1 Basic Concepts

This first chapter is a very short chapter. The main focus of this chapter is to introduce some of the basic definitions and concepts that we'll see on a regular basis in the class. During a typical class we usually introduce many of these topics right as they are needed rather than spending time on them here. This is done purely from a time perspective. Time is generally at a premium in the class and so it is simply easier to introduce them right as we need them. However, for those that wish to have many/most of the basic definitions/concepts all in one place we put them here.

In addition we'll take a quick look at direction fields and how we can use them to determine some information about the solution to a differential equation without actually having the solution.

1.1 Definitions

Differential Equation

The first definition that we should cover should be that of **differential equation**. A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

There is one differential equation that everybody probably knows, that is Newton's Second Law of Motion. If an object of mass m is moving with acceleration a and being acted on with force F then Newton's Second Law tells us.

$$F = ma \tag{1.1}$$

To see that this is in fact a differential equation we need to rewrite it a little. First, remember that we can rewrite the acceleration, a, in one of two ways.

$$a = \frac{dv}{dt} \qquad \text{OR} \qquad a = \frac{d^2u}{dt^2} \tag{1.2}$$

Where v is the velocity of the object and u is the position function of the object at any time t. We should also remember at this point that the force, F may also be a function of time, velocity, and/or position.

So, with all these things in mind Newton's Second Law can now be written as a differential equation in terms of either the velocity, v, or the position, u, of the object as follows.

$$m\frac{dv}{dt} = F\left(t, v\right) \tag{1.3}$$

$$m\frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right) \tag{1.4}$$

So, here is our first differential equation. We will see both forms of this in later chapters.

Here are a few more examples of differential equations.

$$ay'' + by' + cy = g\left(t\right) \tag{1.5}$$

$$\sin\left(y\right)\frac{d^2y}{dx^2} = \left(1 - y\right)\frac{dy}{dx} + y^2\mathbf{e}^{-5y} \tag{1.6}$$

$$y^{(4)} + 10y''' - 4y' + 2y = \cos\left(t\right) \tag{1.7}$$

$$\alpha^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1.8}$$

$$a^2 u_{xx} = u_{tt} \tag{1.9}$$

$$\frac{\partial^3 u}{\partial x^2 \partial t} = 1 + \frac{\partial u}{\partial y} \tag{1.10}$$

Order

The **order** of a differential equation is the largest derivative present in the differential equation. In the differential equations listed above Equation 1.3 is a first order differential equation, Equation 1.4, Equation 1.5, Equation 1.6, Equation 1.8, and Equation 1.9 are second order differential equations, Equation 1.10 is a third order differential equation and Equation 1.7 is a fourth order differential equation.

Note that the order does not depend on whether or not you've got ordinary or partial derivatives in the differential equation.

We will be looking almost exclusively at first and second order differential equations in these notes. As you will see most of the solution techniques for second order differential equations can be easily (and naturally) extended to higher order differential equations and we'll discuss that idea later on.

Ordinary and Partial Differential Equations

A differential equation is called an **ordinary differential equation**, abbreviated by **ode,** if it has ordinary derivatives in it. Likewise, a differential equation is called a **partial differential equation**, abbreviated by **pde,** if it has partial derivatives in it. In the differential equations above Equation 1.3 - Equation 1.7 are ode's and Equation 1.8 - Equation 1.10 are pde's.

The vast majority of these notes will deal with ode's. The only exception to this will be the last chapter in which we'll take a brief look at a common and basic solution technique for solving pde's.

Linear Differential Equations

A **linear differential equation** is any differential equation that can be written in the following form.

$$a_n(t)\,y^{(n)}(t) + a_{n-1}(t)\,y^{(n-1)}(t) + \cdots + a_1(t)\,y'(t) + a_0(t)\,y(t) = g(t) \tag{1.11}$$

The important thing to note about linear differential equations is that there are no products of the function, $y(t)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power. Also note that neither the function or its derivatives are "inside" another function, for example, $\sqrt{y'}$ or $\mathbf{e}^{y}$.

The coefficients $a_0(t), \ \ldots \ , a_n(t)$ and $g(t)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, $y(t)$, and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, Equation 1.11 then it is called a **non-linear** differential equation.

In Equation 1.5 - Equation 1.7 above only Equation 1.6 is non-linear, the other two are linear differential equations. We can't classify Equation 1.3 and Equation 1.4 since we do not know what form the function F has. These could be either linear or non-linear depending on F.

Solution

A **solution** to a differential equation on an interval $\alpha < t < \beta$ is any function $y(t)$ which satisfies the differential equation in question on the interval $\alpha < t < \beta$. It is important to note that solutions are often accompanied by intervals and these intervals can impart some important information about the solution. Consider the following example.

Example 1

Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$.

Solution

We'll need the first and second derivative to do this.

$$y'(x) = -\frac{3}{2}x^{-\frac{5}{2}} \qquad y''(x) = \frac{15}{4}x^{-\frac{7}{2}}$$

Plug these as well as the function into the differential equation.

$$\begin{aligned} 4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right) + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{3}{2}}\right) &= 0 \\ 15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}} &= 0 \\ 0 &= 0 \end{aligned}$$

So, $y(x) = x^{-\frac{3}{2}}$ does satisfy the differential equation and hence is a solution. Why then did we include the condition that $x > 0$? We did not use this condition anywhere in the work showing that the function would satisfy the differential equation.

To see why recall that

$$y(x) = x^{-\frac{3}{2}} = \frac{1}{\sqrt{x^3}}$$

In this form it is clear that we'll need to avoid $x = 0$ at the least as this would give division by zero.

Also, there is a general rule of thumb that we're going to run with in this class. This rule of thumb is : Start with real numbers, end with real numbers. In other words, if our differential equation only contains real numbers then we don't want solutions that give complex numbers. So, in order to avoid complex numbers we will also need to avoid negative values of x.

So, we saw in the last example that even though a function may symbolically satisfy a differential equation, because of certain restrictions brought about by the solution we cannot use all values of the independent variable and hence, must make a restriction on the independent variable. This will be the case with many solutions to differential equations.

In the last example, note that there are in fact many more possible solutions to the differential equation given. For instance, all of the following are also solutions

$$y(x) = x^{-\frac{1}{2}}$$
$$y(x) = -9x^{-\frac{3}{2}}$$
$$y(x) = 7x^{-\frac{1}{2}}$$
$$y(x) = -9x^{-\frac{3}{2}} + 7x^{-\frac{1}{2}}$$

We'll leave the details to you to check that these are in fact solutions. Given these examples can you come up with any other solutions to the differential equation? There are in fact an infinite number of solutions to this differential equation.

So, given that there are an infinite number of solutions to the differential equation in the last example (provided you believe us when we say that anyway....) we can ask a natural question. Which is the solution that we want or does it matter which solution we use? This question leads us to the next definition in this section.

Initial Condition(s)

Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated i.c.'s when we're feeling lazy...) are of the form,

$$y(t_0) = y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0) = y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points. As we will see eventually, solutions to "nice enough" differential equations are unique and hence only one solution will meet the given initial conditions.

The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation as we will see.

Example 2

$y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$, $y(4) = \frac{1}{8}$, and $y'(4) = -\frac{3}{64}$.

Solution

As we saw in previous example the function is a solution and we can then note that

$$y(4) = 4^{-\frac{3}{2}} = \frac{1}{\left(\sqrt{4}\right)^3} = \frac{1}{8}$$
$$y'(4) = -\frac{3}{2}4^{-\frac{5}{2}} = -\frac{3}{2}\frac{1}{\left(\sqrt{4}\right)^5} = -\frac{3}{64}$$

and so this solution also meets the initial conditions of $y(4) = \frac{1}{8}$ and $y'(4) = -\frac{3}{64}$. In fact, $y(x) = x^{-\frac{3}{2}}$ is the only solution to this differential equation that satisfies these two initial conditions.

Initial Value Problem

An **Initial Value Problem** (or **IVP**) is a differential equation along with an appropriate number of initial conditions.

Example 3

The following is an IVP.

$$4x^2y'' + 12xy' + 3y = 0 \qquad y(4) = \frac{1}{8}, \quad y'(4) = -\frac{3}{64}$$

Example 4

Here's another IVP.

$$2t\,y' + 4y = 3 \qquad y(1) = -4$$

As we noted earlier the number of initial conditions required will depend on the order of the differential equation.

Interval of Validity

The **interval of validity** for an IVP with initial condition(s

$$y(t_0) = y_0 \qquad \text{and/or } y^{(k)}(t_0) = y_k$$

is the largest possible interval on which the solution is valid and contains t_0. These are easy to define, but can be difficult to find, so we're going to put off saying anything more about these until we get into actually solving differential equations and need the interval of validity.

General Solution

The **general solution** to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

Example 5

$y(t) = \frac{3}{4} + \frac{c}{t^2}$ is the general solution to

$$2t\,y' + 4y = 3$$

We'll leave it to you to check that this function is in fact a solution to the given differential equation. In fact, all solutions to this differential equation will be in this form. This is one of the first differential equations that you will learn how to solve and you will be able to verify this shortly for yourself.

