

11 Vectors

Once again we are completely changing topics from the last chapter. We are going to do a (very) brief introduction to vectors. We'll look at basic notation and concepts involving vectors as well as arithmetic involving vectors. We'll also look at the dot product and cross product of vectors as well as a couple of quick applications of the dot and cross product.

Once we get into the multi-variable Calculus (*i.e.* the topics usually taught in Calculus III) we'll run into vectors on a semi regular basis and so we'll need to be familiar with them and the common notation, concepts and arithmetic involving vectors.

11.1 Basic Concepts

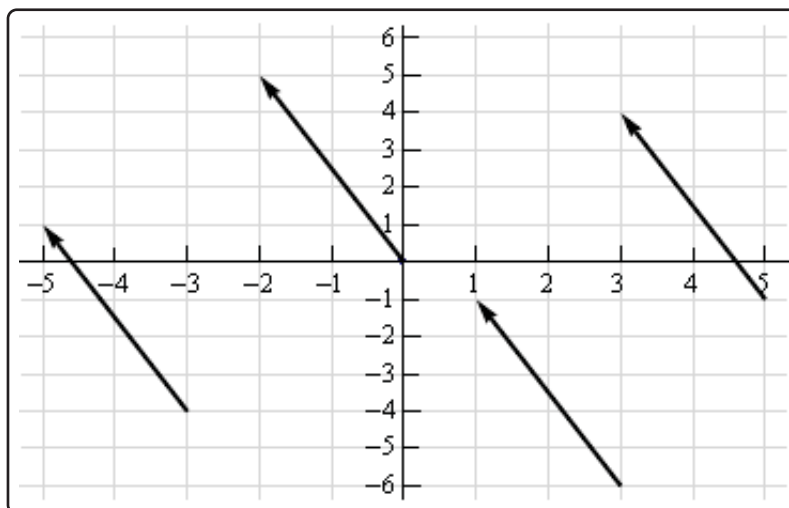
Let's start this section off with a quick discussion on what vectors are used for. Vectors are used to represent quantities that have both a magnitude and a direction. Good examples of quantities that can be represented by vectors are force and velocity. Both of these have a direction and a magnitude.

Let's consider force for a second. A force of say 5 Newtons that is applied in a particular direction can be applied at any point in space. In other words, the point where we apply the force does not change the force itself. Forces are independent of the point of application. To define a force all we need to know is the magnitude of the force and the direction that the force is applied in.

The same idea holds more generally with vectors. Vectors only impart magnitude and direction. They don't impart any information about where the quantity is applied. This is an important idea to always remember in the study of vectors.

In a graphical sense vectors are represented by directed line segments. The length of the line segment is the magnitude of the vector and the direction of the line segment is the direction of the vector. However, because vectors don't impart any information about where the quantity is applied any directed line segment with the same length and direction will represent the same vector.

Consider the sketch below.



Each of the directed line segments in the sketch represents the same vector. In each case the vector starts at a specific point then moves 2 units to the left and 5 units up. The notation that we'll use for this vector is,

$$\vec{v} = \langle -2, 5 \rangle$$

and each of the directed line segments in the sketch are called **representations** of the vector.

Be careful to distinguish vector notation, $\langle -2, 5 \rangle$, from the notation we use to represent coordinates of points, $(-2, 5)$. The vector denotes a magnitude and a direction of a quantity while the point denotes a location in space. So don't mix the notations up!

A representation of the vector $\vec{v} = \langle a_1, a_2 \rangle$ in two dimensional space is any directed line segment, \overrightarrow{AB} , from the point $A = (x, y)$ to the point $B = (x + a_1, y + a_2)$. Likewise a representation of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ in three dimensional space is any directed line segment, \overrightarrow{AB} , from the point $A = (x, y, z)$ to the point $B = (x + a_1, y + a_2, z + a_3)$.

Note that there is very little difference between the two dimensional and three dimensional formulas above. To get from the three dimensional formula to the two dimensional formula all we did is take out the third component/coordinate. Because of this most of the formulas here are given only in their three dimensional version. If we need them in their two dimensional form we can easily modify the three dimensional form.

There is one representation of a vector that is special in some way. The representation of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ that starts at the point $A = (0, 0, 0)$ and ends at the point $B = (a_1, a_2, a_3)$ is called the **position vector** of the point (a_1, a_2, a_3) . So, when we talk about position vectors we are specifying the initial and final point of the vector.

Position vectors are useful if we ever need to represent a point as a vector. As we'll see there are times in which we definitely are going to want to represent points as vectors. In fact, we're going to run into topics that can only be done if we represent points as vectors.

Next, we need to discuss briefly how to generate a vector given the initial and final points of the representation. Given the two points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ the vector with the representation \overrightarrow{AB} is,

$$\vec{v} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

Note that we have to be very careful with direction here. The vector above is the vector that starts at A and ends at B . The vector that starts at B and ends at A , *i.e.* with representation \overrightarrow{BA} is,

$$\vec{w} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

These two vectors are different and so we do need to always pay attention to what point is the starting point and what point is the ending point. When determining the vector between two points we always subtract the initial point from the terminal point.

Example 1

Give the vector for each of the following.

- (a) The vector from $(2, -7, 0)$ to $(1, -3, -5)$.
- (b) The vector from $(1, -3, -5)$ to $(2, -7, 0)$.
- (c) The position vector for $(-90, 4)$

Solution

- (a) The vector from $(2, -7, 0)$ to $(1, -3, -5)$.

Remember that to construct this vector we subtract coordinates of the starting point from the ending point.

$$\langle 1 - 2, -3 - (-7), -5 - 0 \rangle = \langle -1, 4, -5 \rangle$$

- (b) The vector from $(1, -3, -5)$ to $(2, -7, 0)$.

Same thing here.

$$\langle 2 - 1, -7 - (-3), 0 - (-5) \rangle = \langle 1, -4, 5 \rangle$$

Notice that the only difference between the first two is the signs are all opposite. This difference is important as it is this difference that tells us that the two vectors point in opposite directions.

- (c) The position vector for $(-90, 4)$

Not much to this one other than acknowledging that the position vector of a point is nothing more than a vector with the point's coordinates as its components.

$$\langle -90, 4 \rangle$$

We now need to start discussing some of the basic concepts that we will run into on occasion.

Magnitude

The **magnitude**, or **length**, of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ is given by,

$$\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example 2

Determine the magnitude of each of the following vectors.

(a) $\vec{a} = \langle 3, -5, 10 \rangle$

(b) $\vec{u} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$

(c) $\vec{w} = \langle 0, 0 \rangle$

(d) $\vec{i} = \langle 1, 0, 0 \rangle$

Solution

There isn't too much to these other than plug into the formula.

(a) $\|\vec{a}\| = \sqrt{9 + 25 + 100} = \sqrt{134}$

(b) $\|\vec{u}\| = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{1} = 1$

(c) $\|\vec{w}\| = \sqrt{0 + 0} = 0$

(d) $\|\vec{i}\| = \sqrt{1 + 0 + 0} = 1$

We also have the following fact about the magnitude.

$$\text{If } \|\vec{a}\| = 0 \text{ then } \vec{a} = \vec{0}$$

This should make sense. Because we square all the components the only way we can get zero out of the formula was for the components to be zero in the first place.

Unit Vector

Any vector with magnitude of 1, *i.e.* $\|\vec{u}\| = 1$, is called a **unit vector**.

Example 3

Which of the vectors from Example 2 are unit vectors?

Solution

Both the second and fourth vectors had a length of 1 and so they are the only unit vectors from the first example.

Zero Vector

The vector $\vec{w} = \langle 0, 0 \rangle$ that we saw in the first example is called a zero vector since its components are all zero. Zero vectors are often denoted by $\vec{0}$. Be careful to distinguish 0 (the number) from $\vec{0}$ (the vector). The number 0 denotes the origin in space, while the vector $\vec{0}$ denotes a vector that has no magnitude or direction.

Standard Basis Vectors

The fourth vector from the second example, $\vec{i} = \langle 1, 0, 0 \rangle$, is called a **standard basis vector**. In three dimensional space there are three standard basis vectors,

$$\vec{i} = \langle 1, 0, 0 \rangle \quad \vec{j} = \langle 0, 1, 0 \rangle \quad \vec{k} = \langle 0, 0, 1 \rangle$$

In two dimensional space there are two standard basis vectors,

$$\vec{i} = \langle 1, 0 \rangle \quad \vec{j} = \langle 0, 1 \rangle$$

Note that standard basis vectors are also unit vectors.

Warning

We are pretty much done with this section however, before proceeding to the next section we should point out that vectors are not restricted to two dimensional or three dimensional space. Vectors can exist in general n-dimensional space. The general notation for a n-dimensional vector is,

$$\vec{v} = \langle a_1, a_2, a_3, \dots, a_n \rangle$$

and each of the a_i 's are called **components** of the vector.

Because we will be working almost exclusively with two and three dimensional vectors in this course most of the formulas will be given for the two and/or three dimensional cases. However, most of the concepts/formulas will work with general vectors and the formulas are easily (and naturally) modified for general n-dimensional vectors. Also, because it is easier to visualize things in two dimensions most of the figures related to vectors will be two dimensional figures.

So, we need to be careful to not get too locked into the two or three dimensional cases from our discussions in this chapter. We will be working in these dimensions either because it's easier to visualize the situation or because physical restrictions of the problems will enforce a dimension upon us.

11.2 Vector Arithmetic

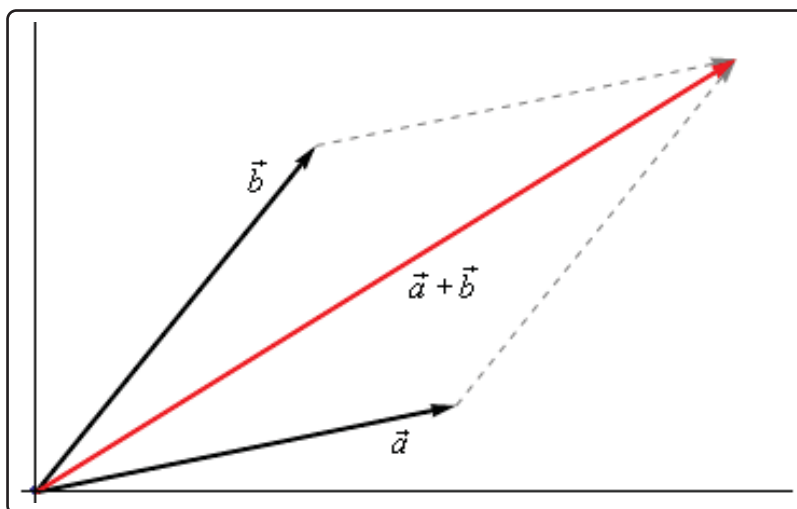
In this section we need to have a brief discussion of vector arithmetic.

We'll start with **addition** of two vectors. So, given the vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the addition of the two vectors is given by the following formula.

Vector Addition

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

The following figure gives the geometric interpretation of the addition of two vectors.



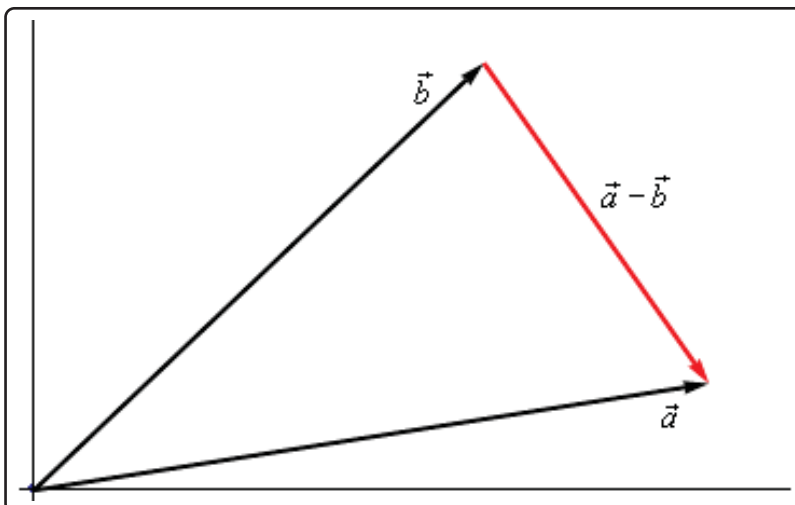
This is sometimes called the **parallelogram law** or **triangle law**.

Computationally, **subtraction** is very similar. Given the vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the difference of the two vectors is given by,

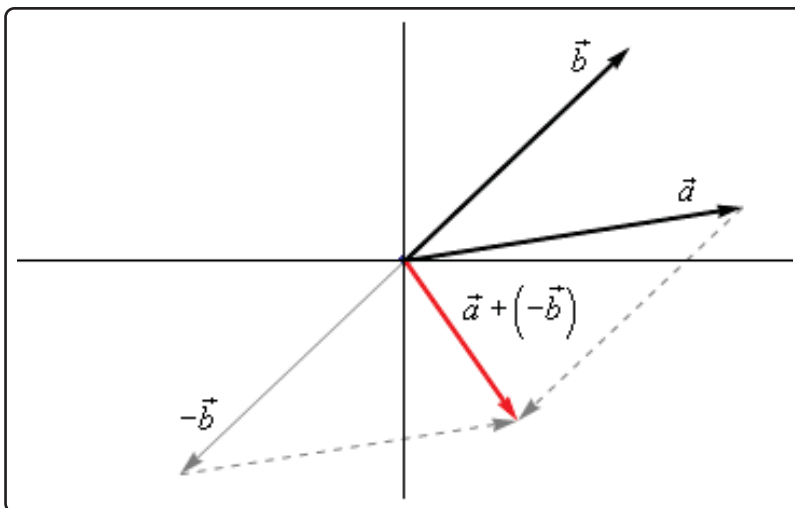
Vector Subtraction

$$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

Here is the geometric interpretation of the difference of two vectors.



It is a little harder to see this geometric interpretation. To help see this let's instead think of subtraction as the addition of \vec{a} and $-\vec{b}$. First, as we'll see in a bit $-\vec{b}$ is the same vector as \vec{b} with opposite signs on all the components. In other words, $-\vec{b}$ goes in the opposite direction as \vec{b} . Here is the vector set up for $\vec{a} + (-\vec{b})$.



As we can see from this figure we can move the vector representing $\vec{a} + (-\vec{b})$ to the position we've got in the first figure showing the difference of the two vectors.

Note that we can't add or subtract two vectors unless they have the same number of components. If they don't have the same number of components then addition and subtraction can't be done.

The next arithmetic operation that we want to look at is **scalar multiplication**. Given the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and any number c the scalar multiplication is,

Scalar Multiplication

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$$

So, we multiply all the components by the constant c . To see the geometric interpretation of scalar multiplication let's take a look at an example.

Example 1

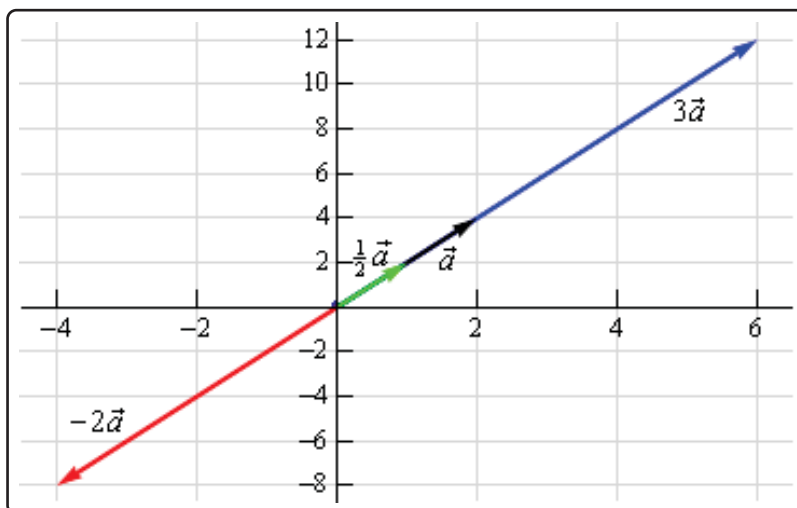
For the vector $\vec{a} = \langle 2, 4 \rangle$ compute $3\vec{a}$, $\frac{1}{2}\vec{a}$ and $-2\vec{a}$. Graph all four vectors on the same axis system.

Solution

Here are the three scalar multiplications.

$$3\vec{a} = \langle 6, 12 \rangle \quad \frac{1}{2}\vec{a} = \langle 1, 2 \rangle \quad -2\vec{a} = \langle -4, -8 \rangle$$

Here is the graph for each of these vectors.



In the previous example we can see that if c is positive all scalar multiplication will do is stretch (if $c > 1$) or shrink (if $c < 1$) the original vector, but it won't change the direction. Likewise, if c is negative scalar multiplication will switch the direction so that the vector will point in exactly the opposite direction and it will again stretch or shrink the magnitude of the vector depending upon the size of c .

There are several nice applications of scalar multiplication that we should now take a look at.

The first is parallel vectors. This is a concept that we will see quite a bit over the next couple of sections. Two vectors are parallel if they have the same direction or are in exactly opposite directions. Now, recall again the geometric interpretation of scalar multiplication. When we performed scalar multiplication we generated new vectors that were parallel to the original vectors (and each other for that matter).

Parallel Vectors

So, let's suppose that \vec{a} and \vec{b} are parallel vectors. If they are parallel then there must be a number c so that,

$$\vec{a} = c\vec{b}$$

So, two vectors are parallel if one is a scalar multiple of the other.

Example 2

Determine if the sets of vectors are parallel or not.

(a) $\vec{a} = \langle 2, -4, 1 \rangle$, $\vec{b} = \langle -6, 12, -3 \rangle$

(b) $\vec{a} = \langle 4, 10 \rangle$, $\vec{b} = \langle 2, -9 \rangle$

Solution

(a) $\vec{a} = \langle 2, -4, 1 \rangle$, $\vec{b} = \langle -6, 12, -3 \rangle$

These two vectors are parallel since $\vec{b} = -3\vec{a}$

(b) $\vec{a} = \langle 4, 10 \rangle$, $\vec{b} = \langle 2, -9 \rangle$

These two vectors aren't parallel. This can be seen by noticing that $4\left(\frac{1}{2}\right) = 2$ and yet $10\left(\frac{1}{2}\right) = 5 \neq -9$. In other words, we can't make \vec{a} be a scalar multiple of \vec{b} .

The next application is best seen in an example.

Example 3

Find a unit vector that points in the same direction as $\vec{w} = \langle -5, 2, 1 \rangle$.

Solution

Okay, what we're asking for is a new parallel vector (points in the same direction) that happens to be a unit vector. We can do this with a scalar multiplication since all scalar multiplication does is change the length of the original vector (along with possibly flipping the direction to the opposite direction).

Here's what we'll do. First let's determine the magnitude of \vec{w} .

$$\|\vec{w}\| = \sqrt{25 + 4 + 1} = \sqrt{30}$$

Now, let's form the following new vector,

$$\vec{u} = \frac{1}{\|\vec{w}\|} \vec{w} = \frac{1}{\sqrt{30}} \langle -5, 2, 1 \rangle = \left\langle -\frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}} \right\rangle$$

The claim is that this is a unit vector. That's easy enough to check

$$\|\vec{u}\| = \sqrt{\frac{25}{30} + \frac{4}{30} + \frac{1}{30}} = \sqrt{\frac{30}{30}} = 1$$

This vector also points in the same direction as \vec{w} since it is only a scalar multiple of \vec{w} and we used a positive multiple.

