

Facts

Given the series $\sum a_n$,

1. If $\sum a_n$ is absolutely convergent and its value is s then any rearrangement of $\sum a_n$ will also have a value of s.
2. If $\sum a_n$ is conditionally convergent and r is any real number then there is a rearrangement of $\sum a_n$ whose value will be r.

Now that we've got the tools under our belt to determine absolute and conditional convergence we can make a few more comments about this.

First, as we showed above in Example 1a an Alternating Harmonic is conditionally convergent and so no matter what value we chose there is some rearrangement of terms that will give that value. Note as well that this fact does not tell us what that rearrangement must be only that it does exist.

Next, we showed in Example 1b that,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$$

is absolutely convergent and so no matter how we rearrange the terms of this series we'll always get the same value. In fact, it can be shown that the value of this series is,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2} = -\frac{\pi^2}{12}$$

10.10 Ratio Test

In this section we are going to take a look at a test that we can use to see if a series is absolutely convergent or not. Recall that if a series is absolutely convergent then we will also know that it's convergent and so we will often use it to simply determine the convergence of a series.

Before proceeding with the test let's do a quick reminder of factorials. This test will be particularly useful for series that contain factorials (and we will see some in the applications) so let's make sure we can deal with them before we run into them in an example.

If n is an integer such that $n \geq 0$ then n factorial is defined as,

$$\begin{aligned} n! &= n\left(n-1\right)\left(n-2\right)\cdots\left(3\right)\left(2\right)\left(1\right) && \text{if } n \geq 1 \\ 0! &= 1 && \text{by definition} \end{aligned}$$

Let's compute a couple real quick.

$$\begin{aligned} 1! &= 1 \\ 2! &= 2\left(1\right) = 2 \\ 3! &= 3\left(2\right)\left(1\right) = 6 \\ 4! &= 4\left(3\right)\left(2\right)\left(1\right) = 24 \\ 5! &= 5\left(4\right)\left(3\right)\left(2\right)\left(1\right) = 120 \end{aligned}$$

In the last computation above, notice that we could rewrite the factorial in a couple of different ways. For instance,

$$\begin{aligned} 5! &= 5\underbrace{\left(4\right)\left(3\right)\left(2\right)\left(1\right)}_{4!} = 5 \cdot 4! \\ 5! &= 5\left(4\right)\underbrace{\left(3\right)\left(2\right)\left(1\right)}_{3!} = 5\left(4\right) \cdot 3! \end{aligned}$$

In general, we can always “strip out” terms from a factorial as follows.

$$\begin{aligned} n! &= n\left(n-1\right)\left(n-2\right)\cdots\left(n-k\right)\left(n-\left(k+1\right)\right)\cdots\left(3\right)\left(2\right)\left(1\right) \\ &= n\left(n-1\right)\left(n-2\right)\cdots\left(n-k\right)\cdot\left(n-\left(k+1\right)\right)! \\ &= n\left(n-1\right)\left(n-2\right)\cdots\left(n-k\right)\cdot\left(n-k-1\right)! \end{aligned}$$

We will need to do this on occasion so don't forget about it.

Also, when dealing with factorials we need to be very careful with parenthesis. For instance, $(2n)! \neq 2\ n!$ as we can see if we write each of the following factorials out.

$$\begin{aligned} \left(2n\right)! &= \left(2n\right)\left(2n-1\right)\left(2n-2\right)\cdots\left(3\right)\left(2\right)\left(1\right) \\ 2\ n! &= 2\left[\left(n\right)\left(n-1\right)\left(n-2\right)\cdots\left(3\right)\left(2\right)\left(1\right)\right] \end{aligned}$$

Again, we will run across factorials with parenthesis so don't drop them. This is often one of the more common mistakes that students make when they first run across factorials.

Okay, we are now ready for the test.

Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

A proof of this test is at the end of the section.

Notice that in the case of $L = 1$ the ratio test is pretty much worthless and we would need to resort to a different test to determine the convergence of the series.

Also, the absolute value bars in the definition of L are absolutely required. If they are not there it will be impossible for us to get the correct answer.

Let's take a look at some examples.

Example 1

Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$$

Solution

With this first example let's be a little careful and make sure that we have everything down correctly. Here are the series terms a_n.

$$a_n = \frac{(-10)^n}{4^{2n+1}(n+1)}$$

Recall that to compute a_{n+1} all that we need to do is substitute $n+1$ for all the n's in a_n.

$$a_{n+1} = \frac{(-10)^{n+1}}{4^{2(n+1)+1}((n+1)+1)} = \frac{(-10)^{n+1}}{4^{2n+3}(n+2)}$$

Now, to define L we will use,

$$L = \lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right|$$

since this will be a little easier when dealing with fractions as we've got here. So,

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(-10)^{n+1}}{4^{2n+3}(n+2)} \frac{4^{2n+1}(n+1)}{(-10)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{-10(n+1)}{4^2(n+2)} \right| \\ &= \frac{10}{16} \lim_{n\to\infty} \frac{n+1}{n+2} \\ &= \frac{10}{16} < 1 \end{aligned}$$

So, $L < 1$ and so by the Ratio Test the series converges absolutely and hence will converge.

As seen in the previous example there is usually a lot of canceling that will happen in these. Make sure that you do this canceling. If you don't do this kind of canceling it can make the limit fairly difficult.

Example 2

Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{n!}{5^n}$$

Solution

Now that we've worked one in detail we won't go into quite the detail with the rest of these. Here is the limit.

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!}{5^{n+1}} \frac{5^n}{n!} \right| = \lim_{n\to\infty} \frac{(n+1)!}{5\, n!}$$

In order to do this limit we will need to eliminate the factorials. We simply can't do the limit with the factorials in it. To eliminate the factorials we will recall from our discussion on factorials above that we can always "strip out" terms from a factorial. If we do that with the numerator (in this case because it's the larger of the two) we get,

$$L = \lim_{n \to \infty} \frac{(n+1)\ \ n!}{5\ n!}$$

at which point we can cancel the $n!$ for the numerator an denominator to get,

$$L = \lim_{n \to \infty} \frac{(n+1)}{5} = \infty > 1$$

So, by the Ratio Test this series diverges.

Example 3

Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$$

Solution

In this case be careful in dealing with the factorials.

$$\begin{aligned} L &= \lim_{n \to \infty} \left| \frac{(n+1)^2}{(2(n+1)-1)!} \frac{(2n-1)!}{n^2} \right| \\ &= \lim_{n \to \infty} \left| \frac{(n+1)^2}{(2n+1)!} \frac{(2n-1)!}{n^2} \right| \\ &= \lim_{n \to \infty} \frac{(n+1)^2}{(2n+1)(2n)(2n-1)!} \frac{(2n-1)!}{n^2} \\ &= \lim_{n \to \infty} \frac{(n+1)^2}{(2n+1)(2n)(n^2)} \\ &= 0 < 1 \end{aligned}$$

So, by the Ratio Test this series converges absolutely and so converges.

Example 4

Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{9^n}{(-2)^{n+1}\,n}$$

Solution

Do not mistake this for a geometric series. The n in the denominator means that this isn't a geometric series. So, let's compute the limit.

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{9^{n+1}}{(-2)^{n+2}\,(n+1)}\,\frac{(-2)^{n+1}\,n}{9^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{9\,n}{(-2)\,(n+1)}\right| \\ &= \frac{9}{2}\lim_{n\to\infty}\frac{n}{n+1} \\ &= \frac{9}{2} > 1 \end{aligned}$$

Therefore, by the Ratio Test this series is divergent.

In the previous example the absolute value bars were required to get the correct answer. If we hadn't used them we would have gotten $L = -\frac{9}{2} < 1$ which would have implied a convergent series!

Now, let's take a look at a couple of examples to see what happens when we get $L = 1$. Recall that the ratio test will not tell us anything about the convergence of these series. In both of these examples we will first verify that we get $L = 1$ and then use other tests to determine the convergence.

Example 5

Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

Solution

Let's first get L.

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)^2+1}\,\frac{n^2+1}{(-1)^n}\right| = \lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1} = 1$$

So, as implied earlier we get $L = 1$ which means the ratio test is no good for determining the convergence of this series. We will need to resort to another test for this series. This series is an alternating series and so let's check the two conditions from that test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n^2+1} = 0$$

$$b_n = \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} = b_{n+1}$$

The two conditions are met and so by the Alternating Series Test this series is convergent. We'll leave it to you to verify this series is also absolutely convergent.

Example 6

Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty}\frac{n+2}{2n+7}$$

Solution

Here's the limit.

$$L = \lim_{n\to\infty}\left|\frac{n+3}{2(n+1)+7}\,\frac{2n+7}{n+2}\right| = \lim_{n\to\infty}\frac{(n+3)(2n+7)}{(2n+9)(n+2)} = 1$$

Again, the ratio test tells us nothing here. We can however, quickly use the divergence test on this. In fact that probably should have been our first choice on this one anyway.

$$\lim_{n\to\infty}\frac{n+2}{2n+7} = \frac{1}{2} \neq 0$$

By the Divergence Test this series is divergent.

So, as we saw in the previous two examples if we get $L = 1$ from the ratio test the series can be either convergent or divergent.

There is one more thing that we should note about the ratio test before we move onto the next section. The last series was a polynomial divided by a polynomial and we saw that we got $L = 1$ from the ratio test. This will always happen with rational expression involving only polynomials or polynomials under radicals. So, in the future it isn't even worth it to try the ratio test on these kinds of problems since we now know that we will get $L = 1$.

Also, in the second to last example we saw an example of an alternating series in which the positive term was a rational expression involving polynomials and again we will always get $L = 1$ in these cases.

Let's close the section out with a proof of the Ratio Test.

Proof of Ratio Test

First note that we can assume without loss of generality that the series will start at $n = 1$ as we've done for all our series test proofs.

Let's start off the proof here by assuming that $L < 1$ and we'll need to show that $\sum a_n$ is absolutely convergent. To do this let's first note that because $L < 1$ there is some number r such that $L < r < 1$.

Now, recall that,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

and because we also have chosen r such that $L < r$ there is some N such that if $n \geq N$ we will have,

$$\left|\frac{a_{n+1}}{a_n}\right| < r \qquad \Rightarrow \qquad |a_{n+1}| < r\,|a_n|$$

Next, consider the following,

$$\begin{aligned}
|a_{N+1}| &< r\,|a_N| \\
|a_{N+2}| &< r\,|a_{N+1}| < r^2\,|a_N| \\
|a_{N+3}| &< r\,|a_{N+2}| < r^3\,|a_N| \\
&\vdots \\
|a_{N+k}| &< r\,|a_{N+k-1}| < r^k\,|a_N|
\end{aligned}$$

So, for $k = 1, 2, 3, \ldots$ we have $|a_{N+k}| < r^k\,|a_N|$. Just why is this important? Well we can now look at the following series.

$$\sum_{k=0}^{\infty} |a_N|\, r^k$$

This is a geometric series and because $0 < r < 1$ we in fact know that it is a convergent series. Also because $|a_{N+k}| < r^k |a_N|$ by the Comparison test the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}|$$

is convergent. However since,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

we know that $\sum_{n=1}^{\infty} |a_n|$ is also convergent since the first term on the right is a finite sum of finite terms and hence finite. Therefore $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Next, we need to assume that $L > 1$ and we'll need to show that $\sum a_n$ is divergent. Recalling that,

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and because $L > 1$ we know that there must be some N such that if $n \geq N$ we will have,

$$\left| \frac{a_{n+1}}{a_n} \right| > 1 \qquad \Rightarrow \qquad |a_{n+1}| > |a_n|$$

However, if $|a_{n+1}| > |a_n|$ for all $n \geq N$ then we know that,

$$\lim_{n \to \infty} |a_n| \neq 0$$

because the terms are getting larger and guaranteed to not be negative. This in turn means that,

$$\lim_{n \to \infty} a_n \neq 0$$

Therefore, by the Divergence Test $\sum a_n$ is divergent.