Actual Solution

The **actual solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Example 6

What is the actual solution to the following IVP

$$2t\,y' + 4y = 3 \qquad y(1) = -4$$

Solution

This is actually easier to do than it might at first appear. From the previous example we already know (well that is provided you believe our solution to this example...) that all solutions to the differential equation are of the form.

$$y(t) = \frac{3}{4} + \frac{c}{t^2}$$

All that we need to do is determine the value of c that will give us the solution that we're after. To find this all we need do is use our initial condition as follows.

$$-4 = y(1) = \frac{3}{4} + \frac{c}{1^2} \qquad \Rightarrow \qquad c = -4 - \frac{3}{4} = -\frac{19}{4}$$

So, the actual solution to the IVP is.

$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$

From this last example we can see that once we have the general solution to a differential equation finding the actual solution is nothing more than applying the initial condition(s) and solving for the constant(s) that are in the general solution.

Implicit/Explicit Solution

In this case it's easier to define an explicit solution, then tell you what an implicit solution isn't, and then give you an example to show you the difference. So, that's what we'll do.

An **explicit solution** is any solution that is given in the form $y = y(t)$. In other words, the only place that y actually shows up is once on the left side and only raised to the first power. An **implicit solution** is any solution that isn't in explicit form. Note that it is possible to have either general implicit/explicit solutions and actual implicit/explicit solutions.

Example 7

$y^2 = t^2 - 3$ is the actual implicit solution to $y' = \frac{t}{y}, \quad y(2) = -1$

At this point we will ask that you trust us that this is in fact a solution to the differential equation. You will learn how to get this solution in a later section. The point of this example is that since there is a y^2 on the left side instead of a single $y(t)$ this is not an explicit solution!

Example 8

Find an actual explicit solution to $y' = \frac{t}{y}, \quad y(2) = -1$.

Solution

We already know from the previous example that an implicit solution to this IVP is $y^2 = t^2 - 3$. To find the explicit solution all we need to do is solve for $y(t)$.

$$y(t) = \pm\sqrt{t^2 - 3}$$

Now, we've got a problem here. There are two functions here and we only want one and in fact only one will be correct! We can determine the correct function by reapplying the initial condition. Only one of them will satisfy the initial condition.

In this case we can see that the "-" solution will be the correct one. The actual explicit solution is then

$$y(t) = -\sqrt{t^2 - 3}$$

In this case we were able to find an explicit solution to the differential equation. It should be noted however that it will not always be possible to find an explicit solution.

Also, note that in this case we were only able to get the explicit actual solution because we had the initial condition to help us determine which of the two functions would be the correct solution.

We've now gotten most of the basic definitions out of the way and so we can move onto other topics.

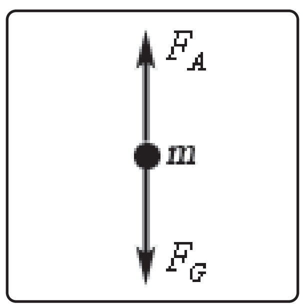
1.2 Direction Fields

This topic is given its own section for a couple of reasons. First, understanding direction fields and what they tell us about a differential equation and its solution is important and can be introduced without any knowledge of how to solve a differential equation and so can be done here before we get into solving them. So, having some information about the solution to a differential equation without actually having the solution is a nice idea that needs some investigation.

Next, since we need a differential equation to work with, this is a good section to show you that differential equations occur naturally in many cases and how we get them. Almost every physical situation that occurs in nature can be described with an appropriate differential equation. The differential equation may be easy or difficult to arrive at depending on the situation and the assumptions that are made about the situation and we may not ever be able to solve it, however it will exist.

The process of describing a physical situation with a differential equation is called modeling. We will be looking at modeling several times throughout this class.

One of the simplest physical situations to think of is a falling object. So, let's consider a falling object with mass m and derive a differential equation that, when solved, will give us the velocity of the object at any time, t. We will assume that only gravity and air resistance will act upon the object as it falls. Below is a figure showing the forces that will act upon the object.



Before defining all the terms in this problem we need to set some conventions. We will assume that forces acting in the downward direction are positive forces while forces that act in the upward direction are negative. Likewise, we will assume that an object moving downward (*i.e.* a falling object) will have a positive velocity.

Now, let's take a look at the forces shown in the diagram above. F_G is the force due to gravity and is given by $F_G = mg$ where g is the acceleration due to gravity. In this class we use g = 9.8 m/s^2 or g = 32 ft/s^2 depending on whether we will use the metric or Imperial system. F_A is the force due to air resistance and for this example we will assume that it is proportional to the velocity, v, of the mass. Therefore, the force due to air resistance is then given by $F_A = -\gamma v$, where $\gamma > 0$. Note that the "$-$" is required to get the correct sign on the force. Both γ and v are positive and the force is acting upward and hence must be negative. The "$-$" will give us the correct sign and hence direction for this force.

Recall from the previous section that Newton's Second Law of motion can be written as

$$m\frac{dv}{dt} = F(t, v)$$

where $F(t, v)$ is the sum of forces that act on the object and may be a function of the time t and the velocity of the object, v. For our situation we will have two forces acting on the object gravity, $F_G = mg$. acting in the downward direction and hence will be positive, and air resistance, $F_A = -\gamma v$, acting in the upward direction and hence will be negative. Putting all of this together into Newton's Second Law gives the following.

$$m\frac{dv}{dt} = mg - \gamma v$$

To simplify the differential equation let's divide out the mass, m.

$$\frac{dv}{dt} = g - \frac{\gamma v}{m} \tag{1.12}$$

This then is a first order linear differential equation that, when solved, will give the velocity, v (in m/s), of a falling object of mass m that has both gravity and air resistance acting upon it.

In order to look at direction fields (that is after all the topic of this section....) it would be helpful to have some numbers for the various quantities in the differential equation. So, let's assume that we have a mass of 2 kg and that $\gamma = 0.392$. Plugging this into Equation 1.12 gives the following differential equation.

$$\frac{dv}{dt} = 9.8 - 0.196v \tag{1.13}$$

Let's take a geometric view of this differential equation. Let's suppose that for some time, t, the velocity just happens to be $v = 30$ m/s. Note that we're not saying that the velocity ever will be 30 m/s. All that we're saying is that let's suppose that by some chance the velocity does happen to be 30 m/s at some time t. So, if the velocity does happen to be 30 m/s at some time t we can plug $v = 30$ into Equation 1.13 to get.

$$\frac{dv}{dt} = 3.92$$

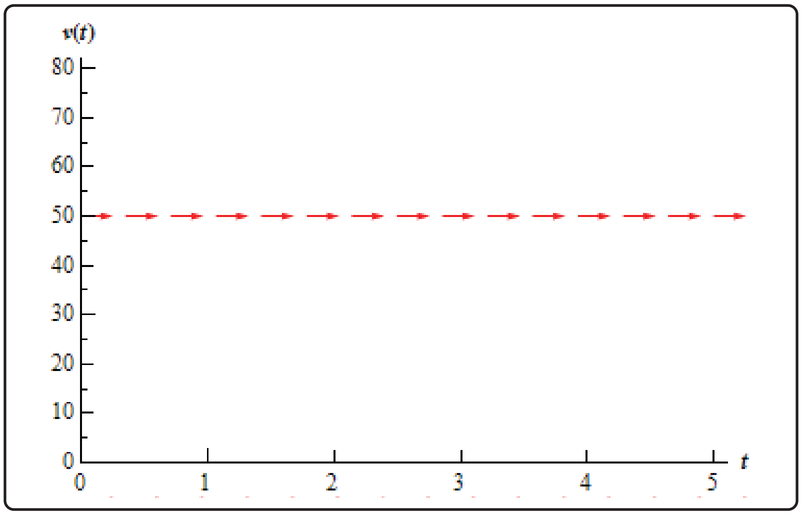
Recall from your Calculus I course that a positive derivative means that the function in question, the velocity in this case, is increasing, so if the velocity of this object is ever 30m/s for any time t the velocity must be increasing at that time.

Also, recall that the value of the derivative at a particular value of t gives the slope of the tangent line to the graph of the function at that time, t. So, if for some time t the velocity happens to be 30 m/s the slope of the tangent line to the graph of the velocity is 3.92.

We could continue in this fashion and pick different values of v and compute the slope of the tangent line for those values of the velocity. However, let's take a slightly more organized approach to this. Let's first identify the values of the velocity that will have zero slope or horizontal tangent lines. These are easy enough to find. All we need to do is set the derivative equal to zero and solve for v.

In the case of our example we will have only one value of the velocity which will have horizontal tangent lines, $v = 50$ m/s. What this means is that IF (again, there's that word if), for some time t, the velocity happens to be 50 m/s then the tangent line at that point will be horizontal. What the slope of the tangent line is at times before and after this point is not known yet and has no bearing on the slope at this particular time, t.