So, in general, given a vector \vec{w} , $\vec{u} = \frac{\vec{w}}{\|\vec{w}\|}$ will be a unit vector that points in the same direction as \vec{w} .

Standard Basis Vectors Revisited

In the previous section we introduced the idea of standard basis vectors without really discussing why they were important. We can now do that. Let's start with the vector

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

We can use the addition of vectors to break this up as follows,

$$\begin{aligned} \vec{a} &= \langle a_1, a_2, a_3 \rangle \\ &= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \end{aligned}$$

Using scalar multiplication we can further rewrite the vector as,

$$\begin{aligned} \vec{a} &= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \end{aligned}$$

Finally, notice that these three new vectors are simply the three standard basis vectors for three dimensional space.

$$\langle a_1, a_2, a_3 \rangle = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

So, we can take any vector and write it in terms of the standard basis vectors. From this point on we will use the two notations interchangeably so make sure that you can deal with both notations.

Example 4

If $\vec{a} = \langle 3, -9, 1 \rangle$ and $\vec{w} = -\vec{i} + 8\vec{k}$ compute $2\vec{a} - 3\vec{w}$.

Solution

In order to do the problem we'll convert to one notation and then perform the indicated operations.

$$\begin{aligned} 2\vec{a} - 3\vec{w} &= 2 \langle 3, -9, 1 \rangle - 3 \langle -1, 0, 8 \rangle \\ &= \langle 6, -18, 2 \rangle - \langle -3, 0, 24 \rangle \\ &= \langle 9, -18, -22 \rangle \end{aligned}$$

We will leave this section with some basic properties of vector arithmetic.

Properties

If \vec{v} , \vec{w} and \vec{u} are vectors (each with the same number of components) and a and b are two numbers then we have the following properties.

$$\vec{v} + \vec{w} = \vec{w} + \vec{v} \qquad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

$$\vec{v} + \vec{0} = \vec{v} \qquad 1\vec{v} = \vec{v}$$

$$a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w} \qquad (a + b)\vec{v} = a\vec{v} + b\vec{v}$$

The proofs of these are pretty much just “computation” proofs so we'll prove one of them and leave the others to you to prove.

Proof of $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$

We'll start with the two vectors, $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$ and $\vec{w} = \langle w_1, w_2, \dots, w_n \rangle$ and yes we did mean for these to each have n components. The theorem works for general vectors so we may as well do the proof for general vectors.

Now, as noted above this is pretty much just a “computational” proof. What that means is that we'll compute the left side and then do some basic arithmetic on the result to show that we can make the left side look like the right side. Here is the work.

$$\begin{aligned} a(\vec{v} + \vec{w}) &= a(\langle v_1, v_2, \dots, v_n \rangle + \langle w_1, w_2, \dots, w_n \rangle) \\ &= a\langle v_1 + w_1, v_2 + w_2, \dots, v_n + w_n \rangle \\ &= \langle a(v_1 + w_1), a(v_2 + w_2), \dots, a(v_n + w_n) \rangle \\ &= \langle av_1 + aw_1, av_2 + aw_2, \dots, av_n + aw_n \rangle \\ &= \langle av_1, av_2, \dots, av_n \rangle + \langle aw_1, aw_2, \dots, aw_n \rangle \\ &= a\langle v_1, v_2, \dots, v_n \rangle + a\langle w_1, w_2, \dots, w_n \rangle = a\vec{v} + a\vec{w} \end{aligned}$$

11.3 Dot Product

The next topic for discussion is that of the dot product. Let's jump right into the definition of the dot product. Given the two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the dot product is,

Dot Product

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \quad (11.1)$$

Sometimes the dot product is called the **scalar product**. The dot product is also an example of an **inner product** and so on occasion you may hear it called an inner product.

Example 1

Compute the dot product for each of the following.

(a) $\vec{v} = 5\vec{i} - 8\vec{j}$, $\vec{w} = \vec{i} + 2\vec{j}$

(b) $\vec{a} = \langle 0, 3, -7 \rangle$, $\vec{b} = \langle 2, 3, 1 \rangle$

Solution

Not much to do with these other than use the formula.

(a) $\vec{v} \cdot \vec{w} = 5 - 16 = -11$

(b) $\vec{a} \cdot \vec{b} = 0 + 9 - 7 = 2$

Here are some properties of the dot product.

Properties

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$(c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w})$$

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

$$\vec{v} \cdot \vec{0} = 0$$

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$$

$$\text{If } \vec{v} \cdot \vec{v} = 0 \text{ then } \vec{v} = \vec{0}$$

The proofs of these properties are mostly “computational” proofs and so we’re only going to do a couple of them and leave the rest to you to prove.

Proof of $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

We’ll start with the three vectors, $\vec{u} = \langle u_1, u_2, \dots, u_n \rangle$, $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$ and $\vec{w} = \langle w_1, w_2, \dots, w_n \rangle$ and yes we did mean for these to each have n components. The theorem works for general vectors so we may as well do the proof for general vectors.

Now, as noted above this is pretty much just a “computational” proof. What that means is that we’ll compute the left side and then do some basic arithmetic on the result to show that we can make the left side look like the right side. Here is the work.

$$\begin{aligned} \vec{u} \cdot (\vec{v} + \vec{w}) &= \langle u_1, u_2, \dots, u_n \rangle \cdot (\langle v_1, v_2, \dots, v_n \rangle + \langle w_1, w_2, \dots, w_n \rangle) \\ &= \langle u_1, u_2, \dots, u_n \rangle \cdot \langle v_1 + w_1, v_2 + w_2, \dots, v_n + w_n \rangle \\ &= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \dots + u_n(v_n + w_n) \\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \dots + u_nv_n + u_nw_n \\ &= (u_1v_1 + u_2v_2 + \dots + u_nv_n) + (u_1w_1 + u_2w_2 + \dots + u_nw_n) \\ &= \langle u_1, u_2, \dots, u_n \rangle \cdot \langle v_1, v_2, \dots, v_n \rangle + \langle u_1, u_2, \dots, u_n \rangle \cdot \langle w_1, w_2, \dots, w_n \rangle \\ &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \end{aligned}$$

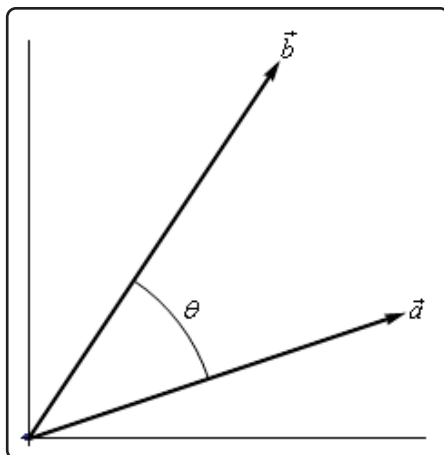
Proof of : If $\vec{v} \cdot \vec{v} = 0$ then $\vec{v} = \vec{0}$

This is a pretty simple proof. Let’s start with $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$ and compute the dot product.

$$\begin{aligned} \vec{v} \cdot \vec{v} &= \langle v_1, v_2, \dots, v_n \rangle \cdot \langle v_1, v_2, \dots, v_n \rangle \\ &= v_1^2 + v_2^2 + \dots + v_n^2 \\ &= 0 \end{aligned}$$

Now, since we know $v_i^2 \geq 0$ for all i then the only way for this sum to be zero is to in fact have $v_i^2 = 0$. This in turn however means that we must have $v_i = 0$ and so we must have had $\vec{v} = \vec{0}$.

There is also a nice geometric interpretation to the dot product. First suppose that θ is the angle between \vec{a} and \vec{b} such that $0 \leq \theta \leq \pi$ as shown in the image below.



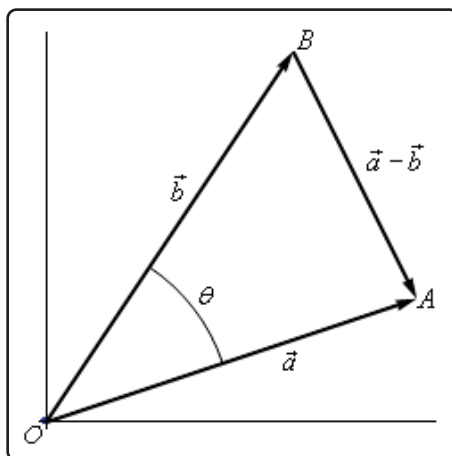
We can then have the following theorem.

Theorem

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta) \quad (11.2)$$

Proof

Let's give a modified version of the sketch above.



The three vectors above form the triangle AOB and note that the length of each side is nothing more than the magnitude of the vector forming that side.

The Law of Cosines tells us that,

$$\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\| \|\vec{b}\| \cos(\theta)$$

Also using the properties of dot products we can write the left side as,

$$\begin{aligned}\|\vec{a} - \vec{b}\|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ &= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \\ &= \|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2\end{aligned}$$

Our original equation is then,

$$\begin{aligned}\|\vec{a} - \vec{b}\|^2 &= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\| \|\vec{b}\| \cos(\theta) \\ \|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2 &= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\| \|\vec{b}\| \cos(\theta) \\ -2\vec{a} \cdot \vec{b} &= -2\|\vec{a}\| \|\vec{b}\| \cos(\theta) \\ \vec{a} \cdot \vec{b} &= \|\vec{a}\| \|\vec{b}\| \cos(\theta)\end{aligned}$$

The formula from this theorem is often used not to compute a dot product but instead to find the angle between two vectors. Note as well that while the sketch of the two vectors in the proof is for two dimensional vectors the theorem is valid for vectors of any dimension (as long as they have the same dimension of course).

Let's see an example of this.

Example 2

Determine the angle between $\vec{a} = \langle 3, -4, -1 \rangle$ and $\vec{b} = \langle 0, 5, 2 \rangle$.

Solution

We will need the dot product as well as the magnitudes of each vector.

$$\vec{a} \cdot \vec{b} = -22 \quad \|\vec{a}\| = \sqrt{26} \quad \|\vec{b}\| = \sqrt{29}$$

The angle is then,

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} = \frac{-22}{\sqrt{26}\sqrt{29}} = -0.8011927$$

$$\theta = \cos^{-1}(-0.8011927) = 2.5 \text{ radians} = 143.24 \text{ degrees}$$

The dot product gives us a very nice method for determining if two vectors are perpendicular and it will give another method for determining when two vectors are parallel. Note as well that often we will use the term **orthogonal** in place of perpendicular.

Orthogonal/Perpendicular Vectors

Now, if two vectors are orthogonal then we know that the angle between them is 90 degrees. From [Equation 11.2](#) this tells us that if two vectors are orthogonal then,

$$\vec{a} \cdot \vec{b} = 0$$

Likewise, if two vectors are parallel then the angle between them is either 0 degrees (pointing in the same direction) or 180 degrees (pointing in the opposite direction). Once again using [Equation 11.2](#) this would mean that one of the following would have to be true.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| (\theta = 0^\circ) \quad \text{OR} \quad \vec{a} \cdot \vec{b} = -\|\vec{a}\| \|\vec{b}\| (\theta = 180^\circ)$$

Example 3

Determine if the following vectors are parallel, orthogonal, or neither.

(a) $\vec{a} = \langle 6, -2, -1 \rangle$, $\vec{b} = \langle 2, 5, 2 \rangle$

(b) $\vec{u} = 2\vec{i} - \vec{j}$, $\vec{v} = -\frac{1}{2}\vec{i} + \frac{1}{4}\vec{j}$

Solution

(a) $\vec{a} = \langle 6, -2, -1 \rangle$, $\vec{b} = \langle 2, 5, 2 \rangle$

First get the dot product to see if they are orthogonal.

$$\vec{a} \cdot \vec{b} = 12 - 10 - 2 = 0$$

The two vectors are orthogonal.

(b) $\vec{u} = 2\vec{i} - \vec{j}$, $\vec{v} = -\frac{1}{2}\vec{i} + \frac{1}{4}\vec{j}$

Again, let's get the dot product first.

$$\vec{u} \cdot \vec{v} = -1 - \frac{1}{4} = -\frac{5}{4}$$

So, they aren't orthogonal. Let's get the magnitudes and see if they are parallel.

$$\|\vec{u}\| = \sqrt{5} \quad \|\vec{v}\| = \sqrt{\frac{5}{16}} = \frac{\sqrt{5}}{4}$$

Now, notice that,

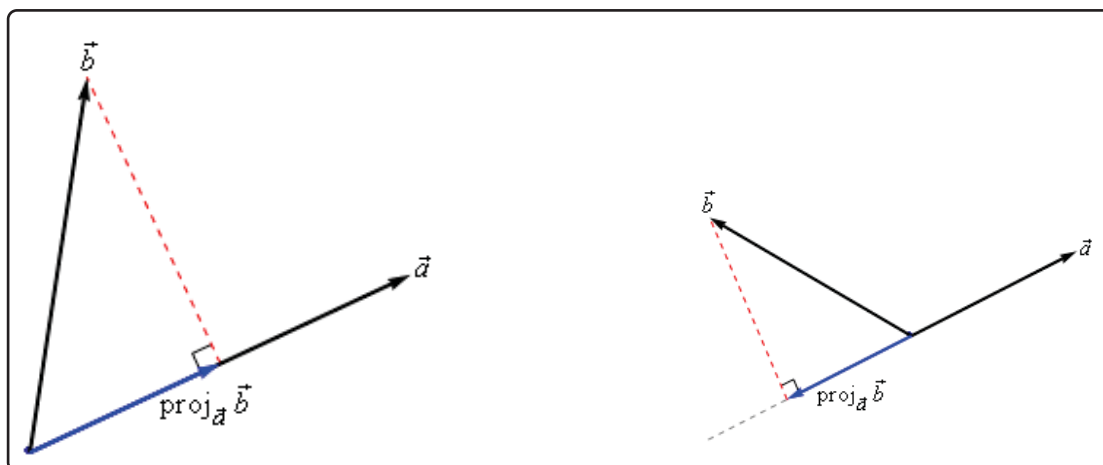
$$\vec{u} \cdot \vec{v} = -\frac{5}{4} = -\sqrt{5} \left(\frac{\sqrt{5}}{4} \right) = -\|\vec{u}\| \|\vec{v}\|$$

So, the two vectors are parallel.

There are several nice applications of the dot product as well that we should look at.

Projections

The best way to understand projections is to see a couple of sketches. So, given two vectors \vec{a} and \vec{b} we want to determine the projection of \vec{b} onto \vec{a} . The projection is denoted by $\text{proj}_{\vec{a}}\vec{b}$. Here are a couple of sketches illustrating the projection.



So, to get the projection of \vec{b} onto \vec{a} we drop straight down from the end of \vec{b} until we hit (and form a right angle) with the line that is parallel to \vec{a} . The projection is then the vector that is parallel to \vec{a} , starts at the same point both of the original vectors started at and ends where the dashed line hits the line parallel to \vec{a} .

There is a nice formula for finding the projection of \vec{b} onto \vec{a} . Here it is,

Vector Projection of \vec{b} onto \vec{a}

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a}$$

Note that we also need to be very careful with notation here. The projection of \vec{a} onto \vec{b} is given by

Vector Projection of \vec{a} onto \vec{b}

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b}$$

We can see that this will be a totally different vector. This vector is parallel to \vec{b} , while $\text{proj}_{\vec{a}}\vec{b}$ is parallel to \vec{a} . So, be careful with notation and make sure you are finding the correct projection.

Here's an example.

Example 4

Determine the projection of $\vec{b} = \langle 2, 1, -1 \rangle$ onto $\vec{a} = \langle 1, 0, -2 \rangle$.

Solution

We need the dot product and the magnitude of \vec{a} .

$$\vec{a} \cdot \vec{b} = 4 \quad \|\vec{a}\|^2 = 5$$

The projection is then,

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a} = \frac{4}{5} \langle 1, 0, -2 \rangle = \left\langle \frac{4}{5}, 0, -\frac{8}{5} \right\rangle$$

For comparison purposes let's do it the other way around as well.

Example 5

Determine the projection of $\vec{a} = \langle 1, 0, -2 \rangle$ onto $\vec{b} = \langle 2, 1, -1 \rangle$.

Solution

We need the dot product and the magnitude of \vec{b} .

$$\vec{a} \cdot \vec{b} = 4 \quad \|\vec{b}\|^2 = 6$$

The projection is then,

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\vec{b} = \frac{4}{6}\langle 2, 1, -1 \rangle = \left\langle \frac{4}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$$

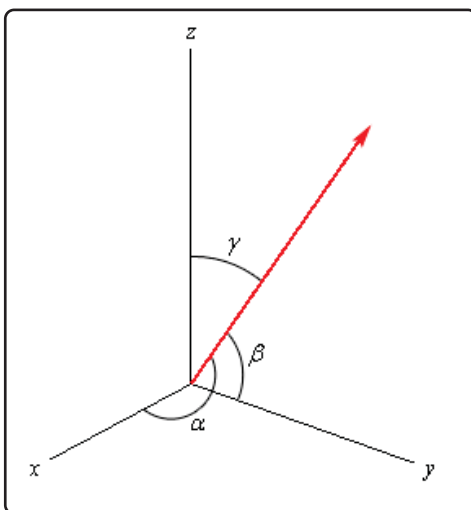
As we can see from the previous two examples the two projections are different so be careful.

Direction Cosines

This application of the dot product requires that we be in three dimensional space unlike all the other applications we've looked at to this point.

Let's start with a vector, \vec{a} , in three dimensional space. This vector will form angles with the x -axis (α), the y -axis (β), and the z -axis (γ). These angles are called **direction angles** and the cosines of these angles are called **direction cosines**.

Here is a sketch of a vector and the direction angles.



The formulas for the direction cosines are,

Direction Cosines

$$\cos(\alpha) = \frac{\vec{a} \cdot \vec{i}}{\|\vec{a}\|} = \frac{a_1}{\|\vec{a}\|} \quad \cos(\beta) = \frac{\vec{a} \cdot \vec{j}}{\|\vec{a}\|} = \frac{a_2}{\|\vec{a}\|} \quad \cos(\gamma) = \frac{\vec{a} \cdot \vec{k}}{\|\vec{a}\|} = \frac{a_3}{\|\vec{a}\|}$$

where \vec{i} , \vec{j} and \vec{k} are the standard basis vectors.

Let's verify the first dot product above. We'll leave the rest to you to verify.

$$\vec{a} \cdot \vec{i} = \langle a_1, a_2, a_3 \rangle \cdot \langle 1, 0, 0 \rangle = a_1$$

Here are a couple of nice facts about the direction cosines.

Facts

1. The vector $\vec{u} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$ is a unit vector.
2. $\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$
3. $\vec{a} = \|\vec{a}\| \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$

Let's do a quick example involving direction cosines.

Example 6

Determine the direction cosines and direction angles for $\vec{a} = \langle 2, 1, -4 \rangle$.

Solution

We will need the magnitude of the vector.

$$\|\vec{a}\| = \sqrt{4 + 1 + 16} = \sqrt{21}$$

The direction cosines and angles are then,

$$\begin{aligned} \cos(\alpha) &= \frac{2}{\sqrt{21}} & \alpha &= 1.119 \text{ radians} = 64.123 \text{ degrees} \\ \cos(\beta) &= \frac{1}{\sqrt{21}} & \beta &= 1.351 \text{ radians} = 77.396 \text{ degrees} \\ \cos(\gamma) &= \frac{-4}{\sqrt{21}} & \gamma &= 2.632 \text{ radians} = 150.794 \text{ degrees} \end{aligned}$$

11.4 Cross Product

In this final section of this chapter we will look at the cross product of two vectors. We should note that the cross product requires both of the vectors to be three dimensional vectors.