Finally, we need to assume that $L = 1$ and show that we could get a series that has any of the three possibilities. To do this we just need a series for each case. We'll leave the details of checking to you but all three of the following series have $L = 1$ and each one exhibits one of the possibilities.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{absolutely convergent}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \text{conditionally convergent}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{divergent}$$

10.11 Root Test

This is the last test for series convergence that we're going to be looking at. As with the Ratio Test this test will also tell whether a series is absolutely convergent or not rather than simple convergence.

Root Test

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

A proof of this test is at the end of the section.

As with the ratio test, if we get $L = 1$ the root test will tell us nothing and we'll need to use another test to determine the convergence of the series. Also note that, generally for the series we'll be dealing with in this class, if $L = 1$ in the Ratio Test then the Root Test will also give $L = 1$.

We will also need the following fact in some of these problems.

Fact

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Let's take a look at a couple of examples.

Example 1

Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$$

Solution

There really isn't much to these problems other than computing the limit and then using the root test. Here is the limit for this problem.

$$L = \lim_{n \to \infty} \left| \frac{n^n}{3^{1+2n}} \right|^{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{3^{\frac{1}{n}+2}} = \frac{\infty}{3^2} = \infty > 1$$

So, by the Root Test this series is divergent.

Example 2

Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \left(\frac{5n - 3n^3}{7n^3 + 2} \right)^n$$

Solution

Again, there isn't too much to this series.

$$L = \lim_{n \to \infty} \left| \left(\frac{5n - 3n^3}{7n^3 + 2} \right)^n \right|^{\frac{1}{n}} = \lim_{n \to \infty} \left| \frac{5n - 3n^3}{7n^3 + 2} \right| = \left| \frac{-3}{7} \right| = \frac{3}{7} < 1$$

Therefore, by the Root Test this series converges absolutely and hence converges.

Note that we had to keep the absolute value bars on the fraction until we'd taken the limit to get the sign correct.

Example 3

Determine if the following series is convergent or divergent.

$$\sum_{n=3}^{\infty} \frac{(-12)^n}{n}$$

Solution

Here's the limit for this series.

$$L = \lim_{n\to\infty} \left|\frac{(-12)^n}{n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{12}{n^{\frac{1}{n}}} = \frac{12}{1} = 12 > 1$$

After using the fact from above we can see that the Root Test tells us that this series is divergent.

Proof of Root Test

First note that we can assume without loss of generality that the series will start at $n = 1$ as we've done for all our series test proofs. Also note that this proof is very similar to the proof of the Ratio Test.

Let's start off the proof here by assuming that $L < 1$ and we'll need to show that $\sum a_n$ is absolutely convergent. To do this let's first note that because $L < 1$ there is some number r such that $L < r < 1$.

Now, recall that,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

and because we also have chosen r such that $L < r$ there is some N such that if $n \geq N$ we will have,

$$|a_n|^{\frac{1}{n}} < r \qquad \Rightarrow \qquad |a_n| < r^n$$

Now the series

$$\sum_{n=0}^{\infty} r^n$$

is a geometric series and because $0 < r < 1$ we in fact know that it is a convergent series. Also, because $|a_n| < r^n$ $n \geq N$ by the Comparison test the series

$$\sum_{n=N}^{\infty} |a_n|$$

is convergent. However since,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$$

we know that $\sum\limits_{n=1}^{\infty} |a_n|$ is also convergent since the first term on the right is a finite sum of finite terms and hence finite. Therefore $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.

Next, we need to assume that $L > 1$ and we'll need to show that $\sum a_n$ is divergent. Recalling that,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

and because $L > 1$ we know that there must be some N such that if $n \geq N$ we will have,

$$|a_n|^{\frac{1}{n}} > 1 \qquad \Rightarrow \qquad |a_n| > 1^n = 1$$

However, if $|a_n| > 1$ for all $n \geq N$ then we know that,

$$\lim_{n\to\infty} |a_n| \neq 0$$

This in turn means that,

$$\lim_{n\to\infty} a_n \neq 0$$

Therefore, by the Divergence Test $\sum a_n$ is divergent.

Finally, we need to assume that $L = 1$ and show that we could get a series that has any of the three possibilities. To do this we just need a series for each case. We'll leave the details of checking to you but all three of the following series have $L = 1$ and each one exhibits one of the possibilities.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{absolutely convergent}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \text{conditionally convergent}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{divergent}$$

10.12 Strategy for Series

Now that we've got all of our tests out of the way it's time to think about organizing all of them into a general set of guidelines to help us determine the convergence of a series.

Note that these are a general set of guidelines and because some series can have more than one test applied to them we will get a different result depending on the path that we take through this set of guidelines. In fact, because more than one test may apply, you should always go completely through the guidelines and identify all possible tests that can be used on a given series. Once this has been done you can identify the test that you feel will be the easiest for you to use.

With that said here is the set of guidelines for determining the convergence of a series.

Strategy for Series

1. With a quick glance does it look like the series terms don't converge to zero in the limit, *i.e.* does $\lim\limits_{n \to \infty} a_n \neq 0$? If so, use the Divergence Test. Note that you should only do the Divergence Test if a quick glance suggests that the series terms may not converge to zero in the limit.
2. Is the series a p-series ($\sum \frac{1}{n^p}$) or a geometric series ($\sum\limits_{n=0}^{\infty} ar^n$ or $\sum\limits_{n=1}^{\infty} ar^{n-1}$)? If so use the fact that p-series will only converge if $p > 1$ and a geometric series will only converge if $|r| < 1$. Remember as well that often some algebraic manipulation is required to get a geometric series into the correct form.
3. Is the series similar to a p-series or a geometric series? If so, try the Comparison Test.
4. Is the series a rational expression involving only polynomials or polynomials under radicals (*i.e.* a fraction involving only polynomials or polynomials under radicals)? If so, try the Comparison Test and/or the Limit Comparison Test. Remember however, that in order to use the Comparison Test and the Limit Comparison Test the series terms all need to be positive.
5. Does the series contain factorials or constants raised to powers involving n? If so, then the Ratio Test may work. Note that if the series term contains a factorial then the only test that we've got that will work is the Ratio Test.
6. Can the series terms be written in the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$? If so, then the Alternating Series Test may work.
7. Can the series terms be written in the form $a_n = (b_n)^n$? If so, then the Root Test may work.
8. If $a_n = f(n)$ for some positive, decreasing function and $\int_a^\infty f(x)\,dx$ is easy to evaluate then the Integral Test may work.

Again, remember that these are only a set of guidelines and not a set of hard and fast rules to use when trying to determine the best test to use on a series. If more than one test can be used try to use the test that will be the easiest for you to use and remember that what is easy for someone else may not be easy for you!

Also, just so we can put all the tests into one place here is a quick listing of all the tests that we've got.

Divergence Test
If $\lim\limits_{n\to\infty} a_n \neq 0$ then $\sum a_n$ will diverge

Integral Test
Suppose that $f(x)$ is a positive, decreasing function on the interval $[k, \infty)$ and that $f(n) = a_n$ then,

1. If $\int_k^\infty f(x)\,dx$ is convergent then so is $\sum\limits_{n=k}^\infty a_n$.
2. If $\int_k^\infty f(x)\,dx$ is divergent then so is $\sum\limits_{n=k}^\infty a_n$.

Comparison Test
Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then,

1. If $\sum b_n$ is convergent then so is $\sum a_n$.
2. If $\sum a_n$ is divergent then so is $\sum b_n$.

Limit Comparison Test
Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n. Define,

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$

If c is positive (*i.e.* $c > 0$) and is finite (*i.e.* $c < \infty$) then either both series converge or both series diverge.

Alternating Series Test
Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. Then if,

1. $\lim\limits_{n\to\infty} b_n = 0$ and,
2. $\{b_n\}$ is eventually a decreasing sequence

the series $\sum a_n$ is convergent

Ratio Test
Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Root Test
Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

10.13 Estimating the Value of a Series

We have now spent quite a few sections determining the convergence of a series, however, with the exception of geometric and telescoping series, we have not talked about finding the value of a series. This is usually a very difficult thing to do and we still aren't going to talk about how to find the value of a series. What we will do is talk about how to estimate the value of a series. Often that is all that you need to know.

Before we get into how to estimate the value of a series let's remind ourselves how series convergence works. It doesn't make any sense to talk about the value of a series that doesn't converge and so we will be assuming that the series we're working with converges. Also, as we'll see the main method of estimating the value of series will come out of this discussion.

So, let's start with the series $\sum\limits_{n=1}^{\infty} a_n$ (the starting point is not important, but we need a starting point to do the work) and let's suppose that the series converges to s. Recall that this means that if we get the partial sums,

$$s_n = \sum_{i=1}^{n} a_i$$

then they will form a convergent sequence and its limit is s. In other words,

$$\lim_{n\to\infty} s_n = s$$

Now, just what does this mean for us? Well, since this limit converges it means that we can make the partial sums, s_n, as close to s as we want simply by taking n large enough. In other words, if we take n large enough then we can say that,

$$s_n \approx s$$

This is one method of estimating the value of a series. We can just take a partial sum and use that as an estimation of the value of the series. There are now two questions that we should ask about this.

First, how good is the estimation? If we don't have an idea of how good the estimation is then it really doesn't do all that much for us as an estimation.

Secondly, is there any way to make the estimate better? Sometimes we can use this as a starting point and make the estimation better. We won't always be able to do this, but if we can that will be nice.

So, let's start with a general discussion about the determining how good the estimation is. Let's first start with the full series and strip out the first n terms.

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{n} a_i + \sum_{i=n+1}^{\infty} a_i \tag{10.5}$$

Note that we converted over to an index of i in order to make the notation consistent with prior notation. Recall that we can use any letter for the index and it won't change the value.

Now, notice that the first series (the n terms that we've stripped out) is nothing more than the partial sum s_n. The second series on the right (the one starting at $i = n + 1$) is called the remainder and denoted by R_n. Finally let's acknowledge that we also know the value of the series since we are assuming it's convergent. Taking this notation into account we can rewrite Equation 10.5 as,

$$s = s_n + R_n$$

We can solve this for the remainder to get,

$$R_n = s - s_n$$

So, the remainder tells us the difference, or error, between the exact value of the series and the value of the partial sum that we are using as the estimation of the value of the series.

Of course, we can't get our hands on the actual value of the remainder because we don't have the actual value of the series. However, we can use some of the tests that we've got for convergence to get a pretty good estimate of the remainder provided we make some assumptions about the series. Once we've got an estimate on the value of the remainder we'll also have an idea on just how good a job the partial sum does of estimating the actual value of the series.

There are several tests that will allow us to get estimates of the remainder. We'll go through each one separately.

Also, when using the tests many of them had preconditions for use (*i.e.* terms had to be positive, terms had to be decreasing *etc.*) and when using the tests we noted that all we really needed was for them to eventually meet the preconditions in order for the test to work. For the following work however, we need the preconditions to always be met for all terms in the series.