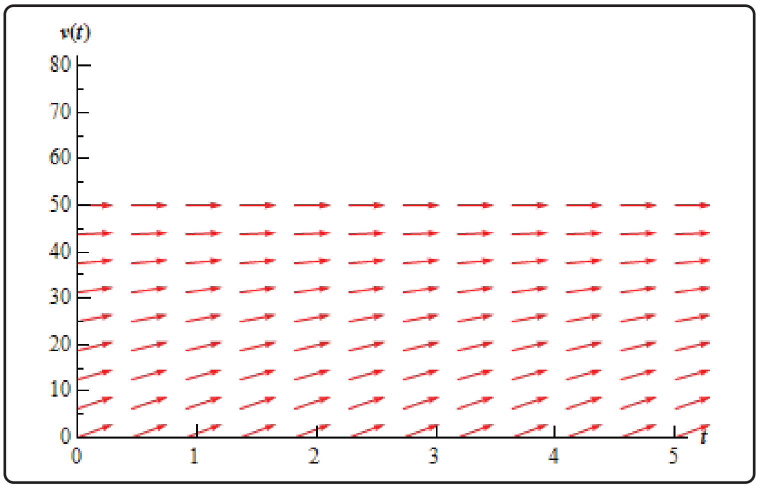
So, if we have $v = 50$, we know that the tangent lines will be horizontal. We denote this on an axis system with horizontal arrows pointing in the direction of increasing t at the level of $v = 50$ as shown in the following figure.



Now, let's get some tangent lines and hence arrows for our graph for some other values of v. At this point the only exact slope that is useful to us is where the slope horizontal. So instead of going after exact slopes for the rest of the graph we are only going to go after general trends in the slope. Is the slope increasing or decreasing? How fast is the slope increasing or decreasing? For this example those types of trends are very easy to get.

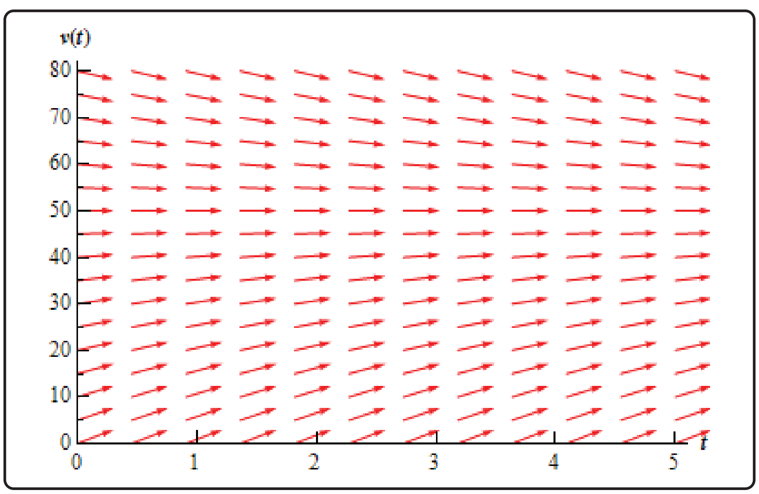
First, notice that the right hand side of Equation 1.13 is a polynomial and hence continuous. This means that it can only change sign if it first goes through zero. So, if the derivative will change signs (no guarantees that it will) it will do so at $v = 50$ and the only place that it may change sign is $v = 50$. This means that for $v > 50$ the slope of the tangent lines to the velocity will have the same sign. Likewise, for $v < 50$ the slopes will also have the same sign. The slopes in these ranges may have (and probably will) have different values, but we do know what their signs must be.

Let's start by looking at $v < 50$. We saw earlier that if $v = 30$ the slope of the tangent line will be 3.92, or positive. Therefore, for all values of $v < 50$ we will have positive slopes for the tangent lines. Also, by looking at Equation 1.13 we can see that as v approaches 50, always staying less than 50, the slopes of the tangent lines will approach zero and hence flatten out. If we move v away from 50, staying less than 50, the slopes of the tangent lines will become steeper. If you want to get an idea of just how steep the tangent lines become you can always pick specific values of v and compute values of the derivative. For instance, we know that at $v = 30$ the derivative is 3.92 and so arrows at this point should have a slope of around 4. Using this information, we can now add in some arrows for the region below $v = 50$ as shown in the graph below.



Now, let's look at $v > 50$. The first thing to do is to find out if the slopes are positive or negative. We will do this the same way that we did in the last bit, *i.e.* pick a value of v, plug this into Equation 1.13 and see if the derivative is positive or negative. Note, that you should NEVER assume that the derivative will change signs where the derivative is zero. It is easy enough to check so you should always do so.

We need to check the derivative so let's use $v = 60$. Plugging this into Equation 1.13 gives the slope of the tangent line as -1.96, or negative. Therefore, for all values of $v > 50$ we will have negative slopes for the tangent lines. As with $v < 50$, by looking at Equation 1.13 we can see that as v approaches 50, always staying greater than 50, the slopes of the tangent lines will approach zero and flatten out. While moving v away from 50 again, staying greater than 50, the slopes of the tangent lines will become steeper. We can now add in some arrows for the region above $v = 50$ as shown in the graph below.

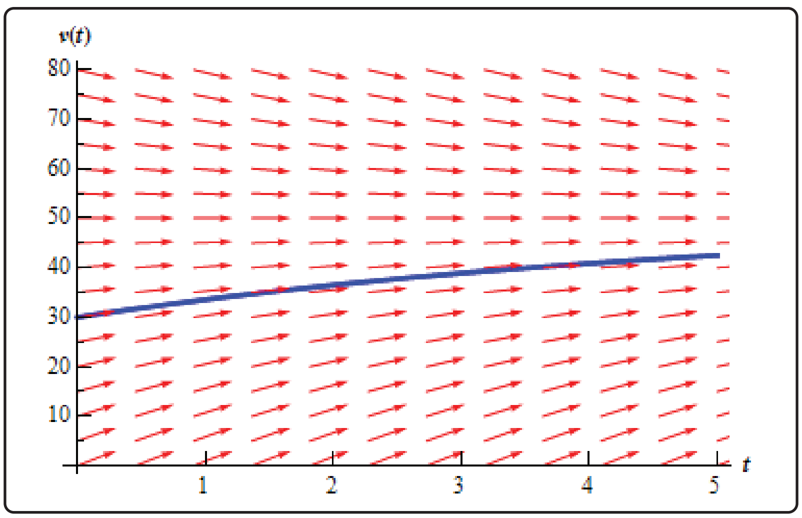


This graph above is called the **direction field** for the differential equation.

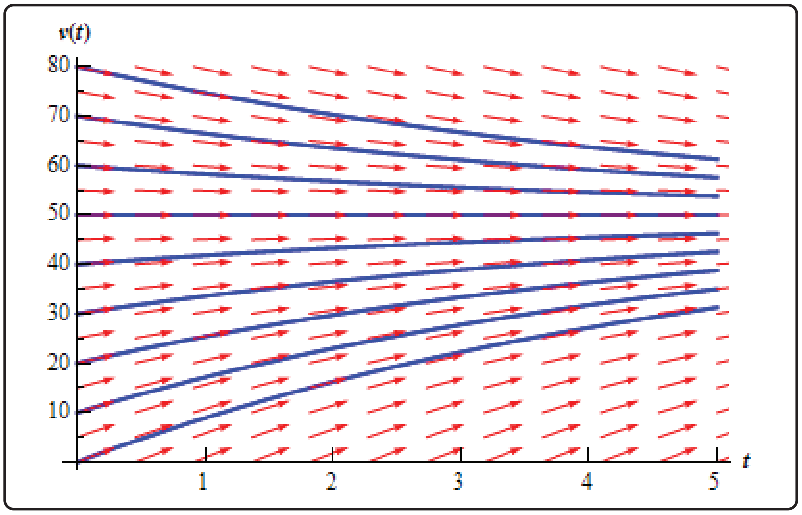
So, just why do we care about direction fields? There are two nice pieces of information that can be readily found from the direction field for a differential equation.

1. **Sketch of solutions**. Since the arrows in the direction fields are in fact tangents to the actual solutions to the differential equations we can use these as guides to sketch the graphs of solutions to the differential equation.
2. **Long Term Behavior**. In many cases we are less interested in the actual solutions to the differential equations as we are in how the solutions behave as t increases. Direction fields, if we can get our hands on them, can be used to find information about this long term behavior of the solution.

So, back to the direction field for our differential equation. Suppose that we want to know what the solution that has the value $v(0) = 30$ looks like. We can go to our direction field and start at 30 on the vertical axis. At this point we know that the solution is increasing and that as it increases the solution should flatten out because the velocity will be approaching the value of $v = 50$. So we start drawing an increasing solution and when we hit an arrow we just make sure that we stay parallel to that arrow. This gives us the figure below.



To get a better idea of how all the solutions are behaving, let's put a few more solutions in. Adding some more solutions gives the figure below. The set of solutions that we've graphed below is often called the **family of solution curves** or the set of **integral curves**. The number of solutions that is plotted when plotting the integral curves varies. You should graph enough solution curves to illustrate how solutions in all portions of the direction field are behaving.



Now, from either the direction field, or the direction field with the solution curves sketched in we can see the behavior of the solution as t increases. For our falling object, it looks like all of the solutions will approach $v = 50$ as t increases.

We will often want to know if the behavior of the solution will depend on the value of $v(0)$. In this case the behavior of the solution will not depend on the value of $v(0)$, but that is probably more of the exception than the rule so don't expect that.

Let's take a look at a more complicated example.