Also, before getting into how to compute these we should point out a major difference between dot products and cross products. The result of a dot product is a number and the result of a cross product is a vector! Be careful not to confuse the two.

So, let's start with the two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the cross product is given by the formula,

Cross Product - Formula

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

This is not an easy formula to remember. There are two ways to derive this formula. Both of them use the fact that the cross product is really the determinant of a 3x3 matrix. If you don't know what that is don't worry about it. You don't need to know anything about matrices or determinants to use either of the methods. The notation for the determinant is as follows,

Cross Product - Determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The first row is the standard basis vectors and must appear in the order given here. The second row is the components of \vec{a} and the third row is the components of \vec{b} . Now, let's take a look at the different methods for getting the formula.

The first method uses the Method of Cofactors. If you don't know the method of cofactors that is fine, the result is all that we need. Here is the formula.

Method of Cofactors

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

where,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

This formula is not as difficult to remember as it might at first appear to be. First, the terms alternate

in sign and notice that the 2x2 is missing the column below the standard basis vector that multiplies it as well as the row of standard basis vectors.

The second method is slightly easier; however, many textbooks don't cover this method as it will only work on 3x3 determinants. This method says to take the determinant as listed above and then copy the first two columns onto the end as shown below.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \begin{vmatrix} \vec{i} & \vec{j} \\ a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

We now have three diagonals that move from left to right and three diagonals that move from right to left. We multiply along each diagonal and add those that move from left to right and subtract those that move from right to left.

This is best seen in an example. We'll also use this example to illustrate a fact about cross products.

Example 1

If $\vec{a} = \langle 2, 1, -1 \rangle$ and $\vec{b} = \langle -3, 4, 1 \rangle$ compute each of the following.

(a) $\vec{a} \times \vec{b}$

(b) $\vec{b} \times \vec{a}$

Solution

(a) $\vec{a} \times \vec{b}$

Here is the computation for this one.

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ -3 & 4 & 1 \end{vmatrix} \begin{vmatrix} \vec{i} & \vec{j} \\ 2 & 1 \\ -3 & 4 \end{vmatrix} \\ &= \vec{i}(1)(1) + \vec{j}(-1)(-3) + \vec{k}(2)(4) - \vec{j}(2)(1) - \vec{i}(-1)(4) - \vec{k}(1)(-3) \\ &= 5\vec{i} + \vec{j} + 11\vec{k} \end{aligned}$$

(b) $\vec{b} \times \vec{a}$

And here is the computation for this one.

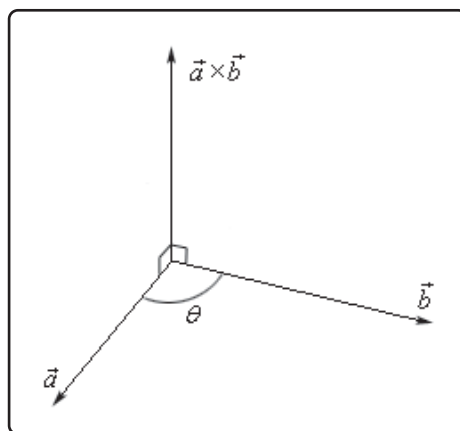
$$\begin{aligned} \vec{b} \times \vec{a} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 4 & 1 \\ 2 & 1 & -1 \end{vmatrix} \begin{vmatrix} \vec{i} & \vec{j} \\ -3 & 4 \\ 2 & 1 \end{vmatrix} \\ &= \vec{i}(4)(-1) + \vec{j}(1)(2) + \vec{k}(-3)(1) - \vec{j}(-3)(-1) - \vec{i}(1)(1) - \vec{k}(4)(2) \\ &= -5\vec{i} - \vec{j} - 11\vec{k} \end{aligned}$$

Notice that switching the order of the vectors in the cross product simply changed all the signs in the result. Note as well that this means that the two cross products will point in exactly opposite directions since they only differ by a sign. We'll formalize up this fact shortly when we list several facts.

There is also a geometric interpretation of the cross product. First we will let θ be the angle between the two vectors \vec{a} and \vec{b} and assume that $0 \leq \theta \leq \pi$, then we have the following fact,

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin(\theta) \quad (11.3)$$

and the following figure.



There should be a natural question at this point. How did we know that the cross product pointed in the direction that we've given it here?

First, as this figure implies, the cross product is orthogonal to both of the original vectors. This will always be the case with one exception that we'll get to in a second.

Second, we knew that it pointed in the upward direction (in this case) by the "right hand rule". This says that if we take our right hand, start at \vec{a} and rotate our fingers towards \vec{b} our thumb will point in the direction of the cross product. Therefore, if we'd sketched in $\vec{b} \times \vec{a}$ above we would have gotten a vector in the downward direction.

Example 2

A plane is defined by any three points that are in the plane. If a plane contains the points $P = (1, 0, 0)$, $Q = (1, 1, 1)$ and $R = (2, -1, 3)$ find a vector that is orthogonal to the plane.

Solution

The one way that we know to get an orthogonal vector is to take a cross product. So, if we could find two vectors that we knew were in the plane and took the cross product of these two vectors we know that the cross product would be orthogonal to both the vectors. However,

since both the vectors are in the plane the cross product would then also be orthogonal to the plane.

So, we need two vectors that are in the plane. This is where the points come into the problem. Since all three points lie in the plane any vector between them must also be in the plane. There are many ways to get two vectors between these points. We will use the following two,

$$\begin{aligned}\vec{PQ} &= \langle 1 - 1, 1 - 0, 1 - 0 \rangle = \langle 0, 1, 1 \rangle \\ \vec{PR} &= \langle 2 - 1, -1 - 0, 3 - 0 \rangle = \langle 1, -1, 3 \rangle\end{aligned}$$

The cross product of these two vectors will be orthogonal to the plane. So, let's find the cross product.

$$\begin{aligned}\vec{PQ} \times \vec{PR} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} \\ 0 & 1 \\ 1 & -1 \end{vmatrix} \\ &= 4\vec{i} + \vec{j} - \vec{k}\end{aligned}$$

So, the vector $4\vec{i} + \vec{j} - \vec{k}$ will be orthogonal to the plane containing the three points.

Now, let's address the one time where the cross product will not be orthogonal to the original vectors. If the two vectors, \vec{a} and \vec{b} , are parallel then the angle between them is either 0 or 180 degrees. From [Equation 11.3](#) this implies that,

$$\|\vec{a} \times \vec{b}\| = 0$$

From a fact about the magnitude we saw in the first section we know that this implies

$$\vec{a} \times \vec{b} = \vec{0}$$

In other words, it won't be orthogonal to the original vectors since we have the zero vector. This does give us another test for parallel vectors however.

Fact

If $\vec{a} \times \vec{b} = \vec{0}$ then \vec{a} and \vec{b} will be parallel vectors.

Let's also formalize up the fact about the cross product being orthogonal to the original vectors.

Fact

Provided $\vec{a} \times \vec{b} \neq \vec{0}$ then $\vec{a} \times \vec{b}$ is orthogonal to both \vec{a} and \vec{b} .

Here are some nice properties about the cross product.

Properties

If \vec{u}, \vec{v} and \vec{w} are vectors and c is a number then,

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

$$(c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$$

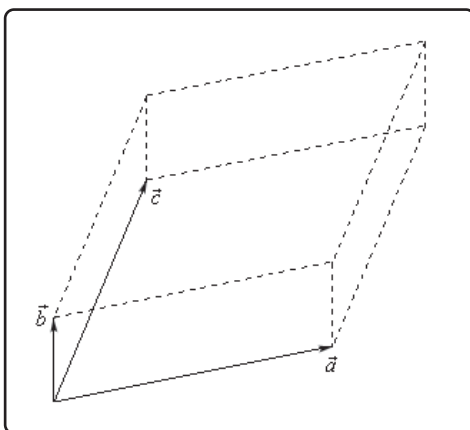
$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The determinant in the last fact is computed in the same way that the cross product is computed. We will see an example of this computation shortly.

There are a couple of geometric applications to the cross product as well. Suppose we have three vectors \vec{a}, \vec{b} and \vec{c} and we form the three dimensional figure shown below.



The area of the parallelogram (two dimensional front of this object) is given by,

$$\text{Area} = \|\vec{a} \times \vec{b}\|$$

and the volume of the parallelepiped (the whole three dimensional object) is given by,

$$\text{Volume} = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|$$

Note that the absolute value bars are required since the quantity could be negative and volume isn't negative.

We can use this volume fact to determine if three vectors lie in the same plane or not. If three vectors lie in the same plane then the volume of the parallelepiped will be zero.

Example 3

Determine if the three vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$ and $\vec{c} = \langle 0, -9, 18 \rangle$ lie in the same plane or not.

Solution

So, as we noted prior to this example all we need to do is compute the volume of the parallelepiped formed by these three vectors. If the volume is zero they lie in the same plane and if the volume isn't zero they don't lie in the same plane.

$$\begin{aligned}\vec{a} \cdot (\vec{b} \times \vec{c}) &= \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & -1 \\ 0 & -9 \end{vmatrix} \\ &= (1)(-1)(18) + (4)(4)(0) + (-7)(2)(-9) - \\ &\quad (4)(2)(18) - (1)(4)(-9) - (-7)(-1)(0) \\ &= -18 + 126 - 144 + 36 \\ &= 0\end{aligned}$$

So, the volume is zero and so they lie in the same plane.

12 Three Dimensional Space

In this chapter we will start looking at three dimensional space (3-D space or \mathbb{R}^3). As with the last chapter this is preparation for multi-variable Calculus (which we'll be starting in the next chapter) as the vast majority of the multi-variable Calculus material assumes we are in three dimensional (or higher dimensional) space.

In this chapter we will discuss the equations of lines and planes in three dimensional space as well as the equations of many of the standard quadric surfaces (*i.e* equations with at least one quadratic term in it).

We will define a vector function and discuss how to perform basic Calculus operations on vector functions. We will also discuss how to get tangent vectors (a vector tangent to a curve), normal vectors (a vector orthogonal/perpendicular) and the curvature of a curve from the vector function that defines the curve. We'll also have a quick discussion of how to get the velocity and acceleration of an object as it travels along a curve defined by a vector function.

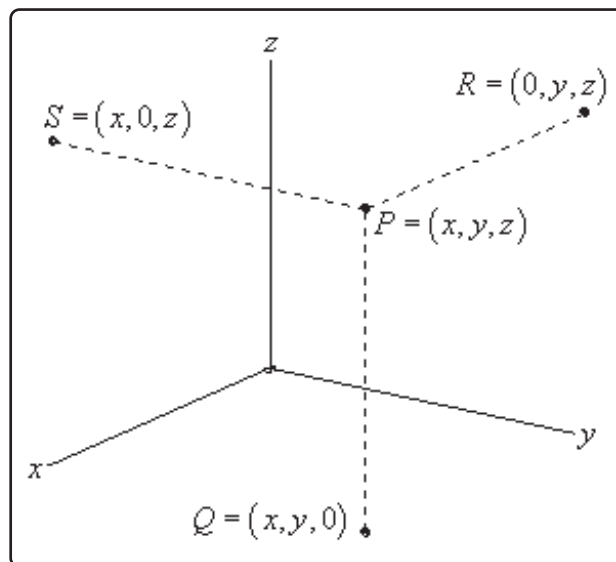
We will close out the chapter with a discussion a couple of alternative coordinates systems for three dimensional space, namely, cylindrical coordinates (a 3D extension of polar coordinates) and spherical coordinates.

12.1 The 3-D Coordinate System

We'll start the chapter off with a fairly short discussion introducing the 3-D coordinate system and the conventions that we'll be using. We will also take a brief look at how the different coordinate systems can change the graph of an equation.

Let's first get some basic notation out of the way. The 3-D coordinate system is often denoted by \mathbb{R}^3 . Likewise, the 2-D coordinate system is often denoted by \mathbb{R}^2 and the 1-D coordinate system is denoted by \mathbb{R} . Also, as you might have guessed then a general n dimensional coordinate system is often denoted by \mathbb{R}^n .

Next, let's take a quick look at the basic coordinate system.



This is the standard placement of the axes in this class. It is assumed that only the positive directions are shown by the axes. If we need the negative axes for any reason we will put them in as needed.

Also note the various points on this sketch. The point P is the general point sitting out in 3-D space. If we start at P and drop straight down until we reach a z -coordinate of zero we arrive at the point Q . We say that Q sits in the xy -plane. The xy -plane corresponds to all the points which have a zero z -coordinate. We can also start at P and move in the other two directions as shown to get points in the xz -plane (this is S with a y -coordinate of zero) and the yz -plane (this is R with an x -coordinate of zero).

Collectively, the xy , xz , and yz -planes are sometimes called the coordinate planes. In the remainder of this class you will need to be able to deal with the various coordinate planes so make sure that you can.

Also, the point Q is often referred to as the projection of P in the xy -plane. Likewise, R is the projection of P in the yz -plane and S is the projection of P in the xz -plane.

Many of the formulas that you are used to working with in \mathbb{R}^2 have natural extensions in \mathbb{R}^3 . For instance, the distance between two points in \mathbb{R}^2 is given by,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

While the distance between any two points in \mathbb{R}^3 is given by,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Likewise, the general equation for a circle with center (h, k) and radius r is given by,

$$(x - h)^2 + (y - k)^2 = r^2$$

and the general equation for a sphere with center (h, k, l) and radius r is given by,

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

With that said we do need to be careful about just translating everything we know about \mathbb{R}^2 into \mathbb{R}^3 and assuming that it will work the same way. A good example of this is in graphing to some extent. Consider the following example.

Example 1

Graph $x = 3$ in \mathbb{R} , \mathbb{R}^2 and \mathbb{R}^3 .

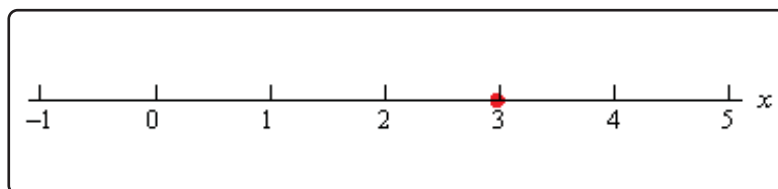
Solution

In \mathbb{R} we have a single coordinate system and so $x = 3$ is a point in a 1-D coordinate system.

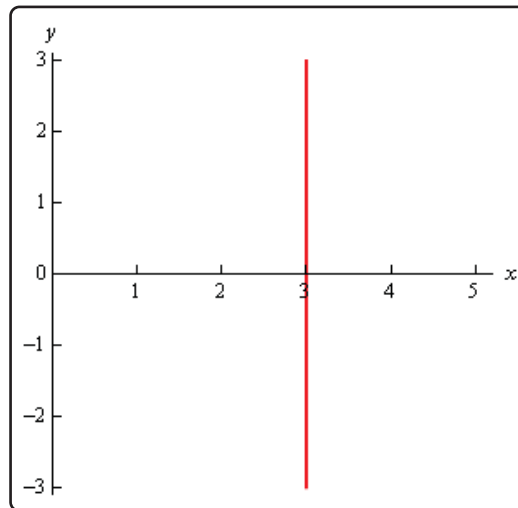
In \mathbb{R}^2 the equation $x = 3$ tells us to graph all the points that are in the form $(3, y)$. This is a vertical line in a 2-D coordinate system.

In \mathbb{R}^3 the equation $x = 3$ tells us to graph all the points that are in the form $(3, y, z)$. If you go back and look at the coordinate plane points this is very similar to the coordinates for the yz -plane except this time we have $x = 3$ instead of $x = 0$. So, in a 3-D coordinate system this is a plane that will be parallel to the yz -plane and pass through the x -axis at $x = 3$.

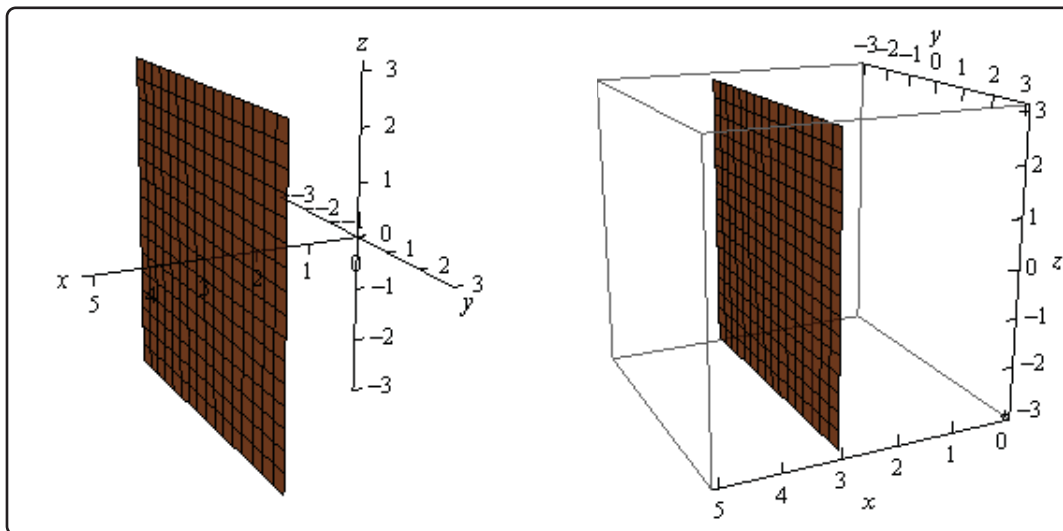
Here is the graph of $x = 3$ in \mathbb{R} .



Here is the graph of $x = 3$ in \mathbb{R}^2 .



Finally, here is the graph of $x = 3$ in \mathbb{R}^3 . Note that we've presented this graph in two different styles. On the left we've got the traditional axis system that we're used to seeing and on the right we've put the graph in a box. Both views can be convenient on occasion to help with perspective and so we'll often do this with 3D graphs and sketches.



Note that at this point we can now write down the equations for each of the coordinate planes as

well using this idea.

$$z = 0 \quad xy \text{ - plane}$$

$$y = 0 \quad xz \text{ - plane}$$

$$x = 0 \quad yz \text{ - plane}$$

Let's take a look at a slightly more general example.

Example 2

Graph $y = 2x - 3$ in \mathbb{R}^2 and \mathbb{R}^3 .

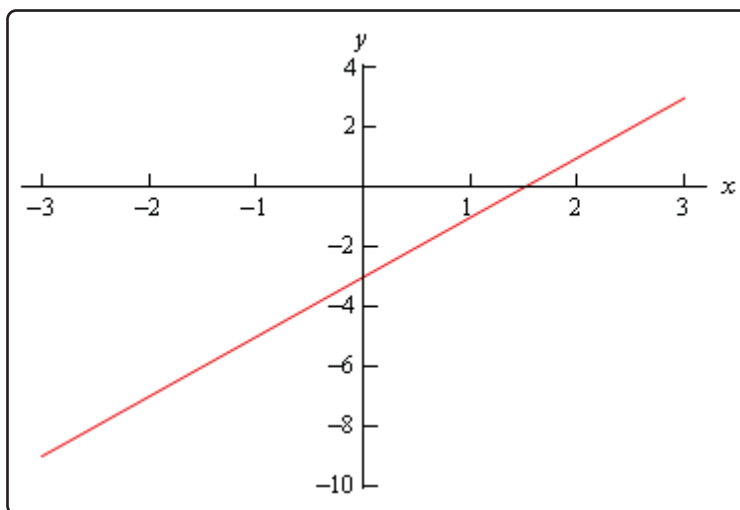
Solution

Note we had to throw out \mathbb{R} for this example since there are two variables which means that we can't be in a 1-D space (1-D space has only one variable!).