If there are a few terms at the start where the preconditions aren't met we'll need to strip those terms out, do the estimate on the series that is left and then add in the terms we stripped out to get a final estimate of the series value.

Integral Test

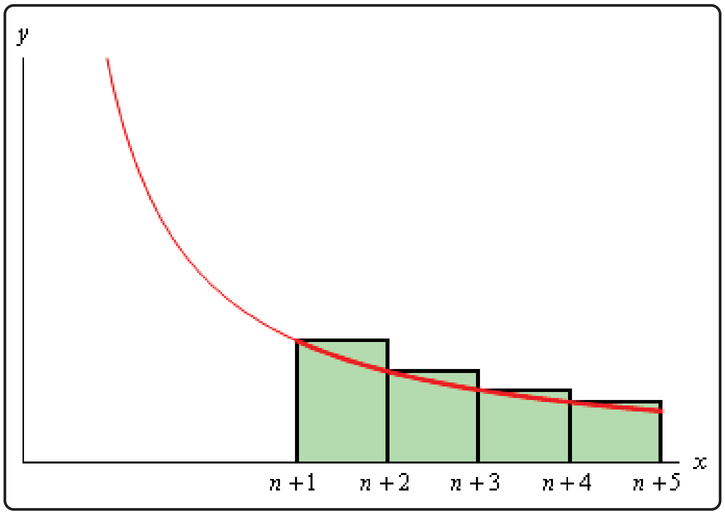
Recall that in this case we will need to assume that the series terms are all positive and be decreasing for all n. We derived the integral test by using the fact that the series could be thought of as an estimation of the area under the curve of $f(x)$ where $f(n) = a_n$. We can do something similar with the remainder.

As we'll soon see if we can get an upper and lower bound on the value of the remainder we can use these bounds to help us get upper and lower bounds on the value of the series. We can in turn use the upper and lower bounds on the series value to actually estimate the value of the series.

So, let's first recall that the remainder is,

$$R_n = \sum_{i=n+1}^{\infty} a_i = a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots$$

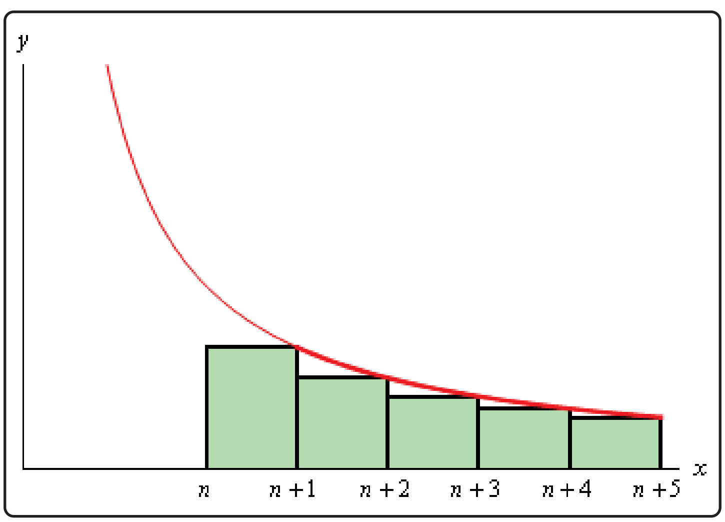
Now, if we start at $x = n + 1$, take rectangles of width 1 and use the left endpoint as the height of the rectangle we can estimate the area under $f(x)$ on the interval $[n + 1, \infty)$ as shown in the sketch below.



We can see that the remainder, R_n, is the area estimation and it will overestimate the exact area. So, we have the following inequality.

$$R_n \geq \int_{n+1}^{\infty} f(x)\, dx \tag{10.6}$$

Next, we could also estimate the area by starting at $x = n$, taking rectangles of width 1 again and then using the right endpoint as the height of the rectangle. This will give an estimation of the area under $f(x)$ on the interval $[n, \infty)$. This is shown in the following sketch.



Again, we can see that the remainder, R_n, is again this estimation and in this case it will underestimate the area. This leads to the following inequality,

$$R_n \le \int_n^\infty f(x)\,dx \tag{10.7}$$

Combining Equation 10.6 and Equation 10.7 gives,

$$\int_{n+1}^\infty f(x)\,dx \le R_n \le \int_n^\infty f(x)\,dx$$

So, provided we can do these integrals we can get both an upper and lower bound on the remainder. This will in turn give us an upper bound and a lower bound on just how good the partial sum, s_n, is as an estimation of the actual value of the series.

In this case we can also use these results to get a better estimate for the actual value of the series as well.

First, we'll start with the fact that

$$s = s_n + R_n$$

Now, if we use Equation 10.6 we get,

$$s = s_n + R_n \ge s_n + \int_{n+1}^\infty f(x)\,dx$$

Likewise if we use Equation 10.7 we get,

$$s = s_n + R_n \le s_n + \int_n^\infty f(x)\,dx$$

Putting these two together gives us,

Estimating Series with Integral Test

$$s_n + \int_{n+1}^\infty f(x)\,dx \le s \le s_n + \int_n^\infty f(x)\,dx \tag{10.8}$$

This gives an upper and a lower bound on the actual value of the series. We could then use as an estimate of the actual value of the series the average of the upper and lower bound.

Let's work an example with this.

Example 1

Using $n = 15$ to estimate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Solution

First, for comparison purposes, we'll note that the actual value of this series is known to be,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.644934068$$

Using $n = 15$ let's first get the partial sum.

$$s_{15} = \sum_{i=1}^{15} \frac{1}{i^2} = 1.580440283$$

Note that this is "close" to the actual value in some sense but isn't really all that close either.

Now, let's compute the integrals. These are fairly simple integrals, so we'll leave it to you to verify the values.

$$\int_{15}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{15} \qquad\qquad \int_{16}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{16}$$

Plugging these into Equation 10.8 gives us,

$$\begin{aligned} 1.580440283 + \frac{1}{16} &\le s \le 1.580440283 + \frac{1}{15} \\ 1.642940283 &\le s \le 1.647106950 \end{aligned}$$

Both the upper and lower bound are now very close to the actual value and if we take the average of the two we get the following estimate of the actual value.

$$s \approx 1.6450236165$$

That is pretty darn close to the actual value.

So, that is how we can use the Integral Test to estimate the value of a series. Let's move on to the next test.

Comparison Test

In this case, unlike with the integral test, we may or may not be able to get an idea of how good a particular partial sum will be as an estimate of the exact value of the series. Much of this will depend on how the comparison test is used.

First, let's remind ourselves on how the comparison test actually works. Given a series $\sum a_n$ let's assume that we've used the comparison test to show that it's convergent. Therefore, we found a second series $\sum b_n$ that converged and $a_n \leq b_n$ for all n. Also recall that we need both a_n and b_n to be positive for all n.

What we want to do is determine how good of a job the partial sum,

$$s_n = \sum_{i=1}^{n} a_i$$

will do in estimating the actual value of the series $\sum a_n$. Again, we will use the remainder to do this. Let's actually write down the remainder for both series.

$$R_n = \sum_{i=n+1}^{\infty} a_i \qquad\qquad T_n = \sum_{i=n+1}^{\infty} b_i$$

Now, since $a_n \leq b_n$ we also know that

$$R_n \leq T_n$$

When using the comparison test it is often the case that the b_n are fairly nice terms and that we might actually be able to get an idea on the size of T_n. For instance, if our second series is a p-series we can use the results from above to get an upper bound on T_n as follows,

Estimating Series with Comparision Test

$$R_n \leq T_n \leq \int_{n}^{\infty} g(x)\,dx \qquad \text{where } g(n) = b_n$$

Also, if the second series is a geometric series then we will be able to compute T_n exactly.

If we are unable to get an idea of the size of T_n then using the comparison test to help with estimates won't do us much good.

Let's take a look at an example.

Example 2

Using $n = 15$ to estimate the value of $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{4^n+1}$.

Solution

To do this we'll first need to go through the comparison test so we can get the second series. So,

$$\frac{2^n}{4^n+1} \le \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n$$

and

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

is a geometric series and converges because $|r| = \frac{1}{2} < 1$.

Now that we've gotten our second series let's get the estimate.

$$s_{15} = \sum_{n=0}^{15} \frac{2^n}{4^n+1} = 1.383062486$$

So, how good is it? Well we know that,

$$R_{15} \le T_{15} = \sum_{n=16}^{\infty} \left(\frac{1}{2}\right)^n$$

will be an upper bound for the error between the actual value and the estimate. Since our second series is a geometric series we can compute this directly as follows.

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{15} \left(\frac{1}{2}\right)^n + \sum_{n=16}^{\infty} \left(\frac{1}{2}\right)^n$$

The series on the left is in the standard form and so we can compute that directly. The first series on the right has a finite number of terms and so can be computed exactly and the second series on the right is the one that we'd like to have the value for. Doing the work gives,

$$\begin{aligned}
\sum_{n=16}^{\infty} \left(\frac{1}{2}\right)^n &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=0}^{15} \left(\frac{1}{2}\right)^n \\
&= \frac{1}{1-\left(\frac{1}{2}\right)} - 1.999969482 \\
&= 0.000030518
\end{aligned}$$

So, according to this if we use

$$s \approx 1.383062486$$

as an estimate of the actual value we will be off from the exact value by no more than 0.000030518 and that's not too bad.

In this case it can be shown that

$$\sum_{n=0}^{\infty} \frac{2^n}{4^n + 1} = 1.383093004$$

and so we can see that the actual error in our estimation is,

$$\text{Error} = \text{Actual} - \text{Estimate} = 1.383093004 - 1.383062486 = 0.000030518$$

Note that in this case the estimate of the error is actually fairly close (and in fact exactly the same) as the actual error. This will not always happen and so we shouldn't expect that to happen in all cases. The error estimate above is simply the upper bound on the error and the actual error will often be less than this value.

Before moving on to the final part of this section let's again note that we will only be able to determine how good the estimate is using the comparison test if we can easily get our hands on the remainder of the second term. The reality is that we won't always be able to do this.

Alternating Series Test

Both of the methods that we've looked at so far have required the series to contain only positive terms. If we allow series to have negative terms in it the process is usually more difficult. However, with that said there is one case where it isn't too bad. That is the case of an alternating series.

Once again we will start off with a convergent series $\sum a_n = \sum (-1)^n b_n$ which in this case happens to be an alternating series that satisfies the conditions of the alternating series test, so we know that $b_n \geq 0$ and is decreasing for all n. Also note that we could have any power on the "-1" we just used n for the sake of convenience. We want to know how good of an estimation of the actual series value will the partial sum, s_n, be. As with the prior cases we know that the remainder, R_n, will be the error in the estimation and so if we can get a handle on that we'll know approximately how good the estimation is.

From the proof of the Alternating Series Test we can see that s will lie between s_n and s_{n+1} for any n and so,

$$|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}$$

Therefore,

Estimating Series with Alternating Series Test

$$|R_n| = |s - s_n| \leq b_{n+1}$$

We needed absolute value bars because we won't know ahead of time if the estimation is larger or smaller than the actual value and we know that the b_n's are positive.

Let's take a look at an example.