Example 1

Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation. Determine how the solutions behave as $t \to \infty$ and if this behavior depends on the value of $y(0)$ describe this dependency.

$$y' = \left(y^2 - y - 2\right)(1 - y)^2$$

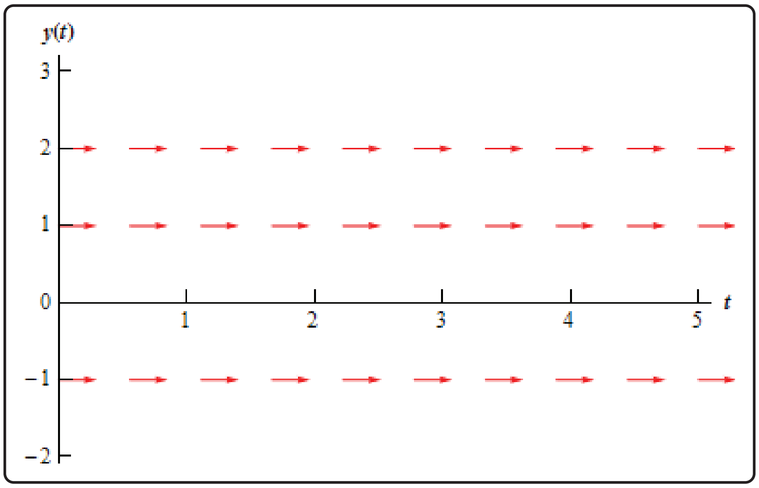
Solution

First, do not worry about where this differential equation came from. To be honest, we just made it up. It may, or may not describe an actual physical situation.

This differential equation looks somewhat more complicated than the falling object example from above. However, with the exception of a little more work, it is not much more complicated. The first step is to determine where the derivative is zero.

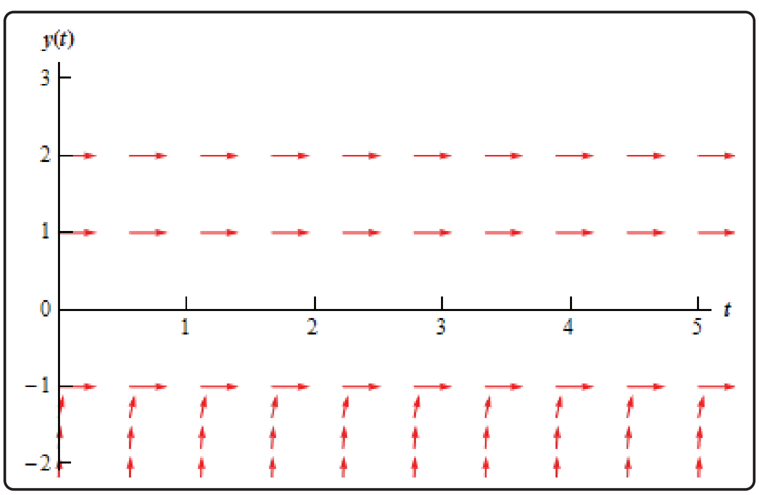
$$\begin{aligned}0 &= \left(y^2 - y - 2\right)(1 - y)^2 \\ 0 &= (y - 2)(y + 1)(1 - y)^2\end{aligned}$$

We can now see that we have three values of y in which the derivative, and hence the slope of tangent lines, will be zero. The derivative will be zero at $y = -1$, 1, and 2. So, let's start our direction field with drawing horizontal tangents for these values. This is shown in the figure below.



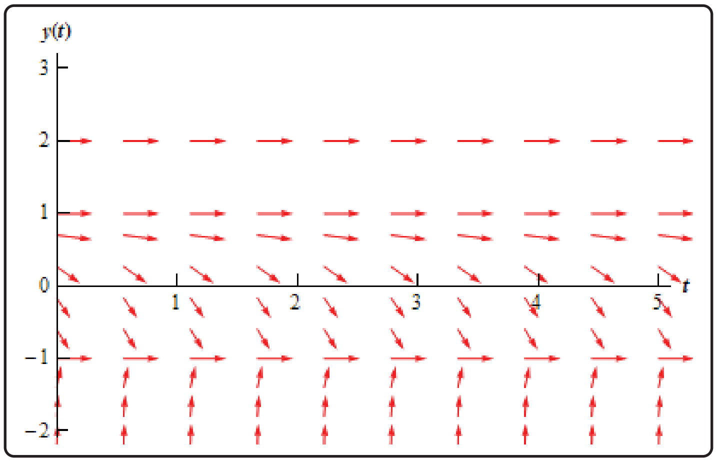
Now, we need to add arrows to the four regions that the graph is now divided into. For each of these regions I will pick a value of y in that region and plug it into the right hand side of the differential equation to see if the derivative is positive or negative in that region. Again, to get an accurate direction fields you should pick a few more values over the whole range to see how the arrows are behaving over the whole range.

$\underline{y < -1}$
In this region we can use $y = -2$ as the test point. At this point we have $y' = 36$. So, tangent lines in this region will have very steep and positive slopes. Also as $y \to -1$ the slopes will flatten out while staying positive. The figure below shows the direction fields with arrows in this region.



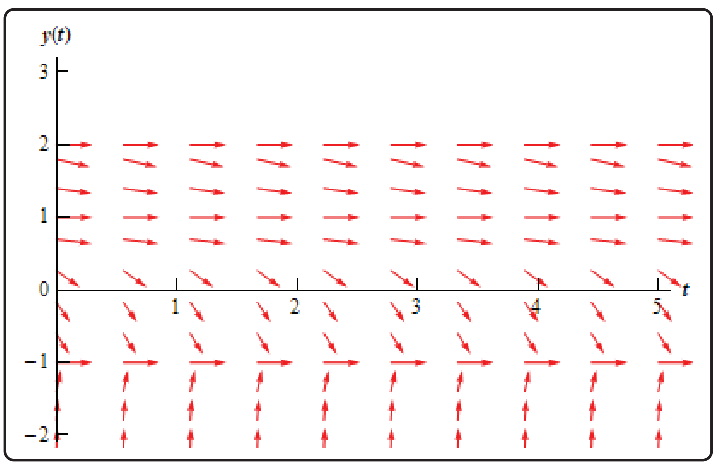
$-1 < y < 1$

In this region we can use $y = 0$ as the test point. At this point we have $y' = -2$. Therefore, tangent lines in this region will have negative slopes and apparently not be very steep. So what do the arrows look like in this region? As $y \to 1$ staying less than 1 of course, the slopes should be negative and approach zero. As we move away from 1 and towards -1 the slopes will start to get steeper (and stay negative), but eventually flatten back out, again staying negative, as $y \to -1$ since the derivative must approach zero at that point. The figure below shows the direction fields with arrows added to this region.



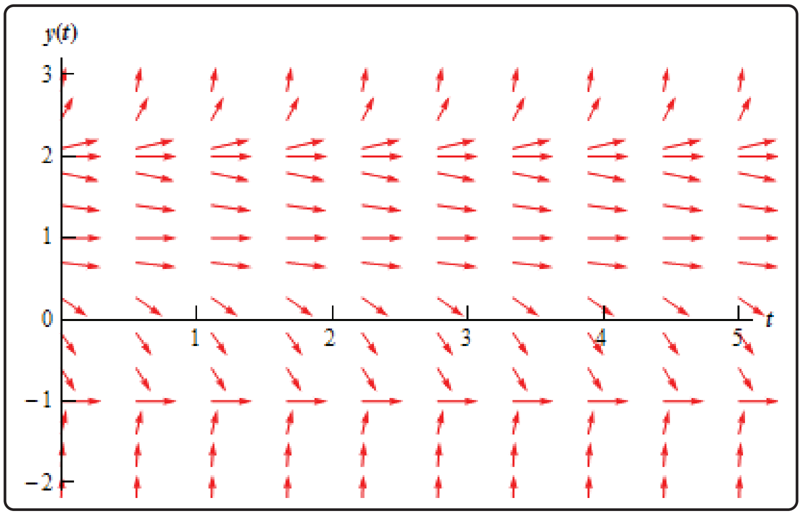
$1 < y < 2$

In this region we will use $y = 1.5$ as the test point. At this point we have $y' = -0.3125$. Tangent lines in this region will also have negative slopes and apparently not be as steep as the previous region. Arrows in this region will behave essentially the same as those in the previous region. Near $y = 1$ and $y = 2$ the slopes will flatten out and as we move from one to the other the slopes will get somewhat steeper before flattening back out. The figure below shows the direction fields with arrows added to this region.

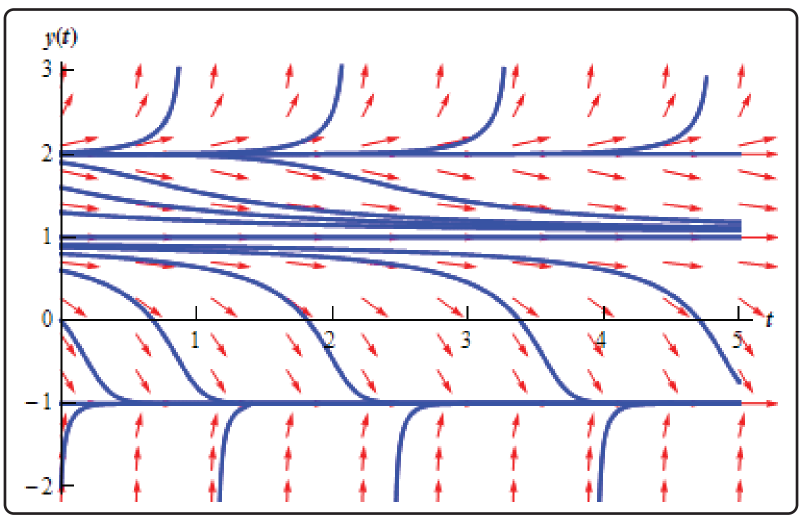


$\underline{y > 2}$
In this last region we will use $y = 3$ as the test point. At this point we have $y' = 16$. So, as we saw in the first region tangent lines will start out fairly flat near $y = 2$ and then as we move way from $y = 2$ they will get fairly steep.