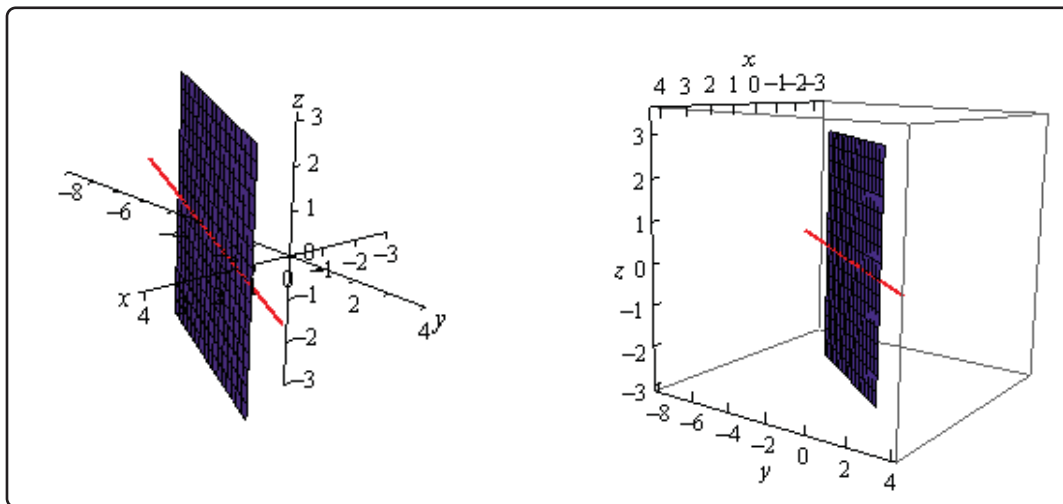
In \mathbb{R}^2 this is a line with slope 2 and a y intercept of -3.

However, in \mathbb{R}^3 this is not necessarily a line. Because we have not specified a value of z we are forced to let z take any value. This means that at any particular value of z we will get a copy of this line. So, the graph is then a vertical plane that lies over the line given by $y = 2x - 3$ in the xy -plane.

Here is the graph in \mathbb{R}^2 .



here is the graph in \mathbb{R}^3 .



Notice that if we look to where the plane intersects the xy -plane we will get the graph of the line in \mathbb{R}^2 as noted in the above graph by the red line through the plane.

Let's take a look at one more example of the difference between graphs in the different coordinate systems.

Example 3

Graph $x^2 + y^2 = 4$ in \mathbb{R}^2 and \mathbb{R}^3 .

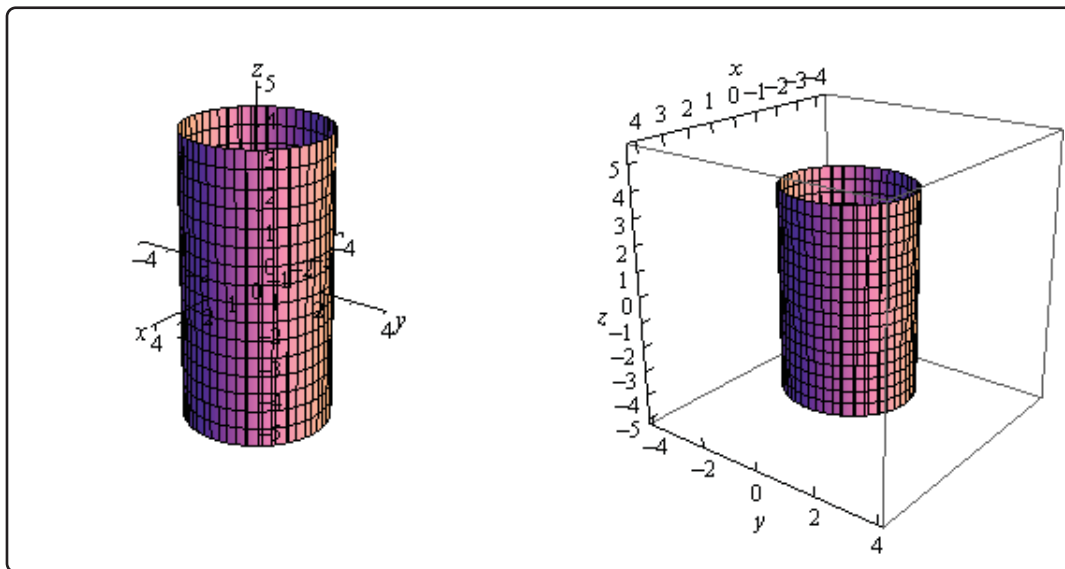
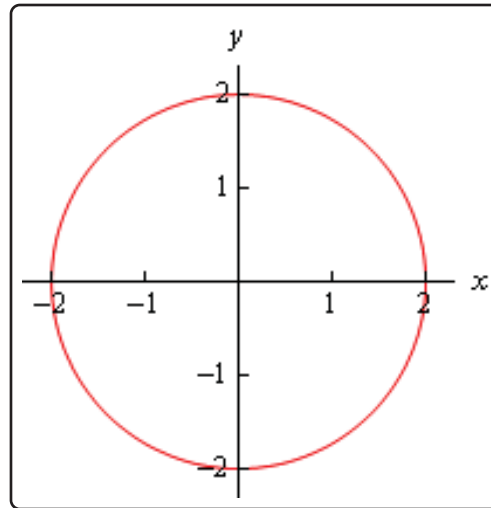
Solution

As with the previous example this won't have a 1-D graph since there are two variables.

In \mathbb{R}^2 this is a circle centered at the origin with radius 2.

In \mathbb{R}^3 however, as with the previous example, this may or may not be a circle. Since we have not specified z in any way we must assume that z can take on any value. In other words, at any value of z this equation must be satisfied and so at any value z we have a circle of radius 2 centered on the z -axis. This means that we have a cylinder of radius 2 centered on the z -axis.

Here are the graphs for this example.



Notice that again, if we look to where the cylinder intersects the xy -plane we will again get the circle from \mathbb{R}^2 .

We need to be careful with the last two examples. It would be tempting to take the results of these and say that we can't graph lines or circles in \mathbb{R}^3 and yet that doesn't really make sense. There is no reason for there to not be graphs of lines or circles in \mathbb{R}^3 . Let's think about the example of the circle. To graph a circle in \mathbb{R}^3 we would need to do something like $x^2 + y^2 = 4$ at $z = 5$. This would be a circle of radius 2 centered on the z -axis at the level of $z = 5$. So, as long as we specify a z we will get a circle and not a cylinder. We will see an easier way to specify circles in a later

section.

We could do the same thing with the line from the second example. However, we will be looking at lines in more generality in the next section and so we'll see a better way to deal with lines in \mathbb{R}^3 there.

The point of the examples in this section is to make sure that we are being careful with graphing equations and making sure that we always remember which coordinate system that we are in.

Another quick point to make here is that, as we've seen in the above examples, many graphs of equations in \mathbb{R}^3 are surfaces. That doesn't mean that we can't graph curves in \mathbb{R}^3 . We can and will graph curves in \mathbb{R}^3 as well as we'll see later in this chapter.

12.2 Equations of Lines

In this section we need to take a look at the equation of a line in \mathbb{R}^3 . As we saw in the previous section the equation $y = mx + b$ does not describe a line in \mathbb{R}^3 , instead it describes a plane. This doesn't mean however that we can't write down an equation for a line in 3-D space. We're just going to need a new way of writing down the equation of a curve.

So, before we get into the equations of lines we first need to briefly look at vector functions. We're going to take a more in depth look at vector functions later. At this point all that we need to worry about is notational issues and how they can be used to give the equation of a curve.

The best way to get an idea of what a vector function is and what its graph looks like is to look at an example. So, consider the following vector function.

$$\vec{r}(t) = \langle t, 1 \rangle$$

A vector function is a function that takes one or more variables, one in this case, and returns a vector. Note as well that a vector function can be a function of two or more variables. However, in those cases the graph may no longer be a curve in space.

The vector that the function gives can be a vector in whatever dimension we need it to be. In the example above it returns a vector in \mathbb{R}^2 . When we get to the real subject of this section, equations of lines, we'll be using a vector function that returns a vector in \mathbb{R}^3

Now, we want to determine the graph of the vector function above. In order to find the graph of our function we'll think of the vector that the vector function returns as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \langle a, b \rangle$, is a vector that starts at the origin and ends at the point (a, b) .

So, to get the graph of a vector function all we need to do is plug in some values of the variable and then plot the point that corresponds to each position vector we get out of the function and play connect the dots. Here are some evaluations for our example.

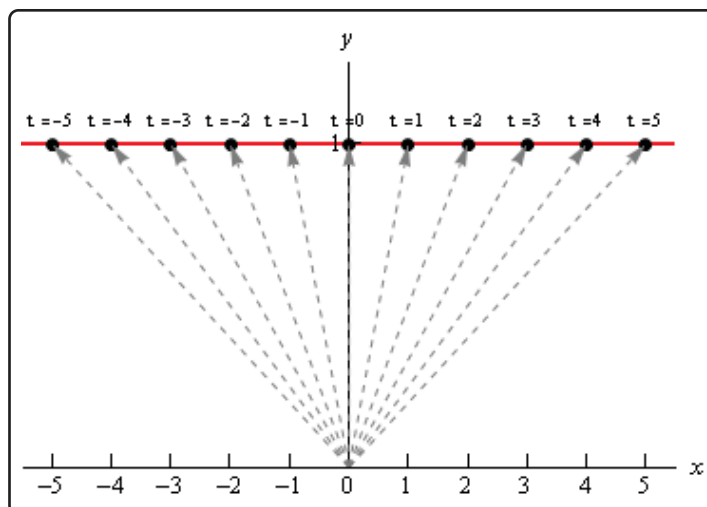
$$\vec{r}(-3) = \langle -3, 1 \rangle \quad \vec{r}(-1) = \langle -1, 1 \rangle \quad \vec{r}(2) = \langle 2, 1 \rangle \quad \vec{r}(5) = \langle 5, 1 \rangle$$

So, each of these are position vectors representing points on the graph of our vector function. The points,

$$(-3, 1) \quad (-1, 1) \quad (2, 1) \quad (5, 1)$$

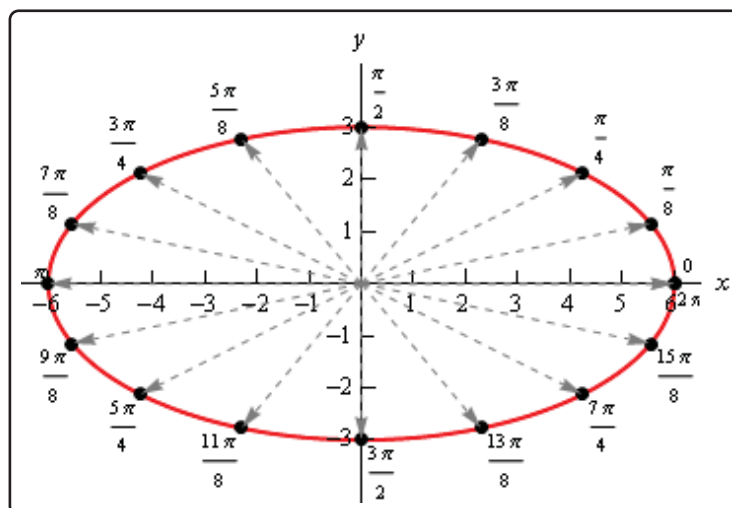
are all points that lie on the graph of our vector function.

If we do some more evaluations and plot all the points we get the following sketch.



In this sketch we've included the position vector (in gray and dashed) for several evaluations as well as the t (above each point) we used for each evaluation. It looks like, in this case the graph of the vector equation is in fact the line $y = 1$.

Here's another quick example. Here is the graph of $\vec{r}(t) = \langle 6 \cos(t), 3 \sin(t) \rangle$.



In this case we get an ellipse. It is important to not come away from this section with the idea that vector functions only graph out lines. We'll be looking at lines in this section, but the graphs of vector functions do not have to be lines as the example above shows.

We'll leave this brief discussion of vector functions with another way to think of the graph of a vector function. Imagine that a pencil/pen is attached to the end of the position vector and as we increase the variable the resulting position vector moves and as it moves the pencil/pen on the end sketches out the curve for the vector function.

Okay, we now need to move into the actual topic of this section. We want to write down the equation of a line in \mathbb{R}^3 and as suggested by the work above we will need a vector function to do this. To

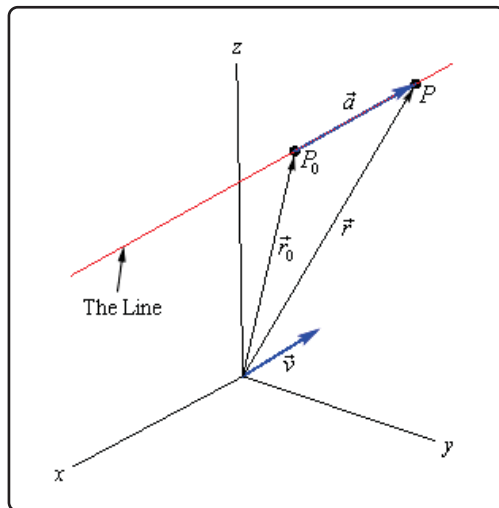
see how we're going to do this let's think about what we need to write down the equation of a line in \mathbb{R}^2 . In two dimensions we need the slope (m) and a point that was on the line in order to write down the equation.

In \mathbb{R}^3 that is still all that we need except in this case the "slope" won't be a simple number as it was in two dimensions. In this case we will need to acknowledge that a line can have a three dimensional slope. So, we need something that will allow us to describe a direction that is potentially in three dimensions. We already have a quantity that will do this for us. Vectors give directions and can be three dimensional objects.

So, let's start with the following information. Suppose that we know a point that is on the line, $P_0 = (x_0, y_0, z_0)$, and that $\vec{v} = \langle a, b, c \rangle$ is some vector that is parallel to the line. Note, in all likelihood, \vec{v} will not be on the line itself. We only need \vec{v} to be parallel to the line. Finally, let $P = (x, y, z)$ be any point on the line.

Now, since our "slope" is a vector let's also represent the two points on the line as vectors. We'll do this with position vectors. So, let \vec{r}_0 and \vec{r} be the position vectors for P_0 and P respectively. Also, for no apparent reason, let's define \vec{a} to be the vector with representation $\overrightarrow{P_0P}$.

We now have the following sketch with all these points and vectors on it.



Now, we've shown the parallel vector, \vec{v} , as a position vector but it doesn't need to be a position vector. It can be anywhere, a position vector, on the line or off the line, it just needs to be parallel to the line.

Next, notice that we can write \vec{r} as follows,

$$\vec{r} = \vec{r}_0 + \vec{a}$$

If you're not sure about this go back and check out the sketch for vector addition in the [vector arithmetic](#) section. Now, notice that the vectors \vec{a} and \vec{v} are parallel. Therefore there is a number, t , such that

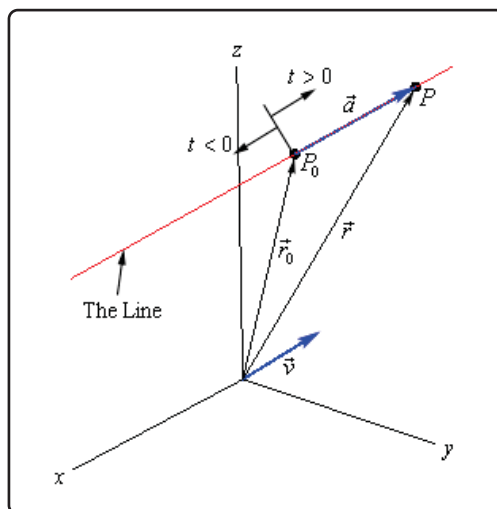
$$\vec{a} = t \vec{v}$$

We now have,

Vector Form of a Line

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

The only part of this equation that is not known is the t . Notice that $t\vec{v}$ will be a vector that lies along the line and it tells us how far from the original point that we should move. If t is positive we move away from the original point in the direction of \vec{v} (right in our sketch) and if t is negative we move away from the original point in the opposite direction of \vec{v} (left in our sketch). As t varies over all possible values we will completely cover the line. The following sketch shows this dependence on t of our sketch.



There are several other forms of the equation of a line. To get the first alternate form let's start with the vector form and do a slight rewrite.

$$\begin{aligned}\vec{r} &= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \\ \langle x, y, z \rangle &= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle\end{aligned}$$

The only way for two vectors to be equal is for the components to be equal. In other words,

Parametric Form of a Line

$$\begin{aligned}x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc\end{aligned}$$

Notice that this is really nothing more than an extension of the [parametric equations](#) we've seen previously. The only difference is that we are now working in three dimensions instead of two dimensions.

To get a point on the line all we do is pick a t and plug into either form of the line. In the vector form of the line we get a position vector for the point and in the parametric form we get the actual coordinates of the point.

There is one more form of the line that we want to look at. If we assume that a , b , and c are all non-zero numbers we can solve each of the equations in the parametric form of the line for t . We can then set all of them equal to each other since t will be the same number in each. Doing this gives the following,

Symmetric Equations of a Line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If one of a , b , or c does happen to be zero we can still write down the symmetric equations. To see this let's suppose that $b = 0$. In this case t will not exist in the parametric equation for y and so we will only solve the parametric equations for x and z for t . We then set those equal and acknowledge the parametric equation for y as follows,

$$\frac{x - x_0}{a} = \frac{z - z_0}{c} \quad y = y_0$$

Let's take a look at an example.

Example 1

Write down the equation of the line that passes through the points $(2, -1, 3)$ and $(1, 4, -3)$. Write down all three forms of the equation of the line.

Solution

To do this we need the vector \vec{v} that will be parallel to the line. This can be any vector as long as it's parallel to the line. In general, \vec{v} won't lie on the line itself. However, in this case it will. All we need to do is let \vec{v} be the vector that starts at the second point and ends at the first point. Since these two points are on the line the vector between them will also lie on the line and will hence be parallel to the line. So,

$$\vec{v} = \langle 1, -5, 6 \rangle$$

Note that the order of the points was chosen to reduce the number of minus signs in the vector. We could just have easily gone the other way.

Once we've got \vec{v} there really isn't anything else to do. To use the vector form we'll need a point on the line. We've got two and so we can use either one. We'll use the first point. Here is the vector form of the line.

$$\vec{r} = \langle 2, -1, 3 \rangle + t \langle 1, -5, 6 \rangle = \langle 2 + t, -1 - 5t, 3 + 6t \rangle$$

Once we have this equation the other two forms follow. Here are the parametric equations of the line.

$$\begin{aligned}x &= 2 + t \\y &= -1 - 5t \\z &= 3 + 6t\end{aligned}$$

Here is the symmetric form.

$$\frac{x - 2}{1} = \frac{y + 1}{-5} = \frac{z - 3}{6}$$

Example 2

Determine if the line that passes through the point $(0, -3, 8)$ and is parallel to the line given by $x = 10 + 3t$, $y = 12t$ and $z = -3 - t$ passes through the xz -plane. If it does give the coordinates of that point.

Solution

To answer this we will first need to write down the equation of the line. We know a point on the line and just need a parallel vector. We know that the new line must be parallel to the line given by the parametric equations in the problem statement. That means that any vector that is parallel to the given line must also be parallel to the new line.