Example 3

Using $n = 15$ to estimate the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

Solution

This is an alternating series and it does converge. In this case the exact value is known and so for comparison purposes,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} = -0.8224670336$$

Now, the estimation is,

$$s_{15} = \sum_{n=1}^{15} \frac{(-1)^n}{n^2} = -0.8245417574$$

From the fact above we know that

$$|R_{15}| = |s - s_{15}| \le b_{16} = \frac{1}{16^2} = 0.00390625$$

So, our estimation will have an error of no more than 0.00390625. In this case the exact value is known and so the actual error is,

$$|R_{15}| = |s - s_{15}| = 0.0020747238$$

In the previous example the estimation had only half the estimated error. It will often be the case that the actual error will be less than the estimated error. Remember that this is only an upper bound for the actual error.

Ratio Test

This will be the final case that we're going to look at for estimating series values and we are going to have to put a couple of fairly stringent restrictions on the series terms in order to do the work. One of the main restrictions we're going to make is to assume that the series terms are positive even though that is not required to actually use the test. We'll also be adding on another restriction in a bit.

In this case we've used the ratio test to show that $\sum a_n$ is convergent. To do this we computed

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

and found that $L < 1$.

As with the previous cases we are going to use the remainder, R_n, to determine how good of an estimation of the actual value the partial sum, s_n, is.

Estimating Series with Ratio Test

To get an estimate of the remainder let's first define the following sequence,

$$r_n = \frac{a_{n+1}}{a_n}$$

We now have two possible cases.

1. If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$ then,

$$R_n \le \frac{a_{n+1}}{1 - r_{n+1}}$$

2. If $\{r_n\}$ is an increasing sequence then,

$$R_n \le \frac{a_{n+1}}{1 - L}$$

Proof

Both parts will need the following work so we'll do it first. We'll start with the remainder.

$$\begin{aligned} R_n &= \sum_{i=n+1}^{\infty} a_i = a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots \\ &= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+1}} + \cdots\right) \end{aligned}$$

Next, we need to do a little work on a couple of these terms.

$$\begin{aligned} R_n &= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}}\frac{a_{n+2}}{a_{n+2}} + \frac{a_{n+4}}{a_{n+1}}\frac{a_{n+2}}{a_{n+2}}\frac{a_{n+3}}{a_{n+3}} + \cdots\right) \\ &= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+2}}{a_{n+1}}\frac{a_{n+3}}{a_{n+2}} + \frac{a_{n+2}}{a_{n+1}}\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+4}}{a_{n+3}} + \cdots\right) \end{aligned}$$

Now use the definition of r_n to write this as,

$$R_n = a_{n+1}\left(1 + r_{n+1} + r_{n+1}r_{n+2} + r_{n+1}r_{n+2}r_{n+3} + \cdots\right)$$

Okay now let's do the proof.

For the first part we are assuming that $\{r_n\}$ is decreasing and so we can estimate the remainder as,

$$\begin{aligned} R_n &= a_{n+1}\left(1 + r_{n+1} + r_{n+1}r_{n+2} + r_{n+1}r_{n+2}r_{n+3} + \cdots\right) \\ &\le a_{n+1}\left(1 + r_{n+1} + r_{n+1}^2 + r_{n+1}^3 + \cdots\right) \\ &= a_{n+1}\sum_{k=0}^{\infty} r_{n+1}^k \end{aligned}$$

Finally, the series here is a geometric series and because $r_{n+1} < 1$ we know that it converges and we can compute its value. So,

$$R_n \le \frac{a_{n+1}}{1 - r_{n+1}}$$

For the second part we are assuming that $\{r_n\}$ is increasing and we know that,

$$\lim_{n\to\infty} |r_n| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

and so we know that $r_n < L$ for all n. The remainder can then be estimated as,

$$\begin{aligned} R_n &= a_{n+1}\left(1 + r_{n+1} + r_{n+1}r_{n+2} + r_{n+1}r_{n+2}r_{n+3} + \cdots\right) \\ &\le a_{n+1}\left(1 + L + L^2 + L^3 + \cdots\right) \\ &= a_{n+1}\sum_{k=0}^{\infty} L^k \end{aligned}$$

This is a geometric series and since we are assuming that our original series converges we also know that $L < 1$ and so the geometric series above converges and we can compute its value. So,

$$R_n \le \frac{a_{n+1}}{1 - L}$$

Note that there are some restrictions on the sequence $\{r_n\}$ and at least one of its terms in order to use these formulas. If the restrictions aren't met then the formulas can't be used.

Let's take a look at an example of this.

Example 4

Using $n = 15$ to estimate the value of $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3^n}$.

Solution

First, let's use the ratio test to verify that this is a convergent series.

$$L = \lim_{n \to \infty} \left| \frac{n+1}{3^{n+1}} \frac{3^n}{n} \right| = \lim_{n \to \infty} \frac{n+1}{3n} = \frac{1}{3} < 1$$

So, it is convergent. Now let's get the estimate.

$$s_{15} = \sum_{n=0}^{15} \frac{n}{3^n} = 0.7499994250$$

To determine an estimate on the remainder, and hence the error, let's first get the sequence $\{r_n\}$.

$$r_n = \frac{n+1}{3^{n+1}} \frac{3^n}{n} = \frac{n+1}{3n} = \frac{1}{3}\left(1 + \frac{1}{n}\right)$$

The last rewrite was just to simplify some of the computations a little. Now, notice that,

$$f(x) = \frac{1}{3}\left(1 + \frac{1}{x}\right) \qquad f'(x) = -\frac{1}{3x^2} < 0$$

Since this function is always decreasing and $f(n) = r_n$ this sequence is decreasing. Also note that $r_{16} = \frac{1}{3}\left(1 + \frac{1}{16}\right) < 1$. Therefore, we can use the first case from the fact above to get,

$$R_{15} \le \frac{a_{16}}{1 - r_{16}} = \frac{\frac{16}{3^{16}}}{1 - \frac{1}{3}\left(1 + \frac{1}{16}\right)} = 0.0000005755187$$

So, it looks like our estimate is probably quite good. In this case the exact value is known.

$$\sum_{n=0}^{\infty} \frac{n}{3^n} = \frac{3}{4}$$

and so we can compute the actual error.

$$|R_{15}| = |s - s_{15}| = 0.000000575$$

This is less than the upper bound, but unlike in the previous example this actual error is quite close to the upper bound.

In the last two examples we've seen that the upper bound computations on the error can sometimes be quite close to the actual error and at other times they can be off by quite a bit. There is usually no way of knowing ahead of time which it will be and without the exact value in hand there will never be a way of determining which it will be.

Notice that this method did require the series terms to be positive, but that doesn't mean that we can't deal with ratio test series if they have negative terms. Often series that we used ratio test on are also alternating series and so if that is the case we can always resort to the previous material to get an upper bound on the error in the estimation, even if we didn't use the alternating series test to show convergence.

Note however that if the series does have negative terms but doesn't happen to be an alternating series then we can't use any of the methods discussed in this section to get an upper bound on the error.

10.14 Power Series

We've spent quite a bit of time talking about series now and with only a couple of exceptions we've spent most of that time talking about how to determine if a series will converge or not. It's now time to start looking at some specific kinds of series and we'll eventually reach the point where we can talk about a couple of applications of series.

In this section we are going to start talking about power series. A **power series about a**, or just **power series**, is any series that can be written in the form,

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where a and c_n are numbers. The c_n's are often called the **coefficients** of the series. The first thing to notice about a power series is that it is a function of x. That is different from any other kind of series that we've looked at to this point. In all the prior sections we've only allowed numbers in the series and now we are allowing variables to be in the series as well. This will not change how things work however. Everything that we know about series still holds.

In the discussion of power series convergence is still a major question that we'll be dealing with. The difference is that the convergence of the series will now depend upon the values of x that we put into the series. A power series may converge for some values of x and not for other values of x.

Before we get too far into power series there is some terminology that we need to get out of the way.

First, as we will see in our examples, we will be able to show that there is a number R so that the power series will converge for, $|x-a| < R$ and will diverge for $|x-a| > R$. This number is called the **radius of convergence** for the series. Note that the series may or may not converge if $|x-a| = R$. What happens at these points will not change the radius of convergence.

Secondly, the interval of all x's, including the endpoints if need be, for which the power series converges is called the **interval of convergence** of the series.

These two concepts are fairly closely tied together. If we know that the radius of convergence of a power series is R then we have the following.

$$\begin{array}{cl} a-R < x < a+R & \text{power series converges} \\ x < a-R \text{ and } x > a+R & \text{power series diverges} \end{array}$$

The interval of convergence must then contain the interval $a-R < x < a+R$ since we know that the power series will converge for these values. We also know that the interval of convergence can't contain x's in the ranges $x < a-R$ and $x > a+R$ since we know the power series diverges for these value of x. Therefore, to completely identify the interval of convergence all that we have to do is determine if the power series will converge for $x = a-R$ or $x = a+R$. If the power series converges for one or both of these values then we'll need to include those in the interval of convergence.

Before getting into some examples let's take a quick look at the convergence of a power series for the case of $x = a$. In this case the power series becomes,

$$\sum_{n=0}^{\infty} c_n (a-a)^n = \sum_{n=0}^{\infty} c_n (0)^n = c_0 (0)^0 + \sum_{n=1}^{\infty} c_n (0)^n = c_0 + \sum_{n=1}^{\infty} 0 = c_0 + 0 = c_0$$

and so the power series converges. Note that we had to strip out the first term since it was the only non-zero term in the series.

It is important to note that no matter what else is happening in the power series we are guaranteed to get convergence for $x = a$. The series may not converge for any other value of x, but it will always converge for $x = a$.

Let's work some examples. We'll put quite a bit of detail into the first example and then not put quite as much detail in the remaining examples.

Example 1

Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x+3)^n$$

Solution

Okay, we know that this power series will converge for $x = -3$, but that's it at this point. To determine the remainder of the x's for which we'll get convergence we can use any of the tests that we've discussed to this point. After application of the test that we choose to work with we will arrive at condition(s) on x that we can use to determine the values of x for which the power series will converge and the values of x for which the power series will diverge. From this we can get the radius of convergence and most of the interval of convergence (with the possible exception of the endpoints).

With all that said, the best tests to use here are almost always the ratio or root test. Most of the power series that we'll be looking at are set up for one or the other. In this case we'll use the ratio test.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{n+1}} \frac{4^n}{(-1)^n (n)(x+3)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{-(n+1)(x+3)}{4n} \right| \end{aligned}$$

Before going any farther with the limit let's notice that since x is not dependent on the limit it can be factored out of the limit. Notice as well that in doing this we'll need to keep the absolute value bars on it since we need to make sure everything stays positive and x could well be a value that will make things negative. The limit is then,

$$\begin{aligned} L &= |x+3| \lim_{n\to\infty} \frac{n+1}{4n} \\ &= \frac{1}{4}|x+3| \end{aligned}$$

So, the ratio test tells us that if $L < 1$ the series will converge, if $L > 1$ the series will diverge, and if $L = 1$ we don't know what will happen. So, we have,

$$\begin{aligned} \frac{1}{4}|x+3| < 1 \qquad &\Rightarrow \qquad |x+3| < 4 \qquad \text{series converges} \\ \frac{1}{4}|x+3| > 1 \qquad &\Rightarrow \qquad |x+3| > 4 \qquad \text{series diverges} \end{aligned}$$

We'll deal with the $L = 1$ case in a bit. Notice that we now have the radius of convergence for this power series. These are exactly the conditions required for the radius of convergence. The radius of convergence for this power series is $R = 4$.

Now, let's get the interval of convergence. We'll get most (if not all) of the interval by solving the first inequality from above.

$$\begin{aligned} -4 < x+3 < 4 \\ -7 < x < 1 \end{aligned}$$

So, most of the interval of validity is given by $-7 < x < 1$. All we need to do is determine if the power series will converge or diverge at the endpoints of this interval. Note that these values of x will correspond to the value of x that will give $L = 1$.