The complete direction field for this differential equation is shown below.



Here is the set of integral curves for this differential equation.



Finally, let's take a look at long term behavior of all solutions. Unlike the first example, the long term behavior in this case will depend on the value of y at $t = 0$. By examining either of the previous two figures we can arrive at the following behavior of solutions as $t \to \infty$.

Value of $y(0)$	Behavior as $t \to \infty$
$y(0) < 1$	$y \to -1$
$1 \le y(0) < 2$	$y \to 1$
$y(0) = 2$	$y \to 2$
$y(0) > 2$	$y \to \infty$

Do not forget to acknowledge what the horizontal solutions are doing. This is often the most missed portion of this kind of problem.

In both of the examples that we've worked to this point the right hand side of the derivative has only contained the function and NOT the independent variable. When the right hand side of the differential equation contains both the function and the independent variable the behavior can be much more complicated and sketching the direction fields by hand can be very difficult. Computer software is very handy in these cases.

In some cases they aren't too difficult to do by hand however. Let's take a look at the following example.

Example 2

Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation.

$$y' = y - x$$

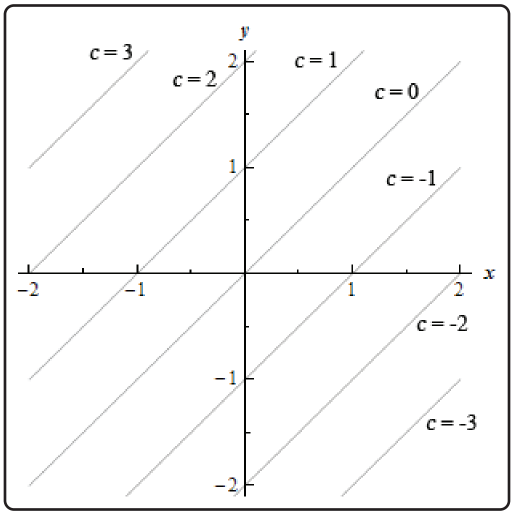
Solution

To sketch direction fields for this kind of differential equation we first identify places where the derivative will be constant. To do this we set the derivative in the differential equation equal to a constant, say c. This gives us a family of equations, called **isoclines**, that we can plot and on each of these curves the derivative will be a constant value of c.

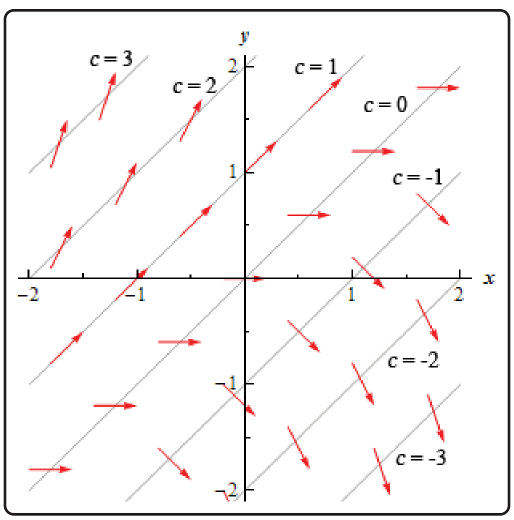
Notice that in the previous examples we looked at the isocline for $c = 0$ to get the direction field started. For our case the family of isoclines is.

$$c = y - x$$

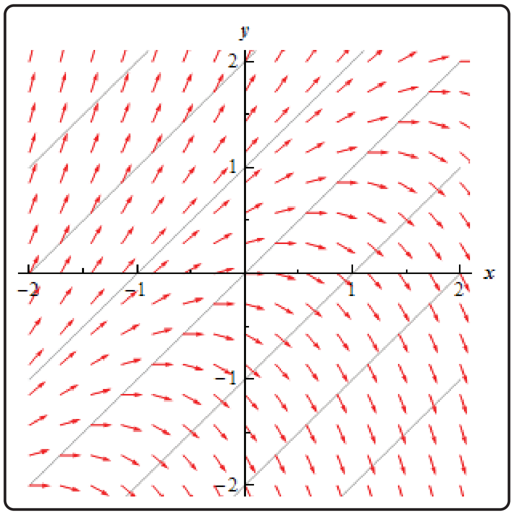
The graph of these curves for several values of c is shown below.



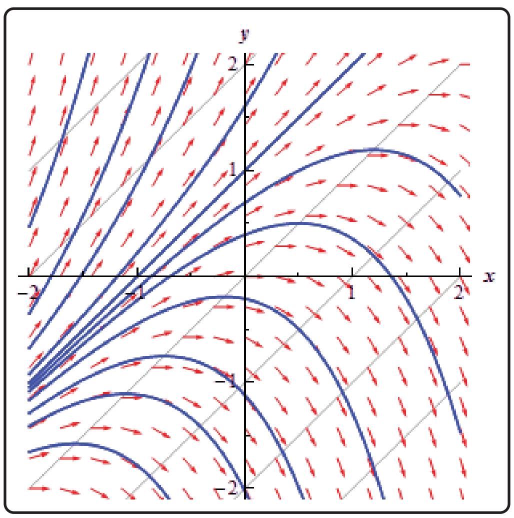
Now, on each of these lines, or isoclines, the derivative will be constant and will have a value of c. On the $c = 0$ isocline the derivative will always have a value of zero and hence the tangents will all be horizontal. On the $c = 1$ isocline the tangents will always have a slope of 1, on the $c = -2$ isocline the tangents will always have a slope of -2, *etc.* Below are a few tangents put in for each of these isoclines.



To add more arrows for those areas between the isoclines start at say, $c = 0$ and move up to $c = 1$ and as we do that we increase the slope of the arrows (tangents) from 0 to 1. This is shown in the figure below.



We can then add in integral curves as we did in the previous examples. This is shown in the figure below.



1.3 Final Thoughts

Before moving on to learning how to solve differential equations we want to give a few final thoughts. Any differential equations course will concern itself with answering one or more of the following questions.

1. **Given a differential equation will a solution exist?** Not all differential equations will have solutions so it's useful to know ahead of time if there is a solution or not. If there isn't a solution why waste our time trying to find something that doesn't exist?

   This question is usually called the **existence question** in a differential equations course.

2. **If a differential equation does have a solution how many solutions are there?** As we will see eventually, it is possible for a differential equation to have more than one solution. We would like to know how many solutions there will be for a given differential equation.

   There is a sub question here as well. What condition(s) on a differential equation are required to obtain a single unique solution to the differential equation?

   Both this question and the sub question are more important than you might realize. Suppose that we derive a differential equation that will give the temperature distribution in a bar of iron at any time t. If we solve the differential equation and end up with two (or more) completely separate solutions we will have problems. Consider the following situation to see this.

   If we subject 10 identical iron bars to identical conditions they should all exhibit the same temperature distribution. So only one of our solutions will be accurate, but we will have no way of knowing which one is the correct solution.

   It would be nice if, during the derivation of our differential equation, we could make sure that our assumptions would give us a differential equation that upon solving will yield a single unique solution.

   This question is usually called the **uniqueness question** in a differential equations course.

3. **If a differential equation does have a solution can we find it?** This may seem like an odd question to ask and yet the answer is not always yes. Just because we know that a solution to a differential equations exists does not mean that we will be able to find it.

In a first course in differential equations (such as this one) the third question is the question that we will concentrate on. We will answer the first two questions for special, and fairly simple, cases, but most of our efforts will be concentrated on answering the third question for as wide a variety of differential equations as possible.

2 First Order Differential Equations

In this chapter we will start our journey of learning how to solve differential equation. However, we won't be solving differential equations in general. What we'll be solving are special cases of differential equations that can be easily solved. To that end we'll see how to solve linear, separable, exact and Bernoulli differential equations. We'll also see how to solve certain differential equations with a proper substitution.

We will also take a look at solving some models of some physical situations. In particular we'll look at mixing problems, population problems and we'll revisit the falling object situation that we first looked at in the last chapter.

In addition, we'll define the equilibrium solutions for a first order differential equation and look at their stability.

We will then close out the chapter by taking a quick look at a method for approximating a solution to a differential equation. Processes such as this can be used to, possibly, get some information about the solution to a differential equation that we are unable to solve.

2.1 Linear Differential Equations

The first special case of first order differential equations that we will look at is the linear first order differential equation. In this case, unlike most of the first order cases that we will look at, we can actually derive a formula for the general solution. The general solution is derived below. However, we would suggest that you do not memorize the formula itself. Instead of memorizing the formula you should memorize and understand the process that I'm going to use to derive the formula. Most problems are actually easier to work by using the process instead of using the formula.