Now recall that in the parametric form of the line the numbers multiplied by t are the components of the vector that is parallel to the line. Therefore, the vector,

$$\vec{v} = \langle 3, 12, -1 \rangle$$

is parallel to the given line and so must also be parallel to the new line.

The equation of new line is then,

$$\vec{r} = \langle 0, -3, 8 \rangle + t \langle 3, 12, -1 \rangle = \langle 3t, -3 + 12t, 8 - t \rangle$$

If this line passes through the xz -plane then we know that the y -coordinate of that point must be zero. So, let's set the y component of the equation equal to zero and see if we can

solve for t . If we can, this will give the value of t for which the point will pass through the xz -plane.

$$-3 + 12t = 0 \quad \Rightarrow \quad t = \frac{1}{4}$$

So, the line does pass through the xz -plane. To get the complete coordinates of the point all we need to do is plug $t = \frac{1}{4}$ into any of the equations. We'll use the vector form.

$$\vec{r} = \left\langle 3 \left(\frac{1}{4} \right), -3 + 12 \left(\frac{1}{4} \right), 8 - \frac{1}{4} \right\rangle = \left\langle \frac{3}{4}, 0, \frac{31}{4} \right\rangle$$

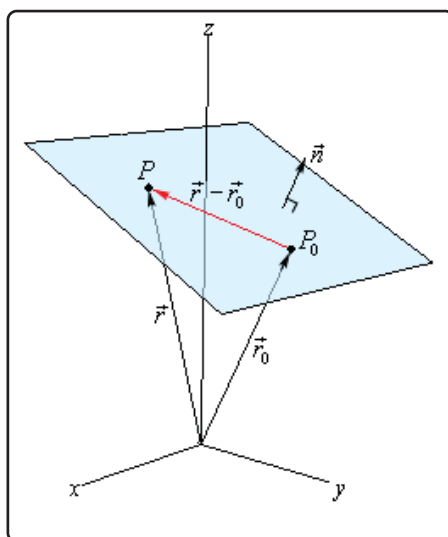
Recall that this vector is the position vector for the point on the line and so the coordinates of the point where the line will pass through the xz -plane are $\left(\frac{3}{4}, 0, \frac{31}{4} \right)$.

12.3 Equations of Planes

In the first section of this chapter we saw a couple of equations of planes. However, none of those equations had three variables in them and were really extensions of graphs that we could look at in two dimensions. We would like a more general equation for planes.

So, let's start by assuming that we know a point that is on the plane, $P_0 = (x_0, y_0, z_0)$. Let's also suppose that we have a vector that is orthogonal (perpendicular) to the plane, $\vec{n} = \langle a, b, c \rangle$. This vector is called the **normal vector**. Now, assume that $P = (x, y, z)$ is any point in the plane. Finally, since we are going to be working with vectors initially we'll let \vec{r}_0 and \vec{r} be the position vectors for P_0 and P respectively.

Here is a sketch of all these vectors.



Notice that we added in the vector $\vec{r} - \vec{r}_0$ which will lie completely in the plane. Also notice that we put the normal vector on the plane, but there is actually no reason to expect this to be the case. We put it here to illustrate the point. It is completely possible that the normal vector does not touch the plane in any way.

Now, because \vec{n} is orthogonal to the plane, it's also orthogonal to any vector that lies in the plane. In particular it's orthogonal to $\vec{r} - \vec{r}_0$. Recall from the [Dot Product](#) section that two orthogonal vectors will have a dot product of zero. In other words,

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \Rightarrow \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

This is called the **vector equation of the plane**.

A slightly more useful form of the equations is as follows. Start with the first form of the vector equation and write down a vector for the difference.

$$\begin{aligned} \langle a, b, c \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \end{aligned}$$

Now, actually compute the dot product to get,

Scalar equation of the plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Often this will be written as,

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

This second form is often how we are given equations of planes. Notice that if we are given the equation of a plane in this form we can quickly get a normal vector for the plane. A normal vector is,

$$\vec{n} = \langle a, b, c \rangle$$

Let's work a couple of examples.

Example 1

Determine the equation of the plane that contains the points $P = (1, -2, 0)$, $Q = (3, 1, 4)$ and $R = (0, -1, 2)$.

Solution

In order to write down the equation of plane we need a point (we've got three so we're cool there) and a normal vector. We need to find a normal vector. Recall however, that we saw how to do this in the [Cross Product](#) section.

We can form the following two vectors from the given points.

$$\vec{PQ} = \langle 2, 3, 4 \rangle \quad \vec{PR} = \langle -1, 1, 2 \rangle$$

These two vectors will lie completely in the plane since we formed them from points that were in the plane. Notice as well that there are many possible vectors to use here, we just chose two of the possibilities.

Now, we know that the cross product of two vectors will be orthogonal to both of these vectors. Since both of these are in the plane any vector that is orthogonal to both of these will also be orthogonal to the plane. Therefore, we can use the cross product as the normal vector.

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} \\ 2 & 3 \\ -1 & 1 \end{vmatrix} = 2\vec{i} - 8\vec{j} + 5\vec{k}$$

The equation of the plane is then,

$$\begin{aligned} 2(x - 1) - 8(y + 2) + 5(z - 0) &= 0 \\ 2x - 8y + 5z &= 18 \end{aligned}$$

We used P for the point but could have used any of the three points.

Example 2

Determine if the plane given by $-x + 2z = 10$ and the line given by $\vec{r} = \langle 5, 2 - t, 10 + 4t \rangle$ are orthogonal, parallel or neither.

Solution

This is not as difficult a problem as it may at first appear to be. We can pick off a vector that is normal to the plane. This is $\vec{n} = \langle -1, 0, 2 \rangle$. We can also get a vector that is parallel to the line. This is $\vec{v} = \langle 0, -1, 4 \rangle$.

Now, if these two vectors are parallel then the line and the plane will be orthogonal. If you think about it this makes some sense. If \vec{n} and \vec{v} are parallel, then \vec{v} is orthogonal to the plane, but \vec{v} is also parallel to the line. So, if the two vectors are parallel the line and plane will be orthogonal.

Let's check this.

$$\vec{n} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 2 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} \\ -1 & 0 \\ 0 & -1 \end{vmatrix} = 2\vec{i} + 4\vec{j} + \vec{k} \neq \vec{0}$$

So, the vectors aren't parallel and so the plane and the line are not orthogonal.

Now, let's check to see if the plane and line are parallel. If the line is parallel to the plane then any vector parallel to the line will be orthogonal to the normal vector of the plane. In other words, if \vec{n} and \vec{v} are orthogonal then the line and the plane will be parallel.

Let's check this.

$$\vec{n} \cdot \vec{v} = 0 + 0 + 8 = 8 \neq 0$$

The two vectors aren't orthogonal and so the line and plane aren't parallel.

So, the line and the plane are neither orthogonal nor parallel.

12.4 Quadric Surfaces

In the previous two sections we've looked at lines and planes in three dimensions (or \mathbb{R}^3) and while these are used quite heavily at times in a Calculus class there are many other surfaces that are also used fairly regularly and so we need to take a look at those.

In this section we are going to be looking at quadric surfaces. Quadric surfaces are the graphs of any equation that can be put into the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where A, \dots, J are constants.

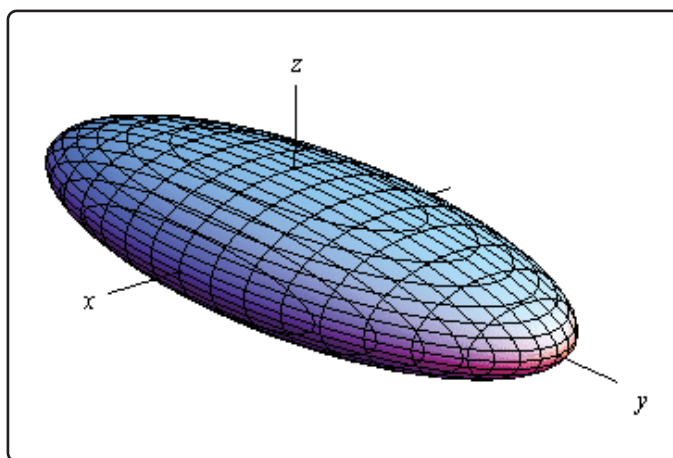
There is no way that we can possibly list all of them, but there are some standard equations so here is a list of some of the more common quadric surfaces.

Ellipsoid

Here is the general equation of an ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical ellipsoid.



If $a = b = c$ then we will have a sphere.

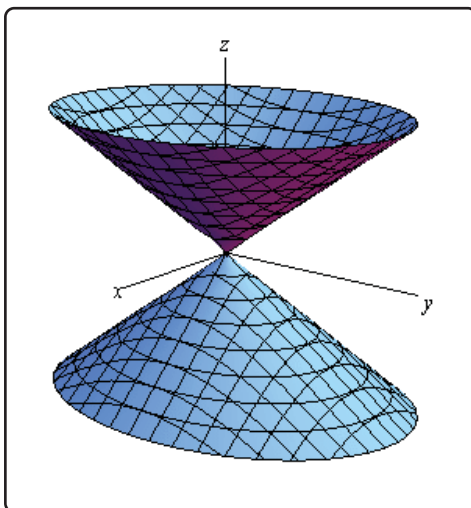
Notice that we only gave the equation for the ellipsoid that has been centered on the origin. Clearly ellipsoids don't have to be centered on the origin. However, in order to make the discussion in this section a little easier we have chosen to concentrate on surfaces that are "centered" on the origin in one way or another.

Cone

Here is the general equation of a cone.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Here is a sketch of a typical cone.



Now, note that while we called this a cone it is more of an hour glass shape rather than what most would call a cone. Of course, the upper and the lower portion of the hour glass really are cones as we would normally think of them.

That brings up the question of what if we really did just want the upper or lower portion (*i.e.* a cone in the traditional sense)? That is easy enough to answer. All we need to do is solve the given equation for z as follows,

$$z^2 = c^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) = \frac{c^2}{a^2} x^2 + \frac{c^2}{b^2} y^2 = A^2 x^2 + B^2 y^2 \quad \rightarrow \quad z = \pm \sqrt{A^2 x^2 + B^2 y^2}$$

We simplified the coefficients a little to make it the equation(s) easier to deal with. Now, we know that square roots always return positive numbers and so we can then see that $z = \sqrt{A^2 x^2 + B^2 y^2}$ will always be positive and so be the equation for just the upper portion of the “cone” above. Likewise, $z = -\sqrt{A^2 x^2 + B^2 y^2}$ will always be negative and so be the equation of just the lower portion of the “cone” above.

Also, note that this is the equation of a cone that will open along the z -axis. To get the equation of a cone that opens along one of the other axes all we need to do is make a slight modification of the equation. This will be the case for the rest of the surfaces that we’ll be looking at in this section as well.

In the case of a cone the variable that sits by itself on one side of the equal sign will determine the axis that the cone opens up along. For instance, a cone that opens up along the x -axis will have the equation,

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2}$$

For most of the following surfaces we will not give the other possible formulas. We will however

acknowledge how each formula needs to be changed to get a change of orientation for the surface.

Cylinder

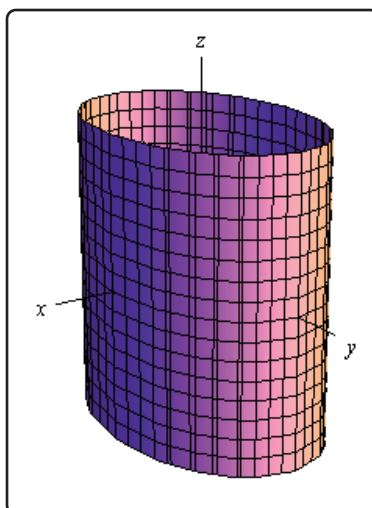
Here is the general equation of a cylinder.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is a cylinder whose cross section is an ellipse. If $a = b$ we have a cylinder whose cross section is a circle. We'll be dealing with those kinds of cylinders more than the general form so the equation of a cylinder with a circular cross section is,

$$x^2 + y^2 = r^2$$

Here is a sketch of typical cylinder with an ellipse cross section.



The cylinder will be centered on the axis corresponding to the variable that does not appear in the equation.

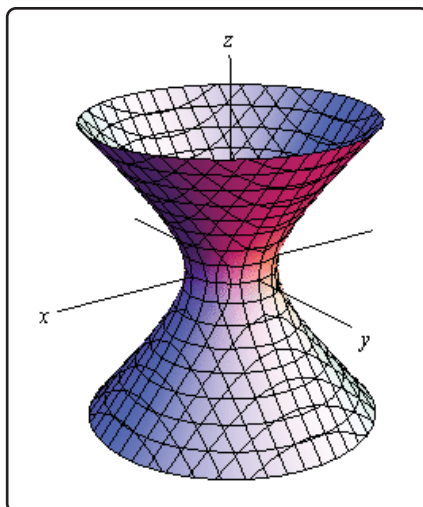
Be careful to not confuse this with a circle. In two dimensions it is a circle, but in three dimensions it is a cylinder.

Hyperboloid of One Sheet

Here is the equation of a hyperboloid of one sheet.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical hyperboloid of one sheet.



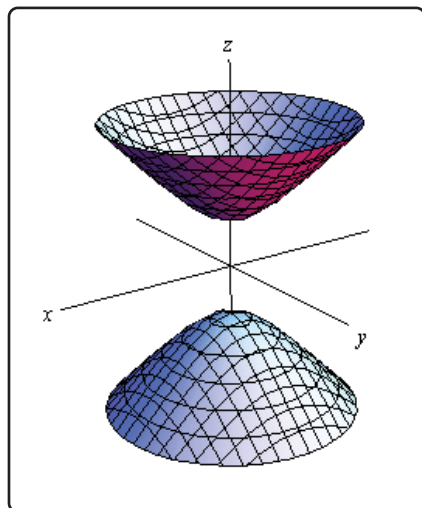
The variable with the negative in front of it will give the axis along which the graph is centered.

Hyperboloid of Two Sheets

Here is the equation of a hyperboloid of two sheets.

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical hyperboloid of two sheets.



The variable with the positive in front of it will give the axis along which the graph is centered.

Notice that the only difference between the hyperboloid of one sheet and the hyperboloid of two sheets is the signs in front of the variables. They are exactly the opposite signs.

Also note that just as we could do with cones, if we solve the equation for z the positive portion will give the equation for the upper part of this while the negative portion will give the equation for the

lower part of this.

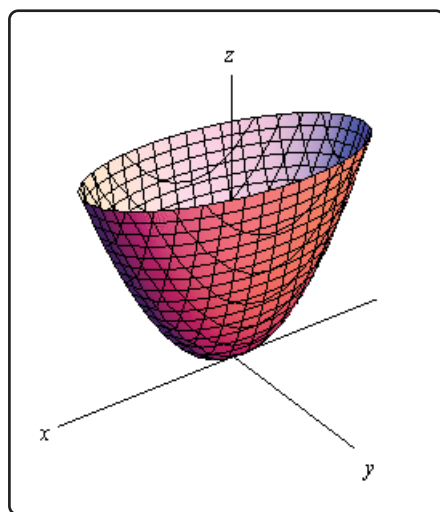
Elliptic Paraboloid

Here is the equation of an elliptic paraboloid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

As with cylinders this has a cross section of an ellipse and if $a = b$ it will have a cross section of a circle. When we deal with these we'll generally be dealing with the kind that have a circle for a cross section.

Here is a sketch of a typical elliptic paraboloid.



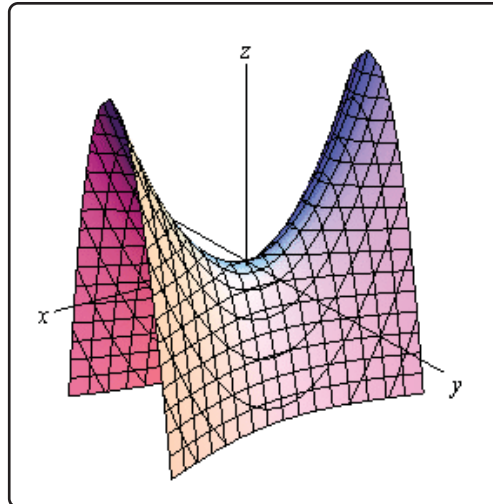
In this case the variable that isn't squared determines the axis upon which the paraboloid opens up. Also, the sign of c will determine the direction that the paraboloid opens. If c is positive then it opens up and if c is negative then it opens down.

Hyperbolic Paraboloid

Here is the equation of a hyperbolic paraboloid.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

Here is a sketch of a typical hyperbolic paraboloid.



These graphs are vaguely saddle shaped and as with the elliptic paraboloid the sign of c will determine the direction in which the surface “opens up”. The graph above is shown for c positive.

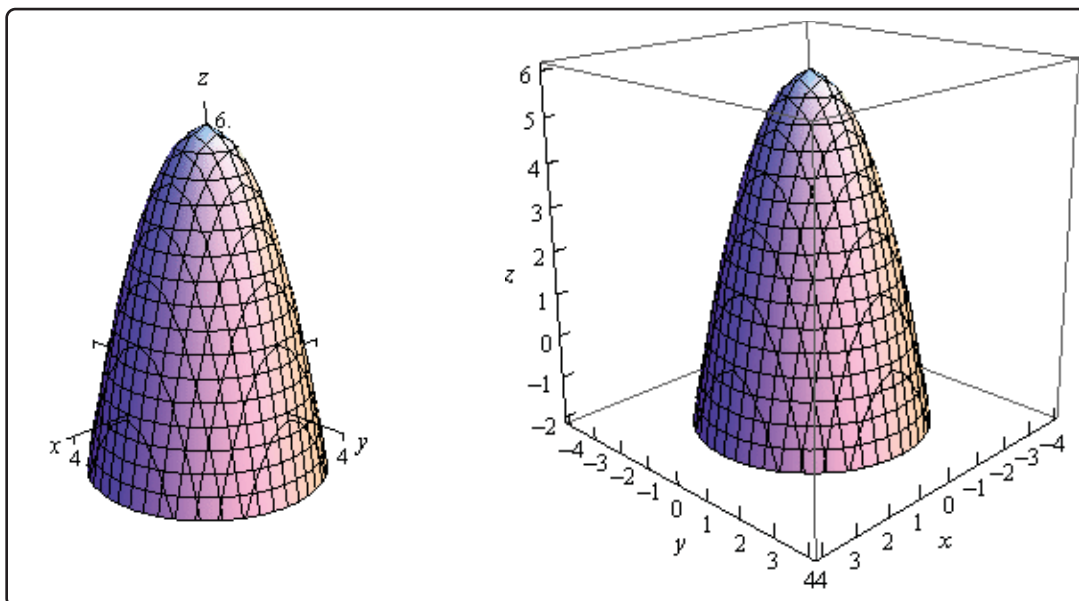
With both of the types of paraboloids discussed above note that the surface can be easily moved up or down by adding/subtracting a constant from the left side.

For instance

$$z = -x^2 - y^2 + 6$$

is an elliptic paraboloid that opens downward (be careful, the “-” is on the x and y instead of the z) and starts at $z = 6$ instead of $z = 0$.

Here are a couple of quick sketches of this surface.