The way to determine convergence at these points is to simply plug them into the original power series and see if the series converges or diverges using any test necessary.

$x = -7$:
In this case the series is,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-4)^n &= \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-1)^n 4^n \\ &= \sum_{n=1}^{\infty} (-1)^n(-1)^n n \qquad\qquad (-1)^n(-1)^n = (-1)^{2n} = 1 \\ &= \sum_{n=1}^{\infty} n \end{aligned}$$

This series is divergent by the Divergence Test since $\lim\limits_{n\to\infty} n = \infty \neq 0$.

$x = 1$:
In this case the series is,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(4)^n = \sum_{n=1}^{\infty} (-1)^n n$$

This series is also divergent by the Divergence Test since $\lim\limits_{n\to\infty} (-1)^n n$ doesn't exist.

So, in this case the power series will not converge for either endpoint. The interval of convergence is then,

$$-7 < x < 1$$

In the previous example the power series didn't converge for either endpoint of the interval. Sometimes that will happen, but don't always expect that to happen. The power series could converge at either both of the endpoints or only one of the endpoints.

Example 2

Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$$

Solution

Let's jump right into the ratio test.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{2^{n+1}(4x-8)^{n+1}}{n+1} \frac{n}{2^n(4x-8)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{2n(4x-8)}{n+1} \right| \\ &= |4x-8| \lim_{n\to\infty} \frac{2n}{n+1} \\ &= 2|4x-8| \end{aligned}$$

So we will get the following convergence/divergence information from this.

$$\begin{aligned} 2|4x-8| &< 1 & &\text{series converges} \\ 2|4x-8| &> 1 & &\text{series diverges} \end{aligned}$$

We need to be careful here in determining the interval of convergence. The interval of convergence requires $|x-a| < R$ and $|x-a| > R$. In other words, we need to factor a 4 out of the absolute value bars in order to get the correct radius of convergence. Doing this gives,

$$\begin{aligned} 8|x-2| &< 1 & \Rightarrow \quad |x-2| &< \frac{1}{8} & &\text{series converges} \\ 8|x-2| &> 1 & \Rightarrow \quad |x-2| &> \frac{1}{8} & &\text{series diverges} \end{aligned}$$

So, the radius of convergence for this power series is $R = \frac{1}{8}$.

Now, let's find the interval of convergence. Again, we'll first solve the inequality that gives convergence above.

$$-\frac{1}{8} < x - 2 < \frac{1}{8}$$
$$\frac{15}{8} < x < \frac{17}{8}$$

Now check the endpoints.

$x = \dfrac{15}{8}$:

The series here is,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{15}{2} - 8\right)^n &= \sum_{n=1}^{\infty} \frac{2^n}{n}\left(-\frac{1}{2}\right)^n \\ &= \sum_{n=1}^{\infty} \frac{2^n}{n}\frac{(-1)^n}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\end{aligned}$$

This is the alternating harmonic series and we know that it converges.

$x = \dfrac{17}{8}$:

The series here is,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{17}{2} - 8\right)^n &= \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{1}{2}\right)^n \\ &= \sum_{n=1}^{\infty} \frac{2^n}{n}\frac{1}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\end{aligned}$$

This is the harmonic series and we know that it diverges.

So, the power series converges for one of the endpoints, but not the other. This will often happen so don't get excited about it when it does. The interval of convergence for this power series is then,

$$\frac{15}{8} \le x < \frac{17}{8}$$

We now need to take a look at a couple of special cases with radius and intervals of convergence.

Example 3

Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=0}^{\infty} n!(2x+1)^n$$

Solution

We'll start this example with the ratio test as we have for the previous ones.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)\,n!\;\;(2x+1)}{n!} \right| \\ &= |2x+1| \lim_{n\to\infty} (n+1) \end{aligned}$$

At this point we need to be careful. The limit is infinite, but there is that term with the x's in front of the limit. We'll have $L = \infty > 1$ provided $x \neq -\frac{1}{2}$.

So, this power series will only converge if $x = -\frac{1}{2}$. If you think about it we actually already knew that however. From our initial discussion we know that every power series will converge for $x = a$ and in this case $a = -\frac{1}{2}$. Remember that we get a from $(x-a)^n$, and notice the coefficient of the x must be a one!

In this case we say the radius of convergence is $R = 0$ and the interval of convergence is $x = -\frac{1}{2}$, and yes we really did mean interval of convergence even though it's only a point.

Example 4

Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{n^n}$$

Solution

In this example the root test seems more appropriate. So,

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(x-6)^n}{n^n} \right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left| \frac{x-6}{n} \right| \\ &= |x-6| \lim_{n\to\infty} \frac{1}{n} \\ &= 0 \end{aligned}$$

So, since $L = 0 < 1$ regardless of the value of x this power series will converge for every x.

In these cases, we say that the radius of convergence is $R = \infty$ and interval of convergence is $-\infty < x < \infty$.

So, let's summarize the last two examples. If the power series only converges for $x = a$ then the radius of convergence is $R = 0$ and the interval of convergence is $x = a$. Likewise, if the power series converges for every x the radius of convergence is $R = \infty$ and interval of convergence is $-\infty < x < \infty$.

Let's work one more example.

Example 5

Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{(-3)^n}$$

Solution

First notice that $a = 0$ in this problem. That's not really important to the problem, but it's worth pointing out so people don't get excited about it.

The important difference in this problem is the exponent on the x. In this case it is $2n$ rather than the standard n. As we will see some power series will have exponents other than an n and so we still need to be able to deal with these kinds of problems.

This one seems set up for the root test again so let's use that.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{x^{2n}}{(-3)^n} \right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left| \frac{x^2}{-3} \right| \\ &= \frac{x^2}{3} \end{aligned}$$

So, we will get convergence if

$$\frac{x^2}{3} < 1 \qquad\qquad \Rightarrow \qquad\qquad x^2 < 3$$

The radius of convergence is NOT 3 however. The radius of convergence requires an exponent of 1 on the x. Therefore,

$$\begin{aligned} \sqrt{x^2} &< \sqrt{3} \\ |x| &< \sqrt{3} \end{aligned}$$

Be careful with the absolute value bars! In this case it looks like the radius of convergence is $R = \sqrt{3}$. Notice that we didn't bother to put down the inequality for divergence this time. The inequality for divergence is just the interval for convergence that the test gives with the inequality switched and generally isn't needed. We will usually skip that part.

Now let's get the interval of convergence. First from the inequality we get,

$$-\sqrt{3} < x < \sqrt{3}$$

Now check the endpoints.

$x = -\sqrt{3}$:

Here the power series is,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\left(-\sqrt{3}\right)^{2n}}{(-3)^n} &= \sum_{n=1}^{\infty} \frac{\left(\left(-\sqrt{3}\right)^2\right)^n}{(-3)^n} \\ &= \sum_{n=1}^{\infty} \frac{(3)^n}{(-1)^n (3)^n} \\ &= \sum_{n=1}^{\infty} (-1)^n \end{aligned}$$

This series is divergent by the Divergence Test since $\lim\limits_{n\to\infty}(-1)^n$ doesn't exist.

$x = \sqrt{3}$:

Because we're squaring the x this series will be the same as the previous step.

$$\sum_{n=1}^{\infty}\frac{\left(\sqrt{3}\right)^{2n}}{(-3)^n} = \sum_{n=1}^{\infty}(-1)^n$$

which is divergent.

The interval of convergence is then,

$$-\sqrt{3} < x < \sqrt{3}$$

10.15 Power Series and Functions

We opened the last section by saying that we were going to start thinking about applications of series and then promptly spent the section talking about convergence again. It's now time to actually start with the applications of series.

With this section we will start talking about how to represent functions with power series. The natural question of why we might want to do this will be answered in a couple of sections once we actually learn how to do this.

Let's start off with one that we already know how to do, although when we first ran across this series we didn't think of it as a power series nor did we acknowledge that it represented a function.

Recall that the geometric series is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad \text{provided } |r| < 1$$

Don't forget as well that if $|r| \geq 1$ the series diverges.

Now, if we take $a = 1$ and $r = x$ this becomes,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{provided } |x| < 1 \tag{10.9}$$

Turning this around we can see that we can represent the function

$$f(x) = \frac{1}{1-x} \tag{10.10}$$

with the power series

$$\sum_{n=0}^{\infty} x^n \qquad \text{provided } |x| < 1 \tag{10.11}$$

This provision is important. We can clearly plug any number other than $x = 1$ into the function, however, we will only get a convergent power series if $|x| < 1$. This means the equality in Equation 10.9 will only hold if $|x| < 1$. For any other value of x the equality won't hold. Note as well that we can also use this to acknowledge that the radius of convergence of this power series is $R = 1$ and the interval of convergence is $|x| < 1$.

This idea of convergence is important here. We will be representing many functions as power series and it will be important to recognize that the representations will often only be valid for a range of x's and that there may be values of x that we can plug into the function that we can't plug into the power series representation.

In this section we are going to concentrate on representing functions with power series where the functions can be related back to Equation 10.10.

In this way we will hopefully become familiar with some of the kinds of manipulations that we will sometimes need to do when working with power series.

So, let's jump into a couple of examples.

Example 1

Find a power series representation for the following function and determine its interval of convergence.

$$g(x) = \frac{1}{1+x^3}$$

Solution

What we need to do here is to relate this function back to Equation 10.10. This is actually easier than it might look. Recall that the x in Equation 10.10 is simply a variable and can represent anything. So, a quick rewrite of $g(x)$ gives,

$$g(x) = \frac{1}{1-(-x^3)}$$

and so the $-x^3$ in $g(x)$ holds the same place as the x in Equation 10.10. Therefore, all we need to do is replace the x in Equation 10.11 and we've got a power series representation for $g(x)$.

$$g(x) = \sum_{n=0}^{\infty} \left(-x^3\right)^n \qquad \text{provided } \left|-x^3\right| < 1$$

Notice that we replaced both the x in the power series and in the interval of convergence.

All we need to do now is a little simplification.

$$g(x) = \sum_{n=0}^{\infty} (-1)^n x^{3n} \qquad \text{provided } |x|^3 < 1 \qquad \Rightarrow \qquad |x| < 1$$

So, in this case the interval of convergence is the same as the original power series. This usually won't happen. More often than not the new interval of convergence will be different from the original interval of convergence.

Example 2

Find a power series representation for the following function and determine its interval of convergence.

$$h(x) = \frac{2x^2}{1+x^3}$$

Solution

This function is similar to the previous function. The difference is the numerator and at first glance that looks to be an important difference. Since Equation 10.10 doesn't have an x in the numerator it appears that we can't relate this function back to that.

However, now that we've worked the first example this one is actually very simple since we can use the result of the answer from that example. To see how to do this let's first rewrite the function a little.

$$h(x) = 2x^2 \frac{1}{1+x^3}$$

Now, from the first example we've already got a power series for the second term so let's use that to write the function as,

$$h(x) = 2x^2 \sum_{n=0}^{\infty} (-1)^n x^{3n} \qquad \text{provided } |x| < 1$$

Notice that the presence of x's outside of the series will NOT affect its convergence and so the interval of convergence remains the same.