So, let's see how to solve a linear first order differential equation. Remember as we go through this process that the goal is to arrive at a solution that is in the form $y = y(t)$. It's sometimes easy to lose sight of the goal as we go through this process for the first time.

In order to solve a linear first order differential equation we MUST start with the differential equation in the form shown below. If the differential equation is not in this form then the process we're going to use will not work.

$$\frac{dy}{dt} + p(t)\,y = g(t) \tag{2.1}$$

Where both $p(t)$ and $g(t)$ are continuous functions. Recall that a quick and dirty definition of a continuous function is that a function will be continuous provided you can draw the graph from left to right without ever picking up your pencil/pen. In other words, a function is continuous if there are no holes or breaks in it.

Now, we are going to assume that there is some magical function somewhere out there in the world, $\mu(t)$, called an **integrating factor**. Do not, at this point, worry about what this function is or where it came from. We will figure out what $\mu(t)$ is once we have the formula for the general solution in hand.

So, now that we have assumed the existence of $\mu(t)$ multiply everything in Equation 2.1 by $\mu(t)$. This will give.

$$\mu(t)\frac{dy}{dt} + \mu(t)\,p(t)\,y = \mu(t)\,g(t) \tag{2.2}$$

Now, this is where the magic of $\mu(t)$ comes into play. We are going to assume that whatever $\mu(t)$ is, it will satisfy the following.

$$\mu(t)\,p(t) = \mu'(t) \tag{2.3}$$

Again do not worry about how we can find a $\mu(t)$ that will satisfy Equation 2.3. As we will see, provided $p(t)$ is continuous we can find it. So substituting Equation 2.3 we now arrive at.

$$\mu(t)\frac{dy}{dt} + \mu'(t)\,y = \mu(t)\,g(t) \tag{2.4}$$

At this point we need to recognize that the left side of Equation 2.4 is nothing more than the following product rule.

$$\mu(t)\frac{dy}{dt} + \mu'(t)\,y = \left(\mu(t)\,y(t)\right)'$$

So we can replace the left side of Equation 2.4 with this product rule. Upon doing this Equation 2.4 becomes

$$\left(\mu\left(t\right)y\left(t\right)\right)' = \mu\left(t\right)g\left(t\right) \tag{2.5}$$

Now, recall that we are after $y(t)$. We can now do something about that. All we need to do is integrate both sides then use a little algebra and we'll have the solution. So, integrate both sides of Equation 2.5 to get.

$$\int \left(\mu\left(t\right)y\left(t\right)\right)' dt = \int \mu\left(t\right)g\left(t\right)\,dt$$

$$\mu\left(t\right)y\left(t\right) + c = \int \mu\left(t\right)g\left(t\right)\,dt \tag{2.6}$$

Note the constant of integration, c, from the left side integration is included here. It is vitally important that this be included. If it is left out you will get the wrong answer every time.

The final step is then some algebra to solve for the solution, $y(t)$.

$$\mu\left(t\right)y\left(t\right) = \int \mu\left(t\right)g\left(t\right)\,dt - c$$

$$y\left(t\right) = \frac{\int \mu\left(t\right)g\left(t\right)\,dt - c}{\mu\left(t\right)}$$

Now, from a notational standpoint we know that the constant of integration, c, is an unknown constant and so to make our life easier we will absorb the minus sign in front of it into the constant and use a plus instead. This will NOT affect the final answer for the solution. So with this change we have.

$$y\left(t\right) = \frac{\int \mu\left(t\right)g\left(t\right)\,dt + c}{\mu\left(t\right)} \tag{2.7}$$

Again, changing the sign on the constant will not affect our answer. If you choose to keep the minus sign you will get the same value of c as we do except it will have the opposite sign. Upon plugging in c we will get exactly the same answer.

There is a lot of playing fast and loose with constants of integration in this section, so you will need to get used to it. When we do this we will always to try to make it very clear what is going on and try to justify why we did what we did.

So, now that we've got a general solution to Equation 2.1 we need to go back and determine just what this magical function $\mu\left(t\right)$ is. This is actually an easier process than you might think. We'll start with Equation 2.3.

$$\mu\left(t\right)p\left(t\right) = \mu'\left(t\right)$$

Divide both sides by $\mu\left(t\right)$,

$$\frac{\mu'\left(t\right)}{\mu\left(t\right)} = p\left(t\right)$$

Now, hopefully you will recognize the left side of this from your Calculus I class as nothing more than the following derivative.

$$\left(\ln\mu\left(t\right)\right)' = p\left(t\right)$$

As with the process above all we need to do is integrate both sides to get.

$$\begin{aligned} \ln \mu(t) + k &= \int p(t)\, dt \\ \ln \mu(t) &= \int p(t)\, dt + k \end{aligned}$$

You will notice that the constant of integration from the left side, k, had been moved to the right side and had the minus sign absorbed into it again as we did earlier. Also note that we're using k here because we've already used c and in a little bit we'll have both of them in the same equation. So, to avoid confusion we used different letters to represent the fact that they will, in all probability, have different values.

Exponentiate both sides to get $\mu(t)$ out of the natural logarithm.

$$\mu(t) = \mathbf{e}^{\int p(t)\, dt + k}$$

Now, it's time to play fast and loose with constants again. It is inconvenient to have the k in the exponent so we're going to get it out of the exponent in the following way.

$$\begin{aligned} \mu(t) &= \mathbf{e}^{\int p(t)\, dt + k} \\ &= \mathbf{e}^{k}\mathbf{e}^{\int p(t)\, dt} \qquad \text{Recall } x^{a+b} = x^a x^b! \end{aligned}$$

Now, let's make use of the fact that k is an unknown constant. If k is an unknown constant then so is $\mathbf{e}^k$ so we might as well just rename it k and make our life easier. This will give us the following.

$$\mu(t) = k\,\mathbf{e}^{\int p(t)\, dt} \tag{2.8}$$

So, we now have a formula for the general solution, Equation 2.7, and a formula for the integrating factor, Equation 2.8. We do have a problem however. We've got two unknown constants and the more unknown constants we have the more trouble we'll have later on. Therefore, it would be nice if we could find a way to eliminate one of them (we'll not be able to eliminate both....).

This is actually quite easy to do. First, substitute Equation 2.8 into Equation 2.7 and rearrange the constants.

$$\begin{aligned} y(t) &= \frac{\int k\,\mathbf{e}^{\int p(t)\, dt} g(t)\, dt + c}{k\,\mathbf{e}^{\int p(t)\, dt}} \\ &= \frac{k\int \mathbf{e}^{\int p(t)\, dt} g(t)\, dt + c}{k\,\mathbf{e}^{\int p(t)\, dt}} \\ &= \frac{\int \mathbf{e}^{\int p(t)\, dt} g(t)\, dt + \frac{c}{k}}{\mathbf{e}^{\int p(t)\, dt}} \end{aligned}$$

So, Equation 2.7 can be written in such a way that the only place the two unknown constants show up is a ratio of the two. Then since both c and k are unknown constants so is the ratio of the two constants. Therefore we'll just call the ratio c and then drop k out of Equation 2.8 since it will just get absorbed into c eventually.

The solution to a linear first order differential equation is then

$$y(t) = \frac{\int \mu(t)\, g(t)\ dt + c}{\mu(t)} \tag{2.9}$$

where,

$$\mu(t) = \mathbf{e}^{\int p(t)\, dt} \tag{2.10}$$

Now, the reality is that Equation 2.9 is not as useful as it may seem. It is often easier to just run through the process that got us to Equation 2.9 rather than using the formula. We will not use this formula in any of our examples. We will need to use Equation 2.10 regularly, as that formula is easier to use than the process to derive it.

Solution Process

The solution process for a first order linear differential equation is as follows.

1. Put the differential equation in the correct initial form, Equation 2.1.
2. Find the integrating factor, $\mu(t)$, using Equation 2.10.
3. Multiply everything in the differential equation by $\mu(t)$ and verify that the left side becomes the product rule $(\mu(t)\, y(t))'$ and write it as such.
4. Integrate both sides, make sure you properly deal with the constant of integration.
5. Solve for the solution $y(t)$.

Let's work a couple of examples. Let's start by solving the differential equation that we derived back in the Direction Field section.

Example 1

Find the solution to the following differential equation.

$$\frac{dv}{dt} = 9.8 - 0.196v$$

Solution

First, we need to get the differential equation in the correct form.

$$\frac{dv}{dt} + 0.196v = 9.8$$

From this we can see that $p(t) = 0.196$ and so $\mu(t)$ is then.

$$\mu(t) = \mathbf{e}^{\int 0.196 dt} = \mathbf{e}^{0.196t}$$

Note that officially there should be a constant of integration in the exponent from the integration. However, we can drop that for exactly the same reason that we dropped the k from Equation 2.8.