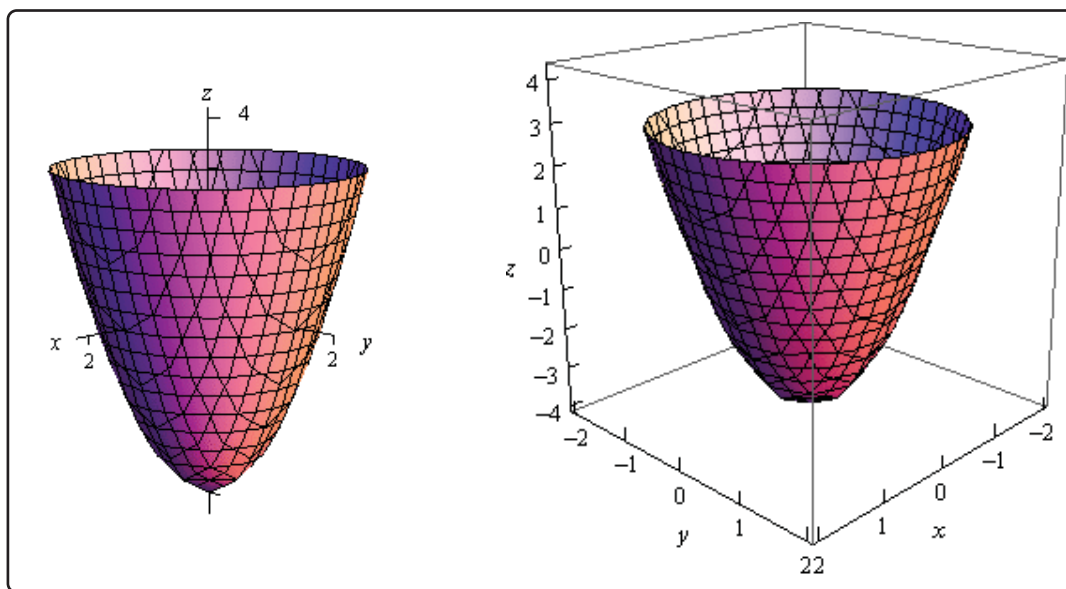


Note that we've given two forms of the sketch here. The sketch on the left has the standard set of axes but it is difficult to see the numbers on the axis. The sketch on the right has been "boxed" and this makes it easier to see the numbers to give a sense of perspective to the sketch. In most sketches that actually involve numbers on the axis system we will give both sketches to help get a feel for what the sketch looks like.

12.5 Functions of Several Variables

In this section we want to go over some of the basic ideas about functions of more than one variable.

First, remember that graphs of functions of two variables, $z = f(x, y)$ are surfaces in three dimensional space. For example, here is the graph of $z = 2x^2 + 2y^2 - 4$.



This is an elliptic paraboloid and is an example of a [quadric surface](#). We saw several of these in the previous section. We will be seeing quadric surfaces fairly regularly later on when we start discussion Multi-variable Calculus.

Another common graph that we'll be seeing quite a bit in this course is the graph of a plane. We have a convention for graphing planes that will make them a little easier to graph and hopefully visualize.

Recall that the [equation of a plane](#) is given by

$$ax + by + cz = d$$

or if we solve this for z we can write it in terms of function notation. This gives,

$$f(x, y) = Ax + By + D$$

To graph a plane we will generally find the intersection points with the three axes and then graph the triangle that connects those three points. This triangle will be a portion of the plane and it will give us a fairly decent idea on what the plane itself should look like. For example, let's graph the plane given by,

$$f(x, y) = 12 - 3x - 4y$$

For purposes of graphing this it would probably be easier to write this as,

$$z = 12 - 3x - 4y \quad \Rightarrow \quad 3x + 4y + z = 12$$

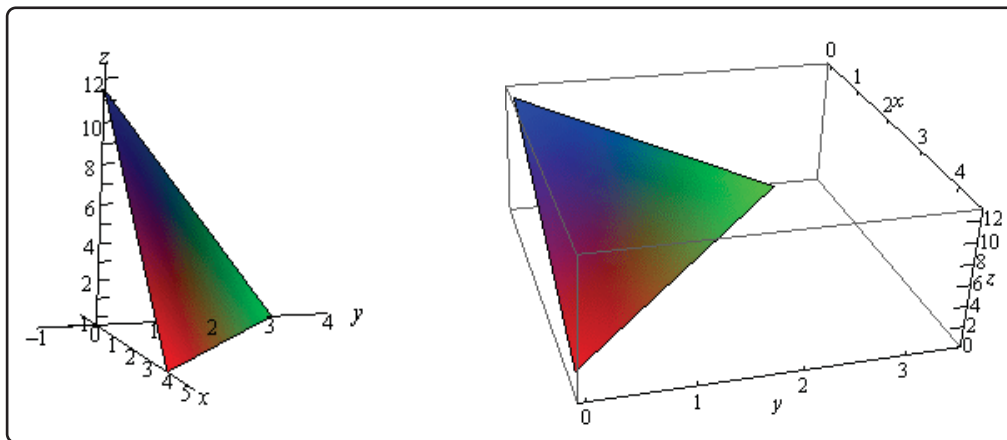
Now, each of the intersection points with the three main coordinate axes is defined by the fact that two of the coordinates are zero. For instance, the intersection with the z -axis is defined by $x = y = 0$. So, the three intersection points are,

$$x - \text{axis} : (4, 0, 0)$$

$$y - \text{axis} : (0, 3, 0)$$

$$z - \text{axis} : (0, 0, 12)$$

Here is the graph of the plane.



Now, to extend this out, graphs of functions of the form $w = f(x, y, z)$ would be four dimensional surfaces. Of course, we can't graph them, but it doesn't hurt to point this out.

We next want to talk about the domains of functions of more than one variable. Recall that domains of functions of a single variable, $y = f(x)$, consisted of all the values of x that we could plug into the function and get back a real number. Now, if we think about it, this means that the domain of a function of a single variable is an interval (or intervals) of values from the number line, or one dimensional space.

The domain of functions of two variables, $z = f(x, y)$, are regions from two dimensional space and consist of all the coordinate pairs, (x, y) , that we could plug into the function and get back a real number.

Example 1

Determine the domain of each of the following.

(a) $f(x, y) = \sqrt{x + y}$

(b) $f(x, y) = \sqrt{x} + \sqrt{y}$

(c) $f(x, y) = \ln(9 - x^2 - 9y^2)$

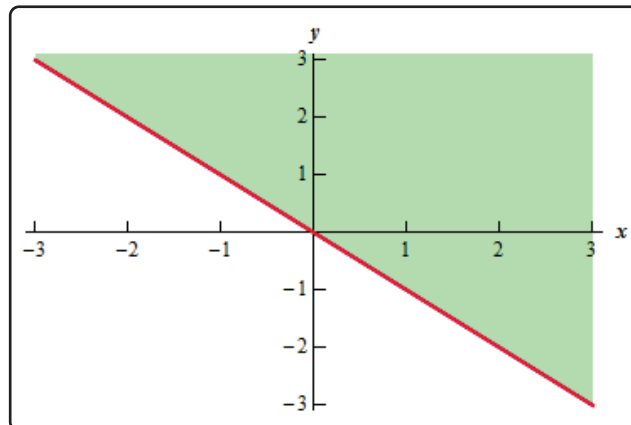
Solution

(a) $f(x, y) = \sqrt{x + y}$

In this case we know that we can't take the square root of a negative number so this means that we must require,

$$x + y \geq 0$$

Here is a sketch of the graph of this region.

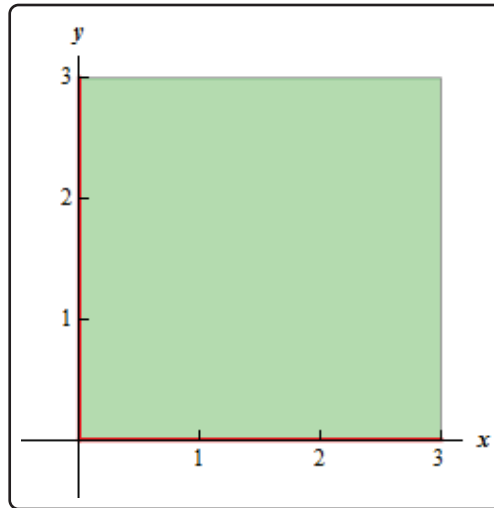


(b) $f(x, y) = \sqrt{x} + \sqrt{y}$

This function is different from the function in the previous part. Here we must require that,

$$x \geq 0 \quad \text{and} \quad y \geq 0$$

and they really do need to be separate inequalities. There is one for each square root in the function. Here is the sketch of this region.

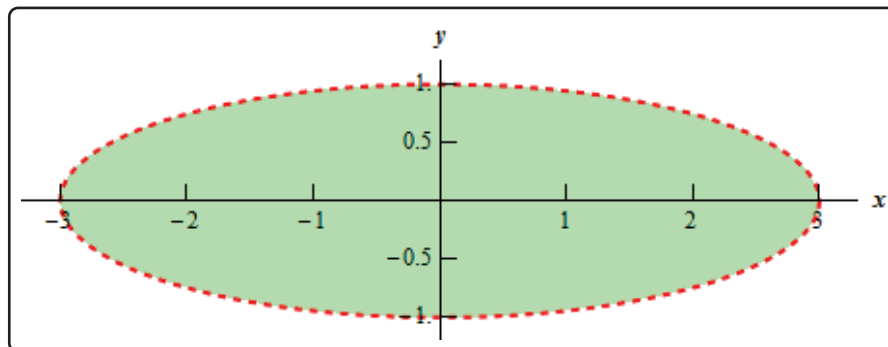


(c) $f(x, y) = \ln(9 - x^2 - 9y^2)$

In this final part we know that we can't take the logarithm of a negative number or zero. Therefore, we need to require that,

$$9 - x^2 - 9y^2 > 0 \quad \Rightarrow \quad \frac{x^2}{9} + y^2 < 1$$

and upon rearranging we see that we need to stay interior to an ellipse for this function. Here is a sketch of this region.



Note that domains of functions of three variables, $w = f(x, y, z)$, will be regions in three dimensional space.

Example 2

Determine the domain of the following function,

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2 - 16}}$$

Solution

In this case we have to deal with the square root and division by zero issues. These will require,

$$x^2 + y^2 + z^2 - 16 > 0 \quad \Rightarrow \quad x^2 + y^2 + z^2 > 16$$

So, the domain for this function is the set of points that lies completely outside a sphere of radius 4 centered at the origin.

The next topic that we should look at is that of **level curves** or **contour curves**. The level curves of the function $z = f(x, y)$ are two dimensional curves we get by setting $z = k$, where k is any number. So the equations of the level curves are $f(x, y) = k$. Note that sometimes the equation will be in the form $f(x, y, z) = 0$ and in these cases the equations of the level curves are $f(x, y, k) = 0$.

You've probably seen level curves (or contour curves, whatever you want to call them) before. If you've ever seen the elevation map for a piece of land, this is nothing more than the contour curves for the function that gives the elevation of the land in that area. Of course, we probably don't have the function that gives the elevation, but we can at least graph the contour curves.

Let's do a quick example of this.

Example 3

Identify the level curves of $f(x, y) = \sqrt{x^2 + y^2}$. Sketch a few of them.

Solution

First, for the sake of practice, let's identify what this surface given by $f(x, y)$ is. To do this let's rewrite it as,

$$z = \sqrt{x^2 + y^2}$$

Recall from the [Quadric Surfaces](#) section that this is the upper portion of the "cone" (or hour glass shaped surface).

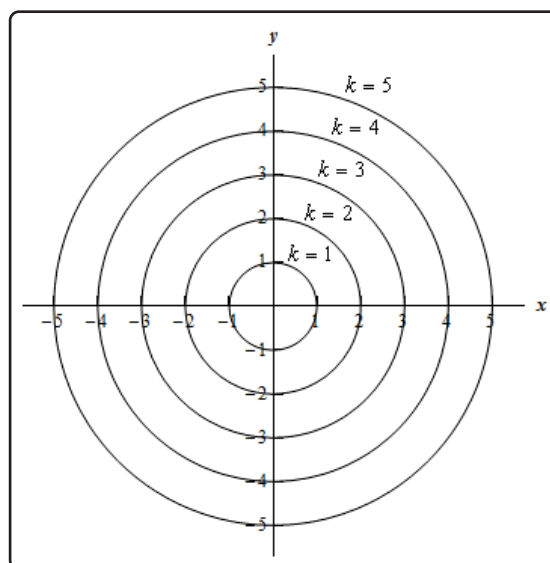
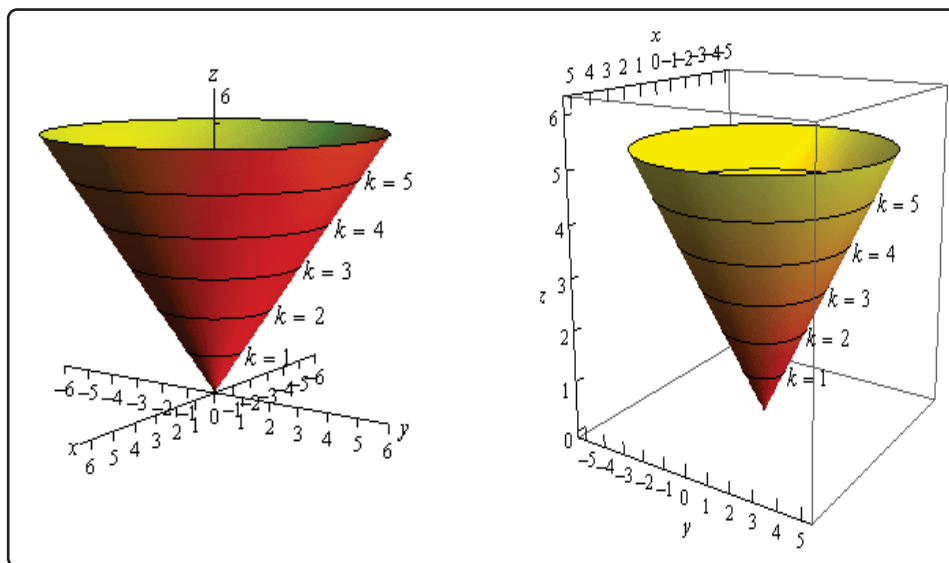
Note that this was not required for this problem. It was done for the practice of identifying the surface and this may come in handy down the road.

Now on to the real problem. The level curves (or contour curves) for this surface are given by the equation are found by substituting $z = k$. In the case of our example this is,

$$k = \sqrt{x^2 + y^2} \quad \Rightarrow \quad x^2 + y^2 = k^2$$

where k is any number. So, in this case, the level curves are circles of radius k with center at the origin.

We can graph these in one of two ways. We can either graph them on the surface itself or we can graph them in a two dimensional axis system. Here is each graph for some values of k .



Note that we can think of contours in terms of the intersection of the surface that is given by $z = f(x, y)$ and the plane $z = k$. The contour will represent the intersection of the surface and the plane.

For functions of the form $f(x, y, z)$ we will occasionally look at **level surfaces**. The equations of level surfaces are given by $f(x, y, z) = k$ where k is any number.

The final topic in this section is that of **traces**. In some ways these are similar to contours. As noted above we can think of contours as the intersection of the surface given by $z = f(x, y)$ and the plane $z = k$. Traces of surfaces are curves that represent the intersection of the surface and the plane given by $x = a$ or $y = b$.

Let's take a quick look at an example of traces.

Example 4

Sketch the traces of $f(x, y) = 10 - 4x^2 - y^2$ for the plane $x = 1$ and $y = 2$.

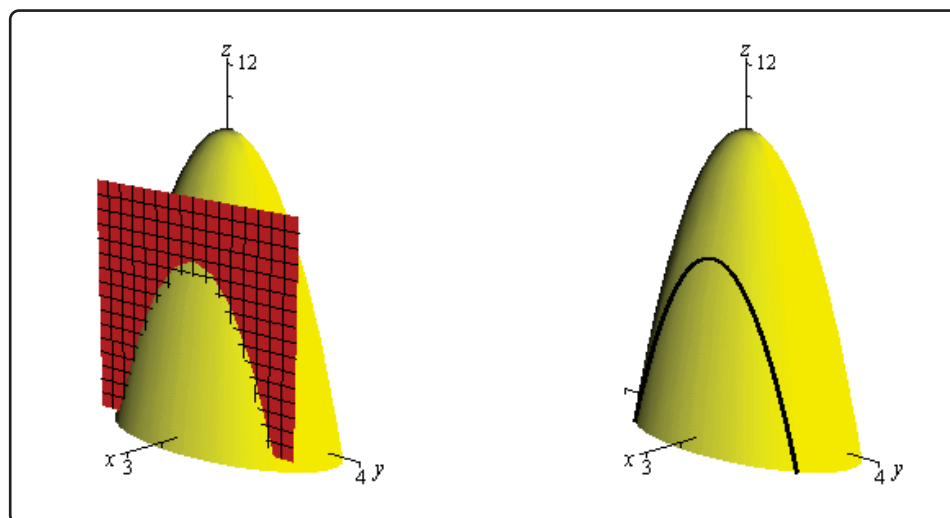
Solution

We'll start with $x = 1$. We can get an equation for the trace by plugging $x = 1$ into the equation. Doing this gives,

$$z = f(1, y) = 10 - 4(1)^2 - y^2 \quad \Rightarrow \quad z = 6 - y^2$$

and this will be graphed in the plane given by $x = 1$.

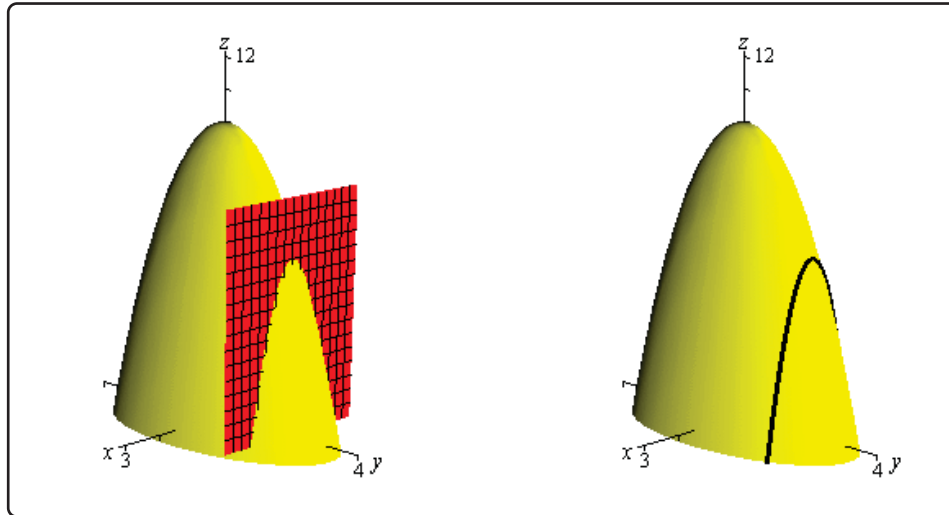
Below are two graphs. The graph on the left is a graph showing the intersection of the surface and the plane given by $x = 1$. On the right is a graph of the surface and the trace that we are after in this part.



For $y = 2$ we will do pretty much the same thing that we did with the first part. Here is the equation of the trace,

$$z = f(x, 2) = 10 - 4x^2 - (2)^2 \quad \Rightarrow \quad z = 6 - 4x^2$$

and here are the sketches for this case.



12.6 Vector Functions

We first saw vector functions back when we were looking at the [Equation of Lines](#). In that section we talked about them because we wrote down the equation of a line in \mathbb{R}^3 in terms of a *vector function* (sometimes called a *vector-valued function*). In this section we want to look a little closer at them and we also want to look at some vector functions in \mathbb{R}^3 other than lines.