The last step is to bring the coefficient into the series and we'll be done. When we do this make sure and combine the x's as well. We typically only want a single x in a power series.

$$h(x) = \sum_{n=0}^{\infty} 2(-1)^n x^{3n+2} \qquad \text{provided } |x| < 1$$

As we saw in the previous example we can often use previous results to help us out. This is an important idea to remember as it can often greatly simplify our work.

Example 3

Find a power series representation for the following function and determine its interval of convergence.

$$f(x) = \frac{x}{5-x}$$

Solution

So, again, we've got an x in the numerator. So, as with the last example let's factor that out and see what we've got left.

$$f(x) = x\,\frac{1}{5-x}$$

If we had a power series representation for

$$g(x) = \frac{1}{5-x}$$

we could get a power series representation for $f(x)$.

So, let's find one. We'll first notice that in order to use Equation 10.10 we'll need the number in the denominator to be a one. That's easy enough to get.

$$g(x) = \frac{1}{5}\,\frac{1}{1-\frac{x}{5}}$$

Now all we need to do to get a power series representation is to replace the x in Equation 10.11 with $\frac{x}{5}$. Doing this gives,

$$g(x) = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n \qquad \text{provided } \left|\frac{x}{5}\right| < 1$$

Now let's do a little simplification on the series.

$$\begin{aligned} g(x) &= \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^n}{5^n} \\ &= \sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} \end{aligned}$$

The interval of convergence for this series is,

$$\left|\frac{x}{5}\right| < 1 \qquad \Rightarrow \qquad \frac{1}{5}|x| < 1 \qquad \Rightarrow \qquad |x| < 5$$

Okay, this was the work for the power series representation for $g(x)$ let's now find a power series representation for the original function. All we need to do for this is to multiply the power series representation for $g(x)$ by x and we'll have it.

$$\begin{aligned} f(x) &= x\,\frac{1}{5-x} \\ &= x\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}} \end{aligned}$$

The interval of convergence doesn't change and so it will be $|x| < 5$.

So, hopefully we now have an idea on how to find the power series representation for some functions. Admittedly all of the functions could be related back to Equation 10.10 but it's a start.

We now need to look at some further manipulation of power series that we will need to do on occasion. We need to discuss differentiation and integration of power series.

Let's start with differentiation of the power series,

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Now, we know that if we differentiate a finite sum of terms all we need to do is differentiate each of the terms and then add them back up. With infinite sums there are some subtleties involved that we need to be careful with but are somewhat beyond the scope of this course.

Nicely enough for us however, it is known that if the power series representation of $f(x)$ has a radius of convergence of $R > 0$ then the term by term differentiation of the power series will also have a radius of convergence of R and (more importantly) will in fact be the power series representation of $f'(x)$ provided we stay within the radius of convergence.

Again, we should make the point that if we aren't dealing with a power series then we may or may not be able to differentiate each term of the series to get the derivative of the series.

So, what all this means for us is that,

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

Note the initial value of this series. It has been changed from $n = 0$ to $n = 1$. This is an acknowledgement of the fact that the derivative of the first term is zero and hence isn't in the derivative. Notice however, that since the $n = 0$ term of the above series is also zero, we could start the series at $n = 0$ if it was required for a particular problem. In general, however, this won't be done in this class.

We can now find formulas for higher order derivatives as well now.

$$\begin{aligned} f''(x) &= \sum_{n=2}^{\infty} n(n-1)\, c_n (x-a)^{n-2} \\ f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2)\, c_n (x-a)^{n-3} \\ &etc. \end{aligned}$$

Once again, notice that the initial value of n changes with each differentiation in order to acknowledge that a term from the original series differentiated to zero.

Let's now briefly talk about integration. Just as with the differentiation, when we've got an infinite series we need to be careful about just integration term by term. Much like with derivatives it turns out that as long as we're working with power series we can just integrate the terms of the series to get the integral of the series itself. In other words,

$$\begin{aligned} \int f(x)\, dx &= \int \sum_{n=0}^{\infty} c_n (x-a)^n \, dx \\ &= \sum_{n=0}^{\infty} \int c_n (x-a)^n \, dx \\ &= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \end{aligned}$$

Notice that we pick up a constant of integration, C, that is outside the series here.

Let's summarize the differentiation and integration ideas before moving on to an example or two.

Fact

If $f(x) = \sum\limits_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence of $R > 0$ then,

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \qquad \int f(x)\, dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

and both of these also have a radius of convergence of R.

Now, let's see how we can use these facts to generate some more power series representations of functions.

Example 4

Find a power series representation for the following function and determine its radius of convergence.

$$g(x) = \frac{1}{(1-x)^2}$$

Solution

To do this problem let's notice that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

Then since we've got a power series representation for

$$\frac{1}{1-x}$$

all that we'll need to do is differentiate that power series to get a power series representation for $g(x)$.

$$\begin{aligned} g(x) &= \frac{1}{(1-x)^2} \\ &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) \\ &= \sum_{n=1}^{\infty} n x^{n-1} \end{aligned}$$

Then since the original power series had a radius of convergence of $R = 1$ the derivative, and hence $g(x)$, will also have a radius of convergence of $R = 1$.

Example 5

Find a power series representation for the following function and determine its radius of convergence.

$$h(x) = \ln(5 - x)$$

Solution

In this case we need to notice that

$$\int \frac{1}{5-x}\,dx = -\ln(5-x)$$

and then recall that we have a power series representation for

$$\frac{1}{5-x}$$

Remember we found a representation for this in Example 3. So,

$$\begin{aligned}
\ln(5-x) &= -\int \frac{1}{5-x}\,dx \\
&= -\int \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}\,dx \\
&= C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,5^{n+1}}
\end{aligned}$$

We can find the constant of integration, C, by plugging in a value of x. A good choice is $x = 0$ since that will make the series easy to evaluate.

$$\begin{aligned}
\ln(5-0) &= C - \sum_{n=0}^{\infty} \frac{0^{n+1}}{(n+1)\,5^{n+1}} \\
\ln(5) &= C
\end{aligned}$$

So, the final answer is,

$$\ln(5-x) = \ln(5) - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,5^{n+1}}$$

Note that it is okay to have the constant sitting outside of the series like this. In fact, there is no way to bring it into the series so don't get excited about it.

Finally, because the power series representation from Example 3 had a radius of convergence of $R = 5$ this series will also have a radius of convergence of $R = 5$.

10.16 Taylor Series

In the previous section we started looking at writing down a power series representation of a function. The problem with the approach in that section is that everything came down to needing to be able to relate the function in some way to

$$\frac{1}{1-x}$$

and while there are many functions out there that can be related to this function there are many more that simply can't be related to this.

So, without taking anything away from the process we looked at in the previous section, what we need to do is come up with a more general method for writing a power series representation for a function.

So, for the time being, let's make two assumptions. First, let's assume that the function $f(x)$ does in fact have a power series representation about $x = a$,

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

Next, we will need to assume that the function, $f(x)$, has derivatives of every order and that we can in fact find them all.

Now that we've assumed that a power series representation exists we need to determine what the coefficients, c_n, are. This is easier than it might at first appear to be. Let's first just evaluate everything at $x = a$. This gives,

$$f(a) = c_0$$

So, all the terms except the first are zero and we now know what c_0 is. Unfortunately, there isn't any other value of x that we can plug into the function that will allow us to quickly find any of the other coefficients. However, if we take the derivative of the function (and its power series) then plug in $x = a$ we get,

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \\ f'(a) &= c_1 \end{aligned}$$

and we now know c_1.

Let's continue with this idea and find the second derivative.

$$\begin{aligned} f''(x) &= 2c_2 + 3(2)c_3(x-a) + 4(3)c_4(x-a)^2 + \cdots \\ f''(a) &= 2c_2 \end{aligned}$$

So, it looks like,

$$c_2 = \frac{f''(a)}{2}$$

Using the third derivative gives,

$$f'''(x) = 3(2)c_3 + 4(3)(2)c_4(x-a) + \cdots$$
$$f'''(a) = 3(2)c_3 \qquad \Rightarrow \qquad c_3 = \frac{f'''(a)}{3(2)}$$

Using the fourth derivative gives,

$$f^{(4)}(x) = 4(3)(2)c_4 + 5(4)(3)(2)c_5(x-a)\cdots$$
$$f^{(4)}(a) = 4(3)(2)c_4 \qquad \Rightarrow \qquad c_4 = \frac{f^{(4)}(a)}{4(3)(2)}$$

Hopefully by this time you've seen the pattern here. It looks like, in general, we've got the following formula for the coefficients.

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This even works for $n = 0$ if you recall that $0! = 1$ and define $f^{(0)}(x) = f(x)$.

So, provided a power series representation for the function $f(x)$ about $x = a$ exists the **Taylor Series for** $f(x)$ **about** $x = a$ is,

Taylor Series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

If we use $a = 0$, so we are talking about the Taylor Series about $x = 0$, we call the series a **Maclaurin Series** for $f(x)$ or,

Maclaurin Series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \end{aligned}$$

Before working any examples of Taylor Series we first need to address the assumption that a Taylor Series will in fact exist for a given function. Let's start out with some notation and definitions that we'll need.

To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the n^{th} **degree Taylor polynomial of** $f(x)$ as,

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Note that this really is a polynomial of degree at most n. If we were to write out the sum without the summation notation this would clearly be an n^{th} degree polynomial. We'll see a nice application of Taylor polynomials in the next section.

Notice as well that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the n^{th} degree Taylor polynomial is just the partial sum for the series.

Next, the **remainder** is defined to be,

$$R_n(x) = f(x) - T_n(x)$$

So, the remainder is really just the *error* between the function $f(x)$ and the n^{th} degree Taylor polynomial for a given n.

With this definition note that we can then write the function as,

$$f(x) = T_n(x) + R_n(x)$$

We now have the following Theorem.

Theorem

Suppose that $f(x) = T_n(x) + R_n(x)$. Then if,

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$ then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

on $|x-a| < R$.

In general, showing that

$$\lim_{n\to\infty} R_n(x) = 0$$

is a somewhat difficult process and so we will be assuming that this can be done for some R in all of the examples that we'll be looking at.

Now let's look at some examples.

Example 1

Find the Taylor Series for $f(x) = \mathbf{e}^x$ about $x = 0$.

Solution

This is actually one of the easier Taylor Series that we'll be asked to compute. To find the Taylor Series for a function we will need to determine a general formula for $f^{(n)}(a)$. This is one of the few functions where this is easy to do right from the start.

To get a formula for $f^{(n)}(0)$ all we need to do is recognize that,

$$f^{(n)}(x) = \mathbf{e}^x \qquad n = 0, 1, 2, 3, \ldots$$

and so,

$$f^{(n)}(0) = \mathbf{e}^0 = 1 \qquad n = 0, 1, 2, 3, \ldots$$

Therefore, the Taylor series for $f(x) = \mathbf{e}^x$ about $x = 0$ is,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example 2

Find the Taylor Series for $f(x) = \mathbf{e}^{-x}$ about $x = 0$.

Solution

There are two ways to do this problem. Both are fairly simple, however one of them requires significantly less work. We'll work both solutions since the longer one has some nice ideas that we'll see in other examples.