Now multiply all the terms in the differential equation by the integrating factor and do some simplification.

$$\begin{aligned} \mathbf{e}^{0.196t}\frac{dv}{dt} + 0.196\mathbf{e}^{0.196t}v &= 9.8\mathbf{e}^{0.196t} \\ \left(\mathbf{e}^{0.196t}v\right)' &= 9.8\mathbf{e}^{0.196t} \end{aligned}$$

Integrate both sides and don't forget the constants of integration that will arise from both integrals.

$$\begin{aligned} \int \left(\mathbf{e}^{0.196t}v\right)' dt &= \int 9.8\mathbf{e}^{0.196t}\, dt \\ \mathbf{e}^{0.196t}v + k &= 50\mathbf{e}^{0.196t} + c \end{aligned}$$

Okay. It's time to play with constants again. We can subtract k from both sides to get.

$$\mathbf{e}^{0.196t}v = 50\mathbf{e}^{0.196t} + c - k$$

Both c and k are unknown constants and so the difference is also an unknown constant. We will therefore write the difference as c. So, we now have

$$\mathbf{e}^{0.196t}v = 50\mathbf{e}^{0.196t} + c$$

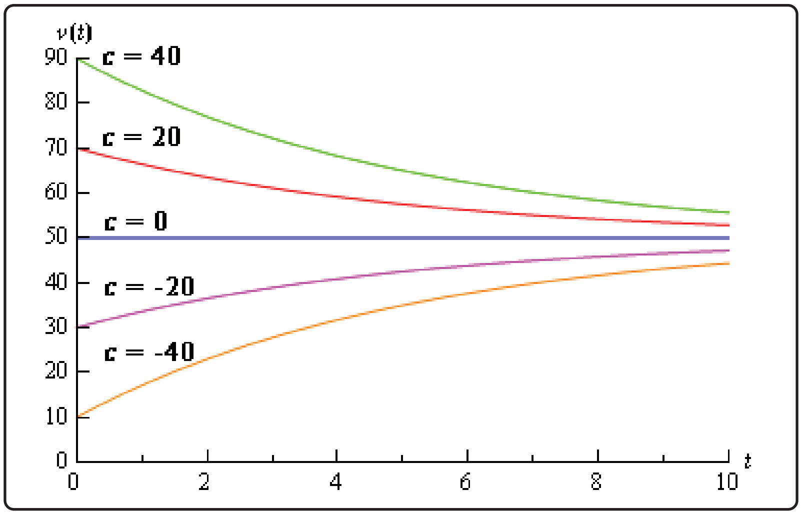
From this point on we will only put one constant of integration down when we integrate both sides knowing that if we had written down one for each integral, as we should, the two would just end up getting absorbed into each other.

The final step in the solution process is then to divide both sides by $\mathbf{e}^{0.196t}$ or to multiply both sides by $\mathbf{e}^{-0.196t}$. Either will work, but we usually prefer the multiplication route. Doing this gives the general solution to the differential equation.

$$v(t) = 50 + c\mathbf{e}^{-0.196t}$$

From the solution to this example we can now see why the constant of integration is so important in this process. Without it, in this case, we would get a single, constant solution, $v(t) = 50$. With the constant of integration we get infinitely many solutions, one for each value of c.

Back in the direction field section where we first derived the differential equation used in the last example we used the direction field to help us sketch some solutions. Let's see if we got them correct. To sketch some solutions all we need to do is to pick different values of c to get a solution. Several of these are shown in the graph below.



So, it looks like we did pretty good sketching the graphs back in the direction field section.

Now, recall from the Definitions section that the Initial Condition(s) will allow us to zero in on a particular solution. Solutions to first order differential equations (not just linear as we will see) will have a single unknown constant in them and so we will need exactly one initial condition to find the value of that constant and hence find the solution that we were after. The initial condition for first order differential equations will be of the form

$$y(t_0) = y_0$$

Recall as well that a differential equation along with a sufficient number of initial conditions is called an Initial Value Problem (IVP).

Example 2

Solve the following IVP.

$$\frac{dv}{dt} = 9.8 - 0.196v \qquad v(0) = 48$$

Solution

To find the solution to an IVP we must first find the general solution to the differential equation and then use the initial condition to identify the exact solution that we are after. So, since this is the same differential equation as we looked at in Example 1, we already have its general solution.

$$v = 50 + c\mathbf{e}^{-0.196t}$$

Now, to find the solution we are after we need to identify the value of c that will give us the solution we are after. To do this we simply plug in the initial condition which will give us an equation we can solve for c. So, let's do this

$$48 = v(0) = 50 + c \qquad \Rightarrow \qquad c = -2$$

So, the actual solution to the IVP is.

$$v = 50 - 2\mathbf{e}^{-0.196t}$$

A graph of this solution can be seen in the figure above.

Let's do a couple of examples that are a little more involved.

Example 3

Solve the following IVP.

$$\cos(x)\, y' + \sin(x)\, y = 2\cos^3(x)\sin(x) - 1 \qquad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}, \qquad 0 \le x < \frac{\pi}{2}$$

Solution

Rewrite the differential equation to get the coefficient of the derivative to be one.

$$y' + \frac{\sin(x)}{\cos(x)} y = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$$
$$y' + \tan(x)\, y = 2\cos^2(x)\sin(x) - \sec(x)$$

Now find the integrating factor.

$$\mu(t) = \mathbf{e}^{\int \tan(x)\,dx} = \mathbf{e}^{\ln|\sec(x)|} = \mathbf{e}^{\ln\ \sec(x)} = \sec(x)$$

Can you do the integral? If not rewrite tangent back into sines and cosines and then use a simple substitution. Note that we could drop the absolute value bars on the secant because of the limits on x. In fact, this is the reason for the limits on x. Note as well that there are two forms of the answer to this integral. They are equivalent as shown below. Which you use is really a matter of preference.

$$\int \tan(x)\ dx = -\ln|\cos(x)| = \ln|\cos(x)|^{-1} = \ln|\sec(x)|$$

Also note that we made use of the following fact.

$$\mathbf{e}^{\ln(f)(x)} = f(x) \tag{2.11}$$

This is an important fact that you should always remember for these problems. We will want to simplify the integrating factor as much as possible in all cases and this fact will help with that simplification.

Now back to the example. Multiply the integrating factor through the differential equation and verify the left side is a product rule. Note as well that we multiply the integrating factor through the rewritten differential equation and NOT the original differential equation. Make sure that you do this. If you multiply the integrating factor through the original differential equation you will get the wrong solution!

$$\begin{aligned} \sec(x)\, y' + \sec(x)\tan(x)\, y &= 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x) \\ (\sec(x)\, y)' &= 2\cos(x)\sin(x) - \sec^2(x) \end{aligned}$$

Integrate both sides.

$$\begin{aligned} \int (\sec(x)\, y(x))'\, dx &= \int 2\cos(x)\sin(x) - \sec^2(x)\ dx \\ \sec(x)\, y(x) &= \int \sin(2x) - \sec^2(x)\ dx \\ \sec(x)\, y(x) &= -\frac{1}{2}\cos(2x) - \tan(x) + c \end{aligned}$$

Note the use of the trig formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ that made the integral easier. Next, solve for the solution.

$$\begin{aligned} y(x) &= -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + c\cos(x) \\ &= -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos(x) \end{aligned}$$

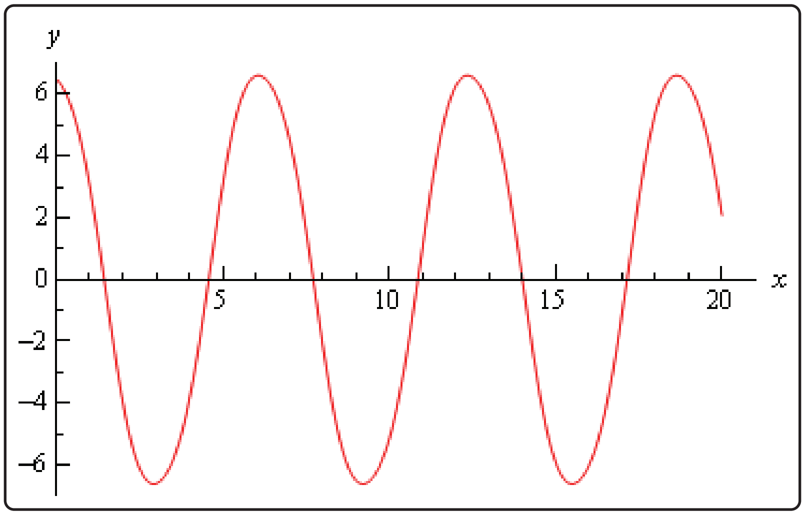
Finally, apply the initial condition to find the value of c.

$$\begin{aligned} 3\sqrt{2} = y\left(\frac{\pi}{4}\right) &= -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right) \\ 3\sqrt{2} &= -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2} \\ c &= 7 \end{aligned}$$

The solution is then.

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

Below is a plot of the solution.



Example 4

Find the solution to the following IVP.

$$t\,y' + 2y = t^2 - t + 1 \qquad y(1) = \frac{1}{2}$$

Solution

First, divide through by the t to get the differential equation into the correct form.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

Now let's get the integrating factor, $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int \frac{2}{t}dt} = \mathbf{e}^{2\ln|t|}$$

Now, we need to simplify $\mu(t)$. However, we can't use Equation 2.11 yet as that requires a coefficient of one in front of the logarithm. So, recall that

$$\ln(x^r) = r\ln(x)$$

and rewrite the integrating factor in a form that will allow us to simplify it.

$$\mu(t) = \mathbf{e}^{2\ln|t|} = \mathbf{e}^{\ln|t|^2} = |t|^2 = t^2$$

We were able to drop the absolute value bars here because we were squaring the t, but often they can't be dropped so be careful with them and don't drop them unless you know that you can. Often the absolute value bars must remain.