A vector function is a function that takes one or more variables and returns a vector. We'll spend most of this section looking at vector functions of a single variable as most of the places where vector functions show up here will be vector functions of single variables. We will however briefly look at vector functions of two variables at the end of this section.

A vector functions of a single variable in \mathbb{R}^2 and \mathbb{R}^3 have the form,

$$\vec{r}(t) = \langle f(t), g(t) \rangle \qquad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

respectively, where $f(t)$, $g(t)$ and $h(t)$ are called the **component functions**.

The main idea that we want to discuss in this section is that of graphing and identifying the graph given by a vector function. Before we do that however, we should talk briefly about the domain of a vector function. The **domain** of a vector function is the set of all t 's for which all the component functions are defined.

Example 1

Determine the domain of the following function.

$$\vec{r}(t) = \langle \cos(t), \ln(4-t), \sqrt{t+1} \rangle$$

Solution

The first component is defined for all t 's. The second component is only defined for $t < 4$. The third component is only defined for $t \geq -1$. Putting all of these together gives the following domain.

$$[-1, 4)$$

This is the largest possible interval for which all three components are defined.

Let's now move into looking at the graph of vector functions. In order to graph a vector function all we do is think of the vector returned by the vector function as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \langle a, b, c \rangle$, is a vector that starts at the origin and ends at the point (a, b, c) .

So, in order to sketch the graph of a vector function all we need to do is plug in some values of t and then plot points that correspond to the resulting position vector we get out of the vector

function.

Because it is a little easier to visualize things we'll start off by looking at graphs of vector functions in \mathbb{R}^2 .

Example 2

Sketch the graph of each of the following vector functions.

(a) $\vec{r}(t) = \langle t, 1 \rangle$

(b) $\vec{r}(t) = \langle t, t^3 - 10t + 7 \rangle$

Solution

(a) $\vec{r}(t) = \langle t, 1 \rangle$

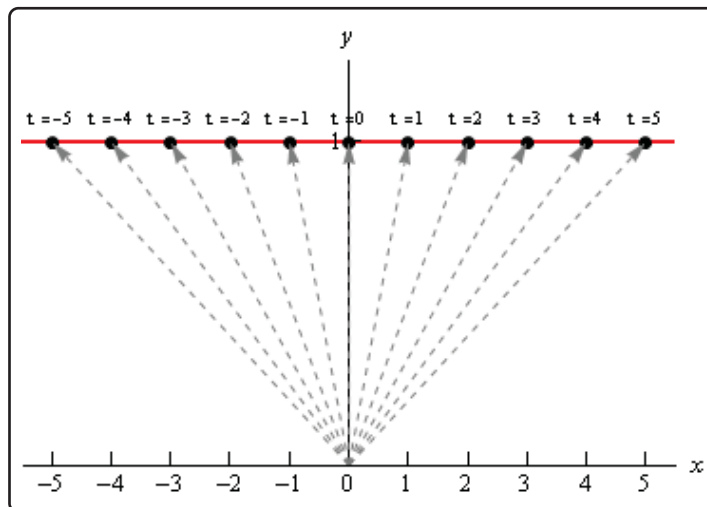
Okay, the first thing that we need to do is plug in a few values of t and get some position vectors. Here are a few,

$$\vec{r}(-3) = \langle -3, 1 \rangle \quad \vec{r}(-1) = \langle -1, 1 \rangle \quad \vec{r}(2) = \langle 2, 1 \rangle \quad \vec{r}(5) = \langle 5, 1 \rangle$$

So, what this tells us is that the following points are all on the graph of this vector function.

$$(-3, 1) \quad (-1, 1) \quad (2, 1) \quad (5, 1)$$

Here is a sketch of this vector function.



In this sketch we've included many more evaluations than just those above. Also note that we've put in the position vectors (in gray and dashed) so you can see how all

this is working. Note however, that in practice the position vectors are generally not included in the sketch.

In this case it looks like we've got the graph of the line $y = 1$.

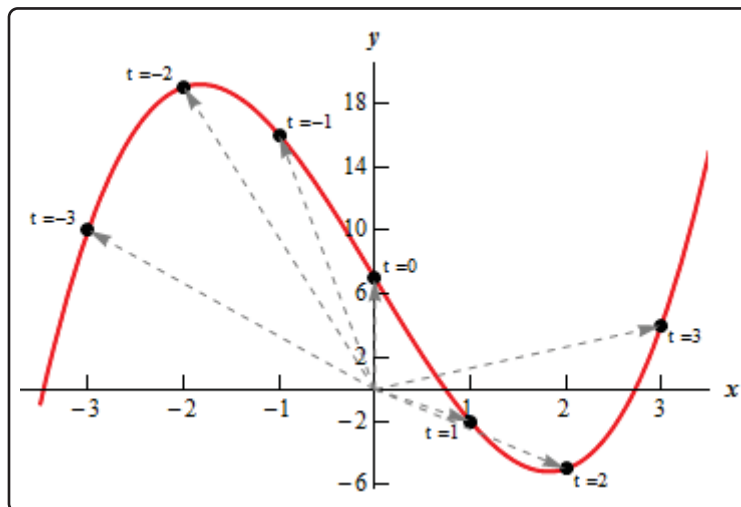
(b) $\vec{r}(t) = \langle t, t^3 - 10t + 7 \rangle$

Here are a couple of evaluations for this vector function.

$$\vec{r}(-3) = \langle -3, 10 \rangle \quad \vec{r}(-1) = \langle -1, 16 \rangle \quad \vec{r}(1) = \langle 1, -2 \rangle \quad \vec{r}(3) = \langle 3, 4 \rangle$$

So, we've got a few points on the graph of this function. However, unlike the first part this isn't really going to be enough points to get a good idea of this graph. In general, it can take quite a few function evaluations to get an idea of what the graph is and it's usually easier to use a computer to do the graphing.

Here is a sketch of this graph. We've put in a few vectors/evaluations to illustrate them, but the reality is that we did have to use a computer to get a good sketch here.



Both of the vector functions in the above example were in the form,

$$\vec{r}(t) = \langle t, g(t) \rangle$$

and what we were really sketching is the graph of $y = g(x)$ as you probably caught onto. Let's graph a couple of other vector functions that do not fall into this pattern.

Example 3

Sketch the graph of each of the following vector functions.

(a) $\vec{r}(t) = \langle 6 \cos(t), 3 \sin(t) \rangle$

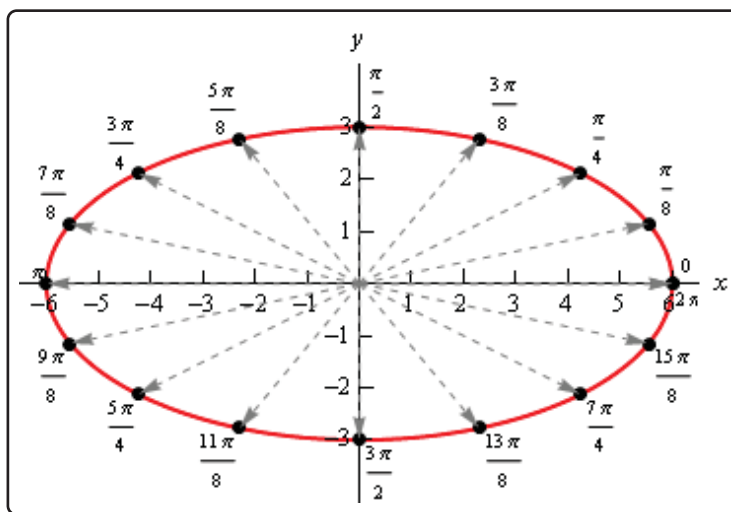
(b) $\vec{r}(t) = \langle t - 2 \sin(t), t^2 \rangle$

Solution

As we saw in the last part of the previous example it can really take quite a few function evaluations to really be able to sketch the graph of a vector function. Because of that we'll be skipping all the function evaluations here and just giving the graph. The main point behind this set of examples is to not get you too locked into the form we were looking at above. The first part will also lead to an important idea that we'll discuss after this example.

So, with that said here are the sketches of each of these.

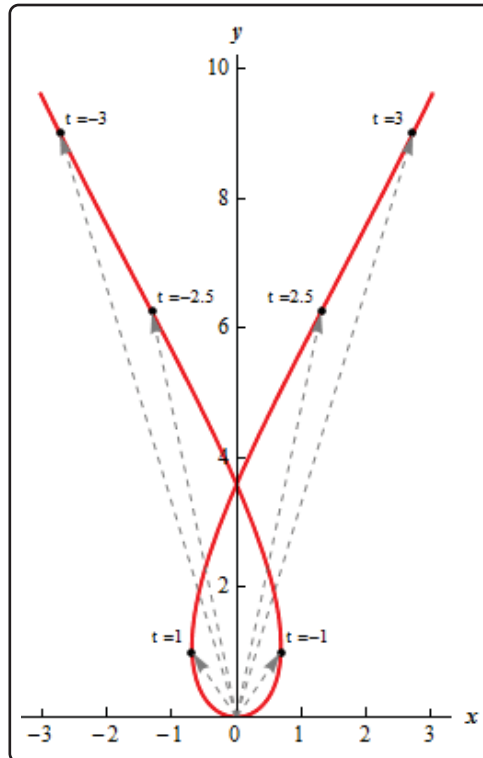
(a) $\vec{r}(t) = \langle 6 \cos(t), 3 \sin(t) \rangle$



So, in this case it looks like we've got an ellipse.

(b) $\vec{r}(t) = \langle t - 2 \sin(t), t^2 \rangle$

Here's the sketch for this vector function.



Before we move on to vector functions in \mathbb{R}^3 let's go back and take a quick look at the first vector function we sketched in the previous example, $\vec{r}(t) = \langle 6 \cos(t), 3 \sin(t) \rangle$. The fact that we got an ellipse here should not come as a surprise to you. We know that the first component function gives the x coordinate and the second component function gives the y coordinates of the point that we graph. If we strip these out to make this clear we get,

$$x = 6 \cos(t) \qquad y = 3 \sin(t)$$

This should look familiar to you. Back when we were looking at [Parametric Equations](#) we saw that this was nothing more than one of the sets of parametric equations that gave an ellipse.

This is an important idea in the study of vector functions. Any vector function can be broken down into a set of parametric equations that represent the same graph. In general, the two dimensional vector function, $\vec{r}(t) = \langle f(t), g(t) \rangle$, can be broken down into the parametric equations,

$$x = f(t) \qquad y = g(t)$$

Likewise, a three dimensional vector function, $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, can be broken down into the parametric equations,

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

Do not get too excited about the fact that we're now looking at parametric equations in \mathbb{R}^3 . They work in exactly the same manner as parametric equations in \mathbb{R}^2 which we're used to dealing with already. The only difference is that we now have a third component.

Let's take a look at a couple of graphs of vector functions.

Example 4

Sketch the graph of the following vector function.

$$\vec{r}(t) = \langle 2 - 4t, -1 + 5t, 3 + t \rangle$$

Solution

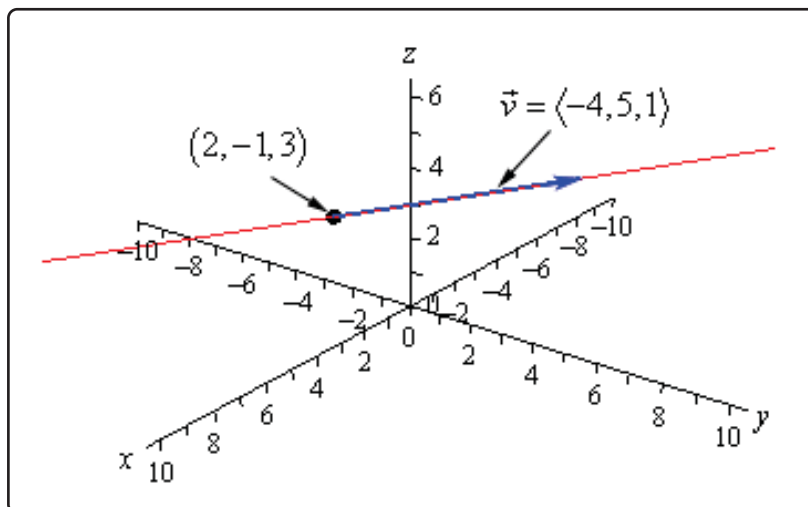
Notice that this is nothing more than a line. It might help if we rewrite it a little.

$$\vec{r}(t) = \langle 2, -1, 3 \rangle + t \langle -4, 5, 1 \rangle$$

In this form we can see that this is the equation of a line that goes through the point $(2, -1, 3)$ and is parallel to the vector $\vec{v} = \langle -4, 5, 1 \rangle$.

To graph this line all that we need to do is plot the point and then sketch in the parallel vector. In order to get the sketch will assume that the vector is on the line and will start at the point in the line. To sketch in the line all we do this is extend the parallel vector into a line.

Here is a sketch.



Example 5

Sketch the graph of the following vector function.

$$\vec{r}(t) = \langle 2 \cos(t), 2 \sin(t), 3 \rangle$$

Solution

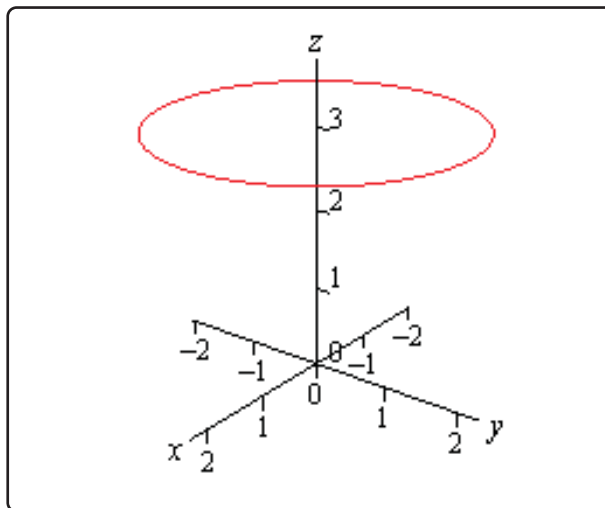
In this case to see what we've got for a graph let's get the parametric equations for the curve.

$$x = 2 \cos(t) \quad y = 2 \sin(t) \quad z = 3$$

If we ignore the z equation for a bit we'll **recall** (hopefully) that the parametric equations for x and y give a circle of radius 2 centered on the origin (or about the z -axis since we are in \mathbb{R}^3).

Now, all the parametric equations here tell us is that no matter what is going on in the graph all the z coordinates must be 3. So, we get a circle of radius 2 centered on the z -axis and at the level of $z = 3$.

Here is a sketch.



Note that it is very easy to modify the above vector function to get a circle centered on the x or y -axis as well. For instance,

$$\vec{r}(t) = \langle 10 \sin(t), -3, 10 \cos(t) \rangle$$

will be a circle of radius 10 centered on the y -axis and at $y = -3$. In other words, as long as two of the terms are a sine and a cosine (with the same coefficient) and the other is a fixed number then we will have a circle that is centered on the axis that is given by the fixed number.

Let's take a look at a modification of this.

Example 6

Sketch the graph of the following vector function.

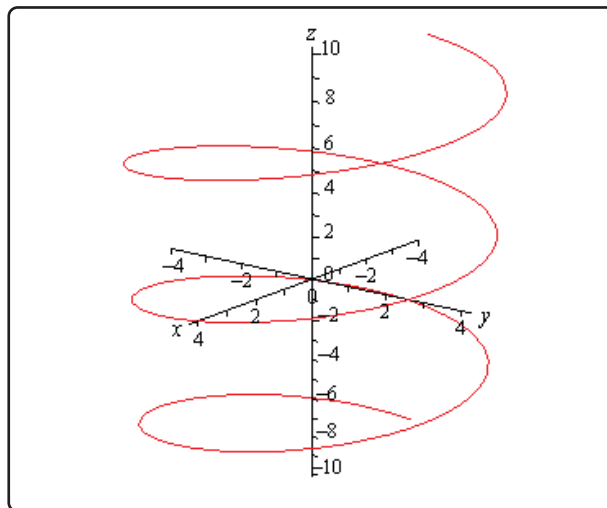
$$\vec{r}(t) = \langle 4 \cos(t), 4 \sin(t), t \rangle$$

Solution

If this one had a constant in the z component we would have another circle. However, in this case we don't have a constant. Instead we've got a t and that will change the curve. However, because the x and y component functions are still a circle in parametric equations our curve should have a circular nature to it in some way.

In fact, the only change is in the z component and as t increases the z coordinate will increase. Also, as t increases the x and y coordinates will continue to form a circle centered on the z -axis. Putting these two ideas together tells us that as we increase t the circle that is being traced out in the x and y directions should also be rising.

Here is a sketch of this curve.



So, we've got a helix (or spiral, depending on what you want to call it) here.

As with circles the component that has the t will determine the axis that the helix rotates about. For instance,

$$\vec{r}(t) = \langle t, 6 \cos(t), 6 \sin(t) \rangle$$

is a helix that rotates around the x -axis.

Also note that if we allow the coefficients on the sine and cosine for both the circle and helix to be different we will get ellipses.

For example,

$$\vec{r}(t) = \langle 9 \cos(t), t, 2 \sin(t) \rangle$$

will be a helix that rotates about the y -axis and is in the shape of an ellipse.

There is a nice formula that we should derive before moving onto vector functions of two variables.

Example 7

Determine the vector equation for the line segment starting at the point $P = (x_1, y_1, z_1)$ and ending at the point $Q = (x_2, y_2, z_2)$.

Solution

It is important to note here that we only want the equation of the line segment that starts at P and ends at Q . We don't want any other portion of the line and we do want the direction of the line segment preserved as we increase t . With all that said, let's not worry about that and just find the vector equation of the line that passes through the two points. Once we have this we will be able to get what we're after.

So, we need a point on the line. We've got two and we will use P . We need a vector that is parallel to the line and since we've got two points we can find the vector between them. This vector will lie on the line and hence be parallel to the line. Also, let's remember that we want to preserve the starting and ending point of the line segment so let's construct the vector using the same "orientation".

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Using this vector and the point P we get the following vector equation of the line.

$$\vec{r}(t) = \langle x_1, y_1, z_1 \rangle + t \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

While this is the vector equation of the line, let's rewrite the equation slightly.

$$\begin{aligned} \vec{r}(t) &= \langle x_1, y_1, z_1 \rangle + t \langle x_2, y_2, z_2 \rangle - t \langle x_1, y_1, z_1 \rangle \\ &= (1 - t) \langle x_1, y_1, z_1 \rangle + t \langle x_2, y_2, z_2 \rangle \end{aligned}$$

This is the equation of the line that contains the points P and Q . We of course just want the line segment that starts at P and ends at Q . We can get this by simply restricting the values of t .

Notice that

$$\vec{r}(0) = \langle x_1, y_1, z_1 \rangle \quad \vec{r}(1) = \langle x_2, y_2, z_2 \rangle$$

So, if we restrict t to be between zero and one we will cover the line segment and we will start and end at the correct point.

So, the vector equation of the line segment that starts at $P = (x_1, y_1, z_1)$ and ends at $Q = (x_2, y_2, z_2)$ is,

$$\vec{r}(t) = (1 - t) \langle x_1, y_1, z_1 \rangle + t \langle x_2, y_2, z_2 \rangle \quad 0 \leq t \leq 1$$

As noted briefly at the beginning of this section we can also have vector functions of two variables. In these cases the graphs of vector function of two variables are surfaces. So, to make sure that we don't forget that let's work an example with that as well.

Example 8

Identify the surface that is described by $\vec{r}(x, y) = x\vec{i} + y\vec{j} + (x^2 + y^2)\vec{k}$.