Solution 1

As with the first example we'll need to get a formula for $f^{(n)}(0)$. However, unlike the first one we've got a little more work to do. Let's first take some derivatives and evaluate them at $x = 0$.

$$\begin{aligned} f^{(0)}(x) &= \mathbf{e}^{-x} & f^{(0)}(0) &= 1 \\ f^{(1)}(x) &= -\mathbf{e}^{-x} & f^{(1)}(0) &= -1 \\ f^{(2)}(x) &= \mathbf{e}^{-x} & f^{(2)}(0) &= 1 \\ f^{(3)}(x) &= -\mathbf{e}^{-x} & f^{(3)}(0) &= -1 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= (-1)^n \mathbf{e}^{-x} & f^{(n)}(0) &= (-1)^n \qquad n = 0, 1, 2, 3 \end{aligned}$$

After a couple of computations we were able to get general formulas for both $f^{(n)}(x)$ and $f^{(n)}(0)$. We often won't be able to get a general formula for $f^{(n)}(x)$ so don't get too excited about getting that formula. Also, as we will see it won't always be easy to get a general formula for $f^{(n)}(a)$.

So, in this case we've got general formulas so all we need to do is plug these into the Taylor Series formula and be done with the problem.

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Solution 2

The previous solution wasn't too bad and we often have to do things in that manner. However, in this case there is a much shorter solution method. In the previous section we used series that we've already found to help us find a new series. Let's do the same thing with this one. We already know a Taylor Series for $\mathbf{e}^x$ about $x = 0$ and in this case the only difference is we've got a "$-x$" in the exponent instead of just an x.

So, all we need to do is replace the x in the Taylor Series that we found in the first example with "$-x$".

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

This is a much shorter method of arriving at the same answer so don't forget about using previously computed series where possible (and allowed of course).

Example 3

Find the Taylor Series for $f(x) = x^4\mathbf{e}^{-3x^2}$ about $x = 0$.

Solution

For this example, we will take advantage of the fact that we already have a Taylor Series for $\mathbf{e}^x$ about $x = 0$. In this example, unlike the previous example, doing this directly would be significantly longer and more difficult.

$$\begin{aligned} x^4\mathbf{e}^{-3x^2} &= x^4 \sum_{n=0}^{\infty} \frac{\left(-3x^2\right)^n}{n!} \\ &= x^4 \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+4}}{n!} \end{aligned}$$

To this point we've only looked at Taylor Series about $x = 0$ (also known as Maclaurin Series) so let's take a look at a Taylor Series that isn't about $x = 0$. Also, we'll pick on the exponential function one more time since it makes some of the work easier. This will be the final Taylor Series for exponentials in this section.

Example 4

Find the Taylor Series for $f(x) = \mathbf{e}^{-x}$ about $x = -4$.

Solution

Finding a general formula for $f^{(n)}(-4)$ is fairly simple.

$$f^{(n)}(x) = (-1)^n\mathbf{e}^{-x} \qquad\qquad f^{(n)}(-4) = (-1)^n\mathbf{e}^{4}$$

The Taylor Series is then,

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n\mathbf{e}^4}{n!}(x+4)^n$$

Okay, we now need to work some examples that don't involve the exponential function since these will tend to require a little more work.

Example 5

Find the Taylor Series for $f(x) = \cos(x)$ about $x = 0$.

Solution

First, we'll need to take some derivatives of the function and evaluate them at $x = 0$.

$$\begin{array}{ll} f^{(0)}(x) = \cos(x) & f^{(0)}(0) = 1 \\ f^{(1)}(x) = -\sin(x) & f^{(1)}(0) = 0 \\ f^{(2)}(x) = -\cos(x) & f^{(2)}(0) = -1 \\ f^{(3)}(x) = \sin(x) & f^{(3)}(0) = 0 \\ f^{(4)}(x) = \cos(x) & f^{(4)}(0) = 1 \\ f^{(5)}(x) = -\sin(x) & f^{(5)}(0) = 0 \\ f^{(6)}(x) = -\cos(x) & f^{(6)}(0) = -1 \\ \vdots & \vdots \end{array}$$

In this example, unlike the previous ones, there is not an easy formula for either the general derivative or the evaluation of the derivative. However, there is a clear pattern to the evaluations. So, let's plug what we've got into the Taylor series and see what we get,

$$\begin{aligned} \cos(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \frac{f^{(5)}(0)}{5!} x^5 + \cdots \\ &= \underbrace{1}_{n=0} + \underbrace{0}_{n=1} - \underbrace{\frac{1}{2!} x^2}_{n=2} + \underbrace{0}_{n=3} + \underbrace{\frac{1}{4!} x^4}_{n=4} + \underbrace{0}_{n=5} - \underbrace{\frac{1}{6!} x^6}_{n=6} + \cdots \end{aligned}$$

So, we only pick up terms with even powers on the x's. This doesn't really help us to get a general formula for the Taylor Series. However, let's drop the zeroes and "renumber" the terms as follows to see what we can get.

$$\cos(x) = \underbrace{1}_{n=0} - \underbrace{\frac{1}{2!} x^2}_{n=1} + \underbrace{\frac{1}{4!} x^4}_{n=2} - \underbrace{\frac{1}{6!} x^6}_{n=3} + \cdots$$

By renumbering the terms as we did we can actually come up with a general formula for the Taylor Series and here it is,

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

This idea of renumbering the series terms as we did in the previous example isn't used all that often, but occasionally is very useful. There is one more series where we need to do it so let's take a look at that so we can get one more example down of renumbering series terms.

Example 6

Find the Taylor Series for $f(x) = \sin(x)$ about $x = 0$.

Solution

As with the last example we'll start off in the same manner.

$$\begin{aligned} f^{(0)}(x) &= \sin(x) & \qquad f^{(0)}(0) &= 0 \\ f^{(1)}(x) &= \cos(x) & f^{(1)}(0) &= 1 \\ f^{(2)}(x) &= -\sin(x) & f^{(2)}(0) &= 0 \\ f^{(3)}(x) &= -\cos(x) & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin(x) & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos(x) & f^{(5)}(0) &= 1 \\ f^{(6)}(x) &= -\sin(x) & f^{(6)}(0) &= 0 \\ &\vdots & &\vdots \end{aligned}$$

So, we get a similar pattern for this one. Let's plug the numbers into the Taylor Series.

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \end{aligned}$$

In this case we only get terms that have an odd exponent on x and as with the last problem once we ignore the zero terms there is a clear pattern and formula. So renumbering the terms as we did in the previous example we get the following Taylor Series.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

We really need to work another example or two in which $f(x)$ isn't about $x = 0$.

Example 7

Find the Taylor Series for $f(x) = \ln(x)$ about $x = 2$.

Solution

Here are the first few derivatives and the evaluations.

$$\begin{aligned}
f^{(0)}(x) &= \ln(x) & f^{(0)}(2) &= \ln 2 \\
f^{(1)}(x) &= \frac{1}{x} & f^{(1)}(2) &= \frac{1}{2} \\
f^{(2)}(x) &= -\frac{1}{x^2} & f^{(2)}(2) &= -\frac{1}{2^2} \\
f^{(3)}(x) &= \frac{2}{x^3} & f^{(3)}(2) &= \frac{2}{2^3} \\
f^{(4)}(x) &= -\frac{2(3)}{x^4} & f^{(4)}(2) &= -\frac{2(3)}{2^4} \\
f^{(5)}(x) &= \frac{2(3)(4)}{x^5} & f^{(5)}(2) &= \frac{2(3)(4)}{2^5} \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \frac{(-1)^{n+1}(n-1)!}{x^n} & f^{(n)}(2) &= \frac{(-1)^{n+1}(n-1)!}{2^n} \qquad n = 1, 2, 3, \ldots
\end{aligned}$$

Note that while we got a general formula here it doesn't work for $n = 0$. This will happen on occasion so don't worry about it when it does.

In order to plug this into the Taylor Series formula we'll need to strip out the $n = 0$ term first.

$$\begin{aligned}
\ln(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= f(2) + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!\ 2^n}(x-2)^n \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,2^n}(x-2)^n
\end{aligned}$$

Notice that we simplified the factorials in this case. You should always simplify them if there are more than one and it's possible to simplify them.

Also, do not get excited about the term sitting in front of the series. Sometimes we need to do that when we can't get a general formula that will hold for all values of n.

Example 8

Find the Taylor Series for $f(x) = \dfrac{1}{x^2}$ about $x = -1$.

Solution

Again, here are the derivatives and evaluations.

$$\begin{aligned}
f^{(0)}(x) &= \frac{1}{x^2} & f^{(0)}(-1) &= \frac{1}{(-1)^2} = 1 \\
f^{(1)}(x) &= -\frac{2}{x^3} & f^{(1)}(-1) &= -\frac{2}{(-1)^3} = 2 \\
f^{(2)}(x) &= \frac{2(3)}{x^4} & f^{(2)}(-1) &= \frac{2(3)}{(-1)^4} = 2(3) \\
f^{(3)}(x) &= -\frac{2(3)(4)}{x^5} & f^{(3)}(-1) &= -\frac{2(3)(4)}{(-1)^5} = 2(3)(4) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \frac{(-1)^n (n+1)!}{x^{n+2}} & f^{(n)}(-1) &= \frac{(-1)^n (n+1)!}{(-1)^{n+2}} = (n+1)!
\end{aligned}$$

Notice that all the negative signs will cancel out in the evaluation. Also, this formula will work for all n, unlike the previous example.

Here is the Taylor Series for this function.

$$\begin{aligned}
\frac{1}{x^2} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n \\
&= \sum_{n=0}^{\infty} \frac{(n+1)!}{n!}(x+1)^n \\
&= \sum_{n=0}^{\infty} (n+1)(x+1)^n
\end{aligned}$$

Now, let's work one of the easier examples in this section. The problem for most students is that it may not appear to be that easy (or maybe it will appear to be too easy) at first glance.

Example 9

Find the Taylor Series for $f(x) = x^3 - 10x^2 + 6$ about $x = 3$.

Solution

Here are the derivatives for this problem.

$$\begin{aligned} f^{(0)}(x) &= x^3 - 10x^2 + 6 & f^{(0)}(3) &= -57 \\ f^{(1)}(x) &= 3x^2 - 20x & f^{(1)}(3) &= -33 \\ f^{(2)}(x) &= 6x - 20 & f^{(2)}(3) &= -2 \\ f^{(3)}(x) &= 6 & f^{(3)}(3) &= 6 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= 0 & f^{(4)}(3) &= 0 \qquad n \geq 4 \end{aligned}$$

This Taylor series will terminate after $n = 3$. This will always happen when we are finding the Taylor Series of a polynomial. Here is the Taylor Series for this one.

$$\begin{aligned} x^3 - 10x^2 + 6 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n \\ &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + 0 \\ &= -57 - 33(x-3) - (x-3)^2 + (x-3)^3 \end{aligned}$$

When finding the Taylor Series of a polynomial we don't do any simplification of the right-hand side. We leave it like it is. In fact, if we were to multiply everything out we just get back to the original polynomial!

While it's not apparent that writing the Taylor Series for a polynomial is useful there are times where this needs to be done. The problem is that they are beyond the scope of this course and so aren't covered here. For example, there is one application to series in the field of Differential Equations where this needs to be done on occasion.

So, we've seen quite a few examples of Taylor Series to this point and in all of them we were able to find general formulas for the series. This won't always be the case. To see an example of one that doesn't have a general formula check out the last example in the next section.

Before leaving this section there are three important Taylor Series that we've derived in this section that we should summarize up in one place. In my class I will assume that you know these formulas from this point on.

Fact

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

10.17 Applications of Series

Now, that we know how to represent function as power series we can now talk about at least a couple of applications of series.