Solution

First, notice that in this case the vector function will in fact be a function of two variables. This will always be the case when we are using vector functions to represent surfaces.

To identify the surface let's go back to parametric equations.

$$x = x \quad y = y \quad z = x^2 + y^2$$

The first two are really only acknowledging that we are picking x and y for free and then determining z from our choices of these two. The last equation is the one that we want. We should recognize that function from the section on [quadric surfaces](#). The third equation is the equation of an elliptic paraboloid and so the vector function represents an elliptic paraboloid.

As a final topic for this section let's generalize the idea from the previous example and note that given any function of one variable ($y = f(x)$ or $x = h(y)$) or any function of two variables ($z = g(x, y)$, $x = g(y, z)$, or $y = g(x, z)$) we can always write down a vector form of the equation.

For a function of one variable this will be,

$$\vec{r}(x) = x\vec{i} + f(x)\vec{j} \quad \vec{r}(y) = h(y)\vec{i} + y\vec{j}$$

and for a function of two variables the vector form will be,

$$\vec{r}(x, y) = x\vec{i} + y\vec{j} + g(x, y)\vec{k} \quad \vec{r}(y, z) = g(y, z)\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r}(x, z) = x\vec{i} + g(x, z)\vec{j} + z\vec{k}$$

depending upon the original form of the function.

For example, the hyperbolic paraboloid $y = 2x^2 - 5z^2$ can be written as the following vector function.

$$\vec{r}(x, z) = x\vec{i} + (2x^2 - 5z^2)\vec{j} + z\vec{k}$$

This is a fairly important idea and we will be doing quite a bit of this kind of thing in multi-variable Calculus.

12.7 Calculus with Vector Functions

In this section we need to talk briefly about limits, derivatives and integrals of vector functions. As you will see, these behave in a fairly predictable manner. We will be doing all of the work in \mathbb{R}^3 but we can naturally extend the formulas/work in this section to \mathbb{R}^n (i.e. n -dimensional space).

Let's start with limits. Here is the limit of a vector function.

Vector Function Limits

$$\begin{aligned}\lim_{t \rightarrow a} \vec{r}(t) &= \lim_{t \rightarrow a} \langle f(t), g(t), h(t) \rangle \\ &= \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle \\ &= \lim_{t \rightarrow a} f(t) \vec{i} + \lim_{t \rightarrow a} g(t) \vec{j} + \lim_{t \rightarrow a} h(t) \vec{k}\end{aligned}$$

So, all that we do is take the limit of each of the component's functions and leave it as a vector.

Example 1

Compute $\lim_{t \rightarrow 1} \vec{r}(t)$ where $\vec{r}(t) = \left\langle t^3, \frac{\sin(3t-3)}{t-1}, e^{2t} \right\rangle$.

Solution

There really isn't all that much to do here.

$$\begin{aligned}\lim_{t \rightarrow 1} \vec{r}(t) &= \left\langle \lim_{t \rightarrow 1} t^3, \lim_{t \rightarrow 1} \frac{\sin(3t-3)}{t-1}, \lim_{t \rightarrow 1} e^{2t} \right\rangle \\ &= \left\langle \lim_{t \rightarrow 1} t^3, \lim_{t \rightarrow 1} \frac{3 \cos(3t-3)}{1}, \lim_{t \rightarrow 1} e^{2t} \right\rangle \\ &= \langle 1, 3, e^2 \rangle\end{aligned}$$

Notice that we had to use [L'Hospital's Rule](#) on the y component.

Now let's take care of derivatives and after seeing how limits work it shouldn't be too surprising that we have the following for derivatives.

Vector Function Derivative

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t) \vec{i} + g'(t) \vec{j} + h'(t) \vec{k}$$

Example 2

Compute $\vec{r}'(t)$ for $\vec{r}(t) = t^6 \vec{i} + \sin(2t) \vec{j} - \ln(t+1) \vec{k}$.

Solution

There really isn't too much to this problem other than taking the derivatives.

$$\vec{r}'(t) = 6t^5 \vec{i} + 2 \cos(2t) \vec{j} - \frac{1}{t+1} \vec{k}$$

Most of the basic facts that we know about derivatives still hold however, just to make it clear here are some facts about derivatives of vector functions.

Facts

$$\frac{d}{dt}(\vec{u} + \vec{v}) = \vec{u}' + \vec{v}'$$

$$\frac{d}{dt}(c\vec{u}) = c\vec{u}'$$

$$\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

$$\frac{d}{dt}(\vec{u} \cdot \vec{v}) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

$$\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$$

$$\frac{d}{dt}(\vec{u}(f(t))) = f'(t)\vec{u}'(f(t))$$

There is also one quick definition that we should get out of the way so that we can use it when we need to.

A **smooth curve** is any curve for which $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$ for any t except possibly at the endpoints. A helix is a smooth curve, for example.

Finally, we need to discuss integrals of vector functions. Using both limits and derivatives as a guide it shouldn't be too surprising that we also have the following for integration for indefinite integrals

Vector Function Indefinite Integral

$$\int \vec{r}(t) dt = \left\langle \int f(t) dt, \int g(t) dt, \int h(t) dt \right\rangle + \vec{c}$$

$$\int \vec{r}(t) dt = \int f(t) dt \vec{i} + \int g(t) dt \vec{j} + \int h(t) dt \vec{k} + \vec{c}$$

and the following for definite integrals.

Vector Function Definite Integral

$$\int_a^b \vec{r}(t) dt = \left\langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right\rangle$$

$$\int_a^b \vec{r}(t) dt = \int_a^b f(t) dt \vec{i} + \int_a^b g(t) dt \vec{j} + \int_a^b h(t) dt \vec{k}$$

With the indefinite integrals we put in a constant of integration to make sure that it was clear that the constant in this case needs to be a vector instead of a regular constant.

Also, for the definite integrals we will sometimes write it as follows,

$$\int_a^b \vec{r}(t) dt = \left(\left\langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right\rangle \right) \Big|_a^b$$

$$\int_a^b \vec{r}(t) dt = \left(\int_a^b f(t) dt \vec{i} + \int_a^b g(t) dt \vec{j} + \int_a^b h(t) dt \vec{k} \right) \Big|_a^b$$

In other words, we will do the indefinite integral and then do the evaluation of the vector as a whole instead of on a component by component basis.

Example 3

Compute $\int \vec{r}(t) dt$ for $\vec{r}(t) = \langle \sin(t), 6, 4t \rangle$.

Solution

All we need to do is integrate each of the components and be done with it.

$$\int \vec{r}(t) dt = \langle -\cos(t), 6t, 2t^2 \rangle + \vec{c}$$

Example 4

Compute $\int_0^1 \vec{r}(t) dt$ for $\vec{r}(t) = \langle \sin(t), 6, 4t \rangle$.

Solution

In this case all that we need to do is reuse the result from the previous example and then do the evaluation.

$$\begin{aligned}\int_0^1 \vec{r}(t) dt &= \left\langle \int_0^1 -\cos(t) dt, \int_0^1 6 dt, \int_0^1 4t dt \right\rangle \\ &= \langle -\cos(1), 6, 2 \rangle - \langle -1, 0, 0 \rangle \\ &= \langle 1 - \cos(1), 6, 2 \rangle\end{aligned}$$

12.8 Tangent, Normal and Binormal Vectors

In this section we want to look at an application of derivatives for vector functions. Actually, there are a couple of applications, but they all come back to needing the first one.

In the past we've used the fact that the derivative of a function was the slope of the tangent line. With vector functions we get exactly the same result, with one exception.

Given the vector function, $\vec{r}(t)$, we call $\vec{r}'(t)$ the **tangent vector** provided it exists and provided $\vec{r}'(t) \neq \vec{0}$. The tangent line to $\vec{r}(t)$ at P is then the line that passes through the point P and is parallel to the tangent vector, $\vec{r}'(t)$. Note that we really do need to require $\vec{r}'(t) \neq \vec{0}$ in order to have a tangent vector. If we had

$$\vec{r}'(t) = \vec{0}$$

we would have a vector that had no magnitude and so couldn't give us the direction of the tangent.

Also, provided $\vec{r}'(t) \neq \vec{0}$, the **unit tangent vector** to the curve is given by,

Unit Tangent Vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

While, the components of the unit tangent vector can be somewhat messy on occasion there are times when we will need to use the unit tangent vector instead of the tangent vector.

Example 1

Find the general formula for the tangent vector and unit tangent vector to the curve given by $\vec{r}(t) = t^2 \vec{i} + 2 \sin(t) \vec{j} + 2 \cos(t) \vec{k}$.

Solution

First, by general formula we mean that we won't be plugging in a specific t and so we will be finding a formula that we can use at a later date if we'd like to find the tangent at any point on the curve. With that said there really isn't all that much to do at this point other than to do the work.

Here is the tangent vector to the curve.

$$\vec{r}'(t) = 2t \vec{i} + 2 \cos(t) \vec{j} - 2 \sin(t) \vec{k}$$

To get the unit tangent vector we need the length of the tangent vector.

$$\begin{aligned}\|\vec{r}'(t)\| &= \sqrt{4t^2 + 4\cos^2(t) + 4\sin^2(t)} \\ &= \sqrt{4t^2 + 4}\end{aligned}$$

The unit tangent vector is then,

$$\begin{aligned}\vec{T}(t) &= \frac{1}{\sqrt{4t^2 + 4}} \left(2t\vec{i} + 2\cos t\vec{j} - 2\sin(t)\vec{k} \right) \\ &= \frac{2t}{\sqrt{4t^2 + 4}}\vec{i} + \frac{2\cos(t)}{\sqrt{4t^2 + 4}}\vec{j} - \frac{2\sin(t)}{\sqrt{4t^2 + 4}}\vec{k}\end{aligned}$$

Example 2

Find the vector equation of the tangent line to the curve given by

$$\vec{r}(t) = t^2\vec{i} + 2\sin(t)\vec{j} + 2\cos(t)\vec{k} \text{ at } t = \frac{\pi}{3}.$$

Solution

First, we need the tangent vector and since this is the function we were working with in the previous example we can just reuse the tangent vector from that example and plug in $t = \frac{\pi}{3}$.

$$\vec{r}'\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}\vec{i} + 2\cos\left(\frac{\pi}{3}\right)\vec{j} - 2\sin\left(\frac{\pi}{3}\right)\vec{k} = \frac{2\pi}{3}\vec{i} + \vec{j} - \sqrt{3}\vec{k}$$

We'll also need the point on the line at $t = \frac{\pi}{3}$ so,

$$\vec{r}\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}\vec{i} + \sqrt{3}\vec{j} + \vec{k}$$

The vector equation of the line is then,

$$\vec{r}(t) = \left\langle \frac{\pi^2}{9}, \sqrt{3}, 1 \right\rangle + t \left\langle \frac{2\pi}{3}, 1, -\sqrt{3} \right\rangle$$

Before moving on let's note a couple of things about the previous example. First, we could have used the unit tangent vector had we wanted to for the parallel vector. However, that would have made for a more complicated equation for the tangent line.

Second, notice that we used $\vec{r}(t)$ to represent the tangent line despite the fact that we used that as well for the function. Do not get excited about that. The $\vec{r}(t)$ here is much like y is with normal functions. With normal functions, y is the generic letter that we used to represent functions and

$\vec{r}(t)$ tends to be used in the same way with vector functions.

Next, we need to talk about the **unit normal** and the **binormal** vectors.

The unit normal vector is defined to be,

Unit Normal Vector

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

The unit normal is orthogonal (or normal, or perpendicular) to the unit tangent vector and hence to the curve as well. We've already seen normal vectors when we were dealing with [Equations of Planes](#). They will show up with some regularity in several Calculus III topics.

The definition of the unit normal vector always seems a little mysterious when you first see it. It follows directly from the following fact.

Fact

Suppose that $\vec{r}(t)$ is a vector such that $\|\vec{r}(t)\| = c$ for all t . Then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

Proof

To prove this fact is pretty simple. From the fact statement and the relationship between the magnitude of a vector and the dot product we have the following.

$$\vec{r}(t) \cdot \vec{r}(t) = \|\vec{r}(t)\|^2 = c^2 \quad \text{for all } t$$

Now, because this is true for all t we can see that,

$$\frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \frac{d}{dt}(c^2) = 0$$

Also, recalling the fact from the previous section about differentiating a dot product we see that,

$$\frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 2\vec{r}'(t) \cdot \vec{r}(t)$$

Or, upon putting all this together we get,

$$2\vec{r}'(t) \cdot \vec{r}(t) = 0 \quad \Rightarrow \quad \vec{r}'(t) \cdot \vec{r}(t) = 0$$

Therefore $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

The definition of the unit normal then falls directly from this. Because $\vec{T}(t)$ is a unit vector we know that $\|\vec{T}(t)\| = 1$ for all t and hence by the Fact $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$. However, because $\vec{T}(t)$ is tangent to the curve, $\vec{T}'(t)$ must be orthogonal, or normal, to the curve as well and so be a normal vector for the curve. All we need to do then is divide by $\|\vec{T}'(t)\|$ to arrive at a unit normal vector.

Next, is the binormal vector. The binormal vector is defined to be,

Binormal Vector

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Because the binormal vector is defined to be the cross product of the unit tangent and unit normal vector we then know that the binormal vector is orthogonal to both the tangent vector and the normal vector.

Example 3

Find the normal and binormal vectors for $\vec{r}(t) = \langle t, 3 \sin(t), 3 \cos(t) \rangle$.

Solution

We first need the unit tangent vector so first get the tangent vector and its magnitude.

$$\begin{aligned}\vec{r}'(t) &= \langle 1, 3 \cos(t), -3 \sin(t) \rangle \\ \|\vec{r}'(t)\| &= \sqrt{1 + 9 \cos^2(t) + 9 \sin^2(t)} = \sqrt{10}\end{aligned}$$

The unit tangent vector is then,

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \cos(t), -\frac{3}{\sqrt{10}} \sin(t) \right\rangle$$

The unit normal vector will now require the derivative of the unit tangent and its magnitude.

$$\begin{aligned}\vec{T}'(t) &= \left\langle 0, -\frac{3}{\sqrt{10}} \sin(t), -\frac{3}{\sqrt{10}} \cos(t) \right\rangle \\ \|\vec{T}'(t)\| &= \sqrt{\frac{9}{10} \sin^2(t) + \frac{9}{10} \cos^2(t)} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}\end{aligned}$$

The unit normal vector is then,

$$\vec{N}(t) = \frac{\sqrt{10}}{3} \left\langle 0, -\frac{3}{\sqrt{10}} \sin(t), -\frac{3}{\sqrt{10}} \cos(t) \right\rangle = \langle 0, -\sin(t), -\cos(t) \rangle$$

Finally, the binormal vector is,

$$\begin{aligned}\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \cos(t) & -\frac{3}{\sqrt{10}} \sin(t) \\ 0 & -\sin(t) & -\cos(t) \end{vmatrix} \\ &= -\frac{3}{\sqrt{10}} \cos^2(t) \vec{i} - \frac{1}{\sqrt{10}} \sin(t) \vec{k} + \frac{1}{\sqrt{10}} \cos(t) \vec{j} - \frac{3}{\sqrt{10}} \sin^2(t) \vec{i} \\ &= -\frac{3}{\sqrt{10}} \vec{i} + \frac{1}{\sqrt{10}} \cos(t) \vec{j} - \frac{1}{\sqrt{10}} \sin(t) \vec{k}\end{aligned}$$

12.9 Arc Length with Vector Functions

In this section we'll recast an old formula into terms of vector functions. We want to determine the length of a vector function,

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

on the interval $a \leq t \leq b$.

We actually already know how to do this. Recall that we can write the vector function into the parametric form,

$$x = f(t) \quad y = g(t) \quad z = h(t)$$

Also, **recall** that with two dimensional parametric curves the arc length is given by,

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

There is a natural extension of this to three dimensions. So, the length of the curve $\vec{r}(t)$ on the interval $a \leq t \leq b$ is,

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

There is a nice simplification that we can make for this. Notice that the integrand (the function we're integrating) is nothing more than the magnitude of the tangent vector,

$$\|\vec{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Therefore, the arc length can be written as,

Arc Length

$$L = \int_a^b \|\vec{r}'(t)\| dt$$

Let's work a quick example of this.

Example 1

Determine the length of the curve $\vec{r}(t) = \langle 2t, 3 \sin(2t), 3 \cos(2t) \rangle$ on the interval $0 \leq t \leq 2\pi$.

Solution

We will first need the tangent vector and its magnitude.

$$\begin{aligned}\vec{r}'(t) &= \langle 2, 6 \cos(2t), -6 \sin(2t) \rangle \\ \|\vec{r}'(t)\| &= \sqrt{4 + 36 \cos^2(2t) + 36 \sin^2(2t)} = \sqrt{4 + 36} = 2\sqrt{10}\end{aligned}$$

The length is then,

$$\begin{aligned}L &= \int_a^b \|\vec{r}'(t)\| dt \\ &= \int_0^{2\pi} 2\sqrt{10} dt \\ &= 4\pi\sqrt{10}\end{aligned}$$

We need to take a quick look at another concept here. We define the **arc length function** as,

Arc Length Function

$$s(t) = \int_0^t \|\vec{r}'(u)\| du$$

Before we look at why this might be important let's work a quick example.

Example 2

Determine the arc length function for $\vec{r}(t) = \langle 2t, 3 \sin(2t), 3 \cos(2t) \rangle$.

Solution

From the previous example we know that,

$$\|\vec{r}'(t)\| = 2\sqrt{10}$$

The arc length function is then,

$$s(t) = \int_0^t 2\sqrt{10} du = \left(2\sqrt{10}u\right)\Big|_0^t = 2\sqrt{10}t$$

Okay, just why would we want to do this? Well let's take the result of the example above and solve it for t .

$$t = \frac{s}{2\sqrt{10}}$$

Now, taking this and plugging it into the original vector function and we can **reparametrize** the function into the form, $\vec{r}(t(s))$. For our function this is,

$$\vec{r}(t(s)) = \left\langle \frac{s}{\sqrt{10}}, 3 \sin\left(\frac{s}{\sqrt{10}}\right), 3 \cos\left(\frac{s}{\sqrt{10}}\right) \right\rangle$$

So, why would we want to do this? Well with the reparameterization we can now tell where we are on the curve after we've traveled a distance of s along the curve. Note as well that we will start the measurement of distance from where we are at $t = 0$.

Example 3

Where on the curve $\vec{r}(t) = \langle 2t, 3 \sin(2t), 3 \cos(2t) \rangle$ are we after traveling a distance of $\frac{\pi\sqrt{10}}{3}$?

Solution

To determine this we need the reparameterization, which we have from above.

$$\vec{r}(t(s)) = \left\langle \frac{s}{\sqrt{10}}, 3 \sin\left(\frac{s}{\sqrt{10}}\right), 3 \cos\left(\frac{s}{\sqrt{10}}\right) \right\rangle$$

Then, to determine where we are all that we need to do is plug in $s = \frac{\pi\sqrt{10}}{3}$ into this and we'll get our location.

$$\vec{r}\left(t\left(\frac{\pi\sqrt{10}}{3}\right)\right) = \left\langle \frac{\pi}{3}, 3 \sin\left(\frac{\pi}{3}\right), 3 \cos\left(\frac{\pi}{3}\right) \right\rangle = \left\langle \frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2} \right\rangle$$

So, after traveling a distance of $\frac{\pi\sqrt{10}}{3}$ along the curve we are at the point $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.