There are in fact many applications of series, unfortunately most of them are beyond the scope of this course. One application of power series (with the occasional use of Taylor Series) is in the field of Ordinary Differential Equations when finding Series Solutions to Differential Equations. If you are interested in seeing how that works you can check out that chapter of my Differential Equations notes.

Another application of series arises in the study of Partial Differential Equations. One of the more commonly used methods in that subject makes use of Fourier Series.

Many of the applications of series, especially those in the differential equations fields, rely on the fact that functions can be represented as a series. In these applications it is very difficult, if not impossible, to find the function itself. However, there are methods of determining the series representation for the unknown function.

While the differential equations applications are beyond the scope of this course there are some applications from a Calculus setting that we can look at.

Example 1

Determine a Taylor Series about $x = 0$ for the following integral.

$$\int \frac{\sin(x)}{x}\, dx$$

Solution

To do this we will first need to find a Taylor Series about $x = 0$ for the integrand. This however isn't terribly difficult. We already have a Taylor Series for sine about $x = 0$ so we'll just use that as follows,

$$\frac{\sin(x)}{x} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$$

We can now do the problem.

$$\begin{aligned}\int \frac{\sin(x)}{x}\, dx &= \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}\, dx \\ &= C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)\ (2n+1)!}\end{aligned}$$

So, while we can't integrate this function in terms of known functions we can come up with a series representation for the integral.

This idea of deriving a series representation for a function instead of trying to find the function itself is used quite often in several fields. In fact, there are some fields where this is one of the main ideas used and without this idea it would be very difficult to accomplish anything in those fields.

Another application of series isn't really an application of infinite series. It's more an application of partial sums. In fact, we've already seen this application in use once in this chapter. In the Estimating the Value of a Series we used a partial sum to estimate the value of a series. We can do the same thing with power series and series representations of functions. The main difference is that we will now be using the partial sum to approximate a function instead of a single value.

We will look at Taylor series for our examples, but we could just as easily use any series representation here. Recall that the **nth degree Taylor Polynomial of** $f(x)$ is given by,

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Let's take a look at example of this.

Example 2

For the function $f(x) = \cos(x)$ plot the function as well as $T_2(x)$, $T_4(x)$, and $T_8(x)$ on the same graph for the interval $[-4, 4]$.

Solution

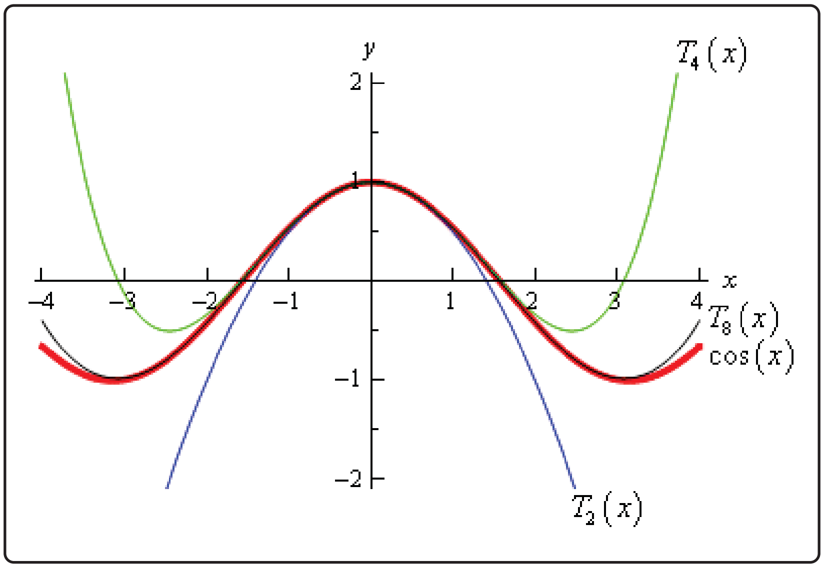
Here is the general formula for the Taylor polynomials for cosine.

$$T_{2n}(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{(2i)!}$$

The three Taylor polynomials that we've got are then,

$$\begin{aligned} T_2(x) &= 1 - \frac{x^2}{2} \\ T_4(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} \\ T_8(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} \end{aligned}$$

Here is the graph of these three Taylor polynomials as well as the graph of cosine.



As we can see from this graph as we increase the degree of the Taylor polynomial it starts to look more and more like the function itself. In fact, by the time we get to $T_8(x)$ the only difference is right at the ends. The higher the degree of the Taylor polynomial the better it approximates the function.

Also, the larger the interval the higher degree Taylor polynomial we need to get a good approximation for the whole interval.

Before moving on let's write down a couple more Taylor polynomials from the previous example. Notice that because the Taylor series for cosine doesn't contain any terms with odd powers on x we get the following Taylor polynomials.

$$T_3(x) = 1 - \frac{x^2}{2}$$
$$T_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$
$$T_9(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}$$

These are identical to those used in the example. Sometimes this will happen although that was not really the point of this. The point is to notice that the nth degree Taylor polynomial may actually have a degree that is less than n. It will never be more than n, but it can be less than n.

The final example in this section really isn't an application of series and probably belonged in the previous section. However, the previous section was getting too long so the example is in this section. This is an example of how to multiply series together and while this isn't an application of series it is something that does have to be done on occasion in the applications. So, in that sense it does belong in this section.

Example 3

Find the first three non-zero terms in the Taylor Series for $f(x) = \mathbf{e}^x \cos(x)$ about $x = 0$.

Solution

Before we start let's acknowledge that the easiest way to do this problem is to simply compute the first 3-4 derivatives, evaluate them at $x = 0$, plug into the formula and we'd be done. However, as we noted prior to this example we want to use this example to illustrate how we multiply series.

We will make use of the fact that we've got Taylor Series for each of these so we can use them in this problem.

$$\mathbf{e}^x \cos(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right)$$

We're not going to completely multiply out these series. We're going to do enough of the multiplication to get an answer. The problem statement says that we want the first three *non-zero* terms. That non-zero bit is important as it is possible that some of the terms will be zero. If none of the terms are zero this would mean that the first three non-zero terms would be the constant term, x term, and x^2 term. However, because some might be zero let's assume that if we get all the terms up through x^4 we'll have enough to get the answer. If we've assumed wrong it will be very easy to fix so don't worry about that.

Now, let's write down the first few terms of each series and we'll stop at the x^4 term in each.

$$\mathbf{e}^x \cos(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right)$$

Note that we do need to acknowledge that these series don't stop. That's the purpose of the "$+\cdots$" at the end of each. Just for a second however, let's suppose that each of these did stop and ask ourselves how we would multiply each out. If this were the case we would take every term in the second and multiply by every term in the first. In other words, we would first multiply every term in the second series by 1, then every term in the second series by x, then by x^2 *etc.*

By stopping each series at x^4 we have now guaranteed that we'll get all terms that have an exponent of 4 or less. Do you see why?

Each of the terms that we neglected to write down have an exponent of at least 5 and so multiplying by 1 or any power of x will result in a term with an exponent that is at a minimum

5. Therefore, none of the neglected terms will contribute terms with an exponent of 4 or less and so weren't needed.

So, let's start the multiplication process.

$$\begin{aligned}
\mathbf{e}^{x}\cos(x) &= \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}+\cdots\right)\\
&= \underbrace{1-\frac{x^2}{2}+\frac{x^4}{24}+\cdots}_{\text{Second Series}\times 1}+\underbrace{x-\frac{x^3}{2}+\frac{x^5}{24}+\cdots}_{\text{Second Series}\times x}+\underbrace{\frac{x^2}{2}-\frac{x^4}{4}+\frac{x^6}{48}+\cdots}_{\text{Second Series}\times {x^2}/{2}}\\
&\qquad +\underbrace{\frac{x^3}{6}-\frac{x^5}{12}+\frac{x^7}{144}+\cdots}_{\text{Second Series}\times {x^3}/{6}}+\underbrace{\frac{x^4}{24}-\frac{x^6}{48}+\frac{x^8}{576}+\cdots}_{\text{Second Series}\times {x^4}/{24}}+\cdots
\end{aligned}$$

Now, collect like terms ignoring everything with an exponent of 5 or more since we won't have all those terms and don't want them either. Doing this gives,

$$\begin{aligned}
\mathbf{e}^{x}\cos(x) &= 1+x+\left(-\frac{1}{2}+\frac{1}{2}\right)x^2+\left(-\frac{1}{2}+\frac{1}{6}\right)x^3+\left(\frac{1}{24}-\frac{1}{4}+\frac{1}{24}\right)x^4+\cdots\\
&= 1+x-\frac{x^3}{3}-\frac{x^4}{6}+\cdots
\end{aligned}$$

There we go. It looks like we over guessed and ended up with four non-zero terms, but that's okay. If we had under guessed and it turned out that we needed terms with x^5 in them all we would need to do at this point is go back and add in those terms to the original series and do a couple quick multiplications. In other words, there is no reason to completely redo all the work.

10.18 Binomial Series

In this final section of this chapter we are going to look at another series representation for a function. Before we do this let's first recall the following theorem.

Binomial Theorem

If n is any positive integer then,

$$\begin{aligned}(a+b)^n &= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n\end{aligned}$$

where,

$$\begin{aligned}\binom{n}{i} &= \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!} \qquad i = 1, 2, 3, \ldots n \\ \binom{n}{0} &= 1\end{aligned}$$

This is useful for expanding $(a+b)^n$ for large n when straight forward multiplication wouldn't be easy to do. Let's take a quick look at an example.

Example 1

Use the Binomial Theorem to expand $(2x-3)^4$

Solution

There really isn't much to do other than plugging into the theorem.

$$\begin{aligned}(2x-3)^4 &= \sum_{i=0}^{4} \binom{4}{i} (2x)^{4-i} (-3)^i \\ &= \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(-3) + \binom{4}{2}(2x)^2(-3)^2 + \binom{4}{3}(2x)(-3)^3 + \binom{4}{4}(-3)^4 \\ &= (2x)^4 + 4(2x)^3(-3) + \frac{4(3)}{2}(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4 \\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81\end{aligned}$$

Now, the Binomial Theorem required that n be a positive integer. There is an extension to this however that allows for any number at all.

Binomial Series

If k is any number and $|x| < 1$ then,

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots\end{aligned}$$

where,

$$\begin{aligned}\binom{k}{n} &= \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \qquad n = 1, 2, 3, \ldots \\ \binom{k}{0} &= 1\end{aligned}$$

So, similar to the binomial theorem except that it's an infinite series and we must have $|x| < 1$ in order to get convergence.

Let's check out an example of this.

Example 2

Write down the first four terms in the binomial series for $\sqrt{9-x}$

Solution

So, in this case $k = \frac{1}{2}$ and we'll need to rewrite the term a little to put it into the form required.

$$\sqrt{9-x} = 3\left(1 - \frac{x}{9}\right)^{\frac{1}{2}} = 3\left(1 + \left(-\frac{x}{9}\right)\right)^{\frac{1}{2}}$$

The first four terms in the binomial series is then,

$$\begin{aligned}\sqrt{9-x} &= 3\left(1 + \left(-\frac{x}{9}\right)\right)^{\frac{1}{2}} \\ &= 3\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(-\frac{x}{9}\right)^n \\ &= 3\left[1 + \left(\frac{1}{2}\right)\left(-\frac{x}{9}\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(-\frac{x}{9}\right)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(-\frac{x}{9}\right)^3 + \cdots\right] \\ &= 3 - \frac{x}{6} - \frac{x^2}{216} - \frac{x^3}{3888} - \cdots\end{aligned}$$