Now, multiply the rewritten differential equation (remember we can't use the original differential equation here...) by the integrating factor.

$$\left(t^2 y\right)' = t^3 - t^2 + t$$

Integrate both sides and solve for the solution.

$$\begin{aligned} t^2 y &= \int t^3 - t^2 + t\,dt \\ &= \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c \\ y\left(t\right) &= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2} \end{aligned}$$

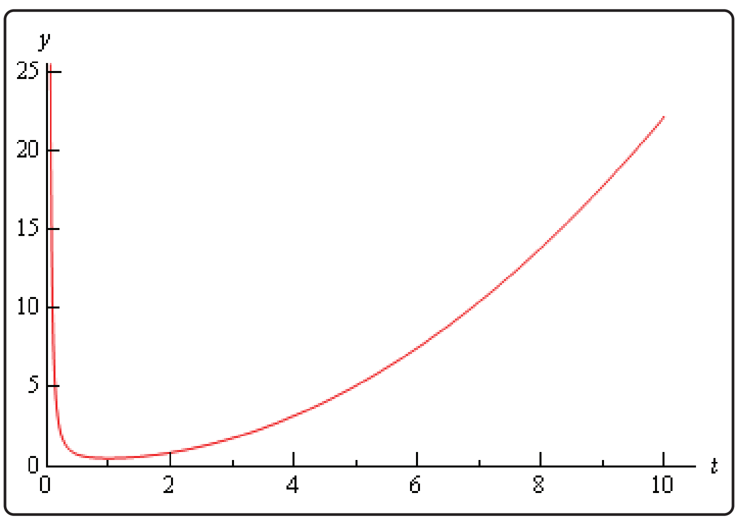
Finally, apply the initial condition to get the value of c.

$$\frac{1}{2} = y\left(1\right) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \qquad \Rightarrow \qquad c = \frac{1}{12}$$

The solution is then,

$$y\left(t\right) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

Here is a plot of the solution.



Example 5

Find the solution to the following IVP.

$$t\,y' - 2y = t^5 \sin(2t) - t^3 + 4t^4 \qquad y(\pi) = \frac{3}{2}\pi^4$$

Solution

First, divide through by t to get the differential equation in the correct form.

$$y' - \frac{2}{t}y = t^4 \sin(2t) - t^2 + 4t^3$$

Now that we have done this we can find the integrating factor, $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int -\frac{2}{t}dt} = \mathbf{e}^{-2\ln|t|}$$

Do not forget that the "$-$" is part of $p(t)$. Forgetting this minus sign can take a problem that is very easy to do and turn it into a very difficult, if not impossible problem so be careful!

Now, we just need to simplify this as we did in the previous example.

$$\mu(t) = \mathbf{e}^{-2\ln|t|} = \mathbf{e}^{\ln|t|^{-2}} = |t|^{-2} = t^{-2}$$

Again, we can drop the absolute value bars since we are squaring the term.

Now multiply the differential equation by the integrating factor (again, make sure it's the rewritten one and not the original differential equation).

$$\left(t^{-2}y\right)' = t^2 \sin(2t) - 1 + 4t$$

Integrate both sides and solve for the solution.

$$\begin{aligned}
t^{-2}y(t) &= \int t^2 \sin(2t)\,dt + \int -1 + 4t\,dt \\
t^{-2}y(t) &= -\frac{1}{2}t^2\cos(2t) + \frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t) - t + 2t^2 + c \\
y(t) &= -\frac{1}{2}t^4\cos(2t) + \frac{1}{2}t^3\sin(2t) + \frac{1}{4}t^2\cos(2t) - t^3 + 2t^4 + ct^2
\end{aligned}$$

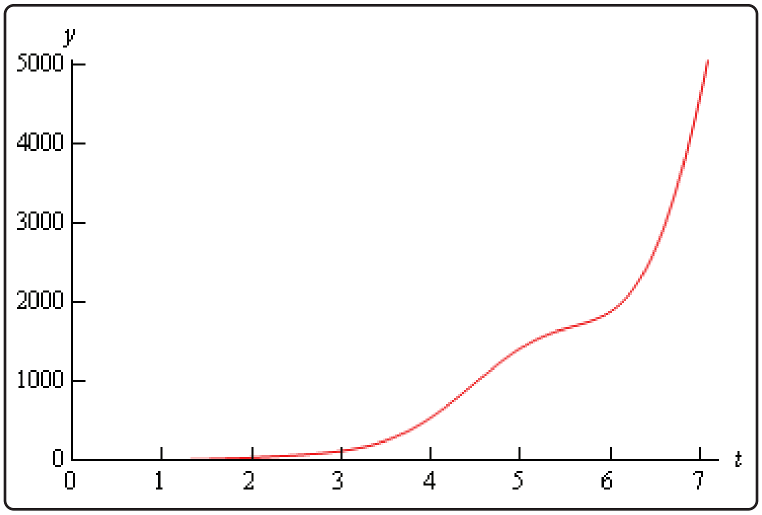
Apply the initial condition to find the value of c.

$$\begin{aligned}
\frac{3}{2}\pi^4 = y(\pi) &= -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^2 - \pi^3 + 2\pi^4 + c\pi^2 = \frac{3}{2}\pi^4 - \pi^3 + \frac{1}{4}\pi^2 + c\pi^2 \\
\pi^3 - \frac{1}{4}\pi^2 &= c\pi^2 \\
c &= \pi - \frac{1}{4}
\end{aligned}$$

The solution is then

$$y(t) = -\frac{1}{2}t^4\cos(2t) + \frac{1}{2}t^3\sin(2t) + \frac{1}{4}t^2\cos(2t) - t^3 + 2t^4 + \left(\pi - \frac{1}{4}\right)t^2$$

Below is a plot of the solution.



Let's work one final example that looks more at interpreting a solution rather than finding a solution.

Example 6

Find the solution to the following IVP and determine all possible behaviors of the solution as $t \to \infty$. If this behavior depends on the value of y_0 give this dependence.

$$2y' - y = 4\sin(3t) \qquad y(0) = y_0$$

Solution

First, divide through by a 2 to get the differential equation in the correct form.

$$y' - \frac{1}{2}y = 2\sin(3t)$$

Now find $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int -\frac{1}{2}dt} = \mathbf{e}^{-\frac{t}{2}}$$

Multiply $\mu(t)$ through the differential equation and rewrite the left side as a product rule.

$$\left(\mathbf{e}^{-\frac{t}{2}}y\right)' = 2\mathbf{e}^{-\frac{t}{2}}\sin(3t)$$

Integrate both sides (the right side requires integration by parts – you can do that right?) and solve for the solution.

$$\begin{aligned}
\mathbf{e}^{-\frac{t}{2}}y &= \int 2\mathbf{e}^{-\frac{t}{2}}\sin(3t)\,dt + c\\
\mathbf{e}^{-\frac{t}{2}}y &= -\frac{24}{37}\mathbf{e}^{-\frac{t}{2}}\cos(3t) - \frac{4}{37}\mathbf{e}^{-\frac{t}{2}}\sin(3t) + c\\
y(t) &= -\frac{24}{37}\cos(3t) - \frac{4}{37}\sin(3t) + c\mathbf{e}^{\frac{t}{2}}
\end{aligned}$$

Apply the initial condition to find the value of c and note that it will contain y_0 as we don't have a value for that.

$$y_0 = y(0) = -\frac{24}{37} + c \qquad \Rightarrow \qquad c = y_0 + \frac{24}{37}$$

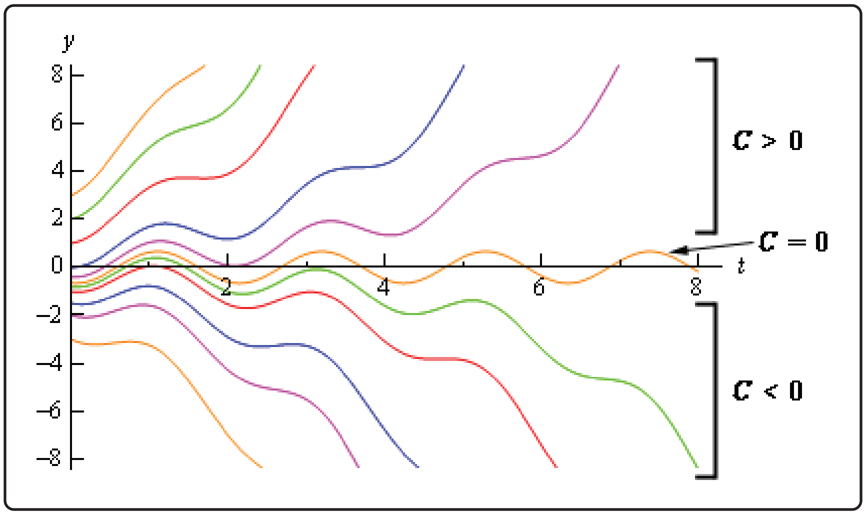
So, the solution is

$$y(t) = -\frac{24}{37}\cos(3t) - \frac{4}{37}\sin(3t) + \left(y_0 + \frac{24}{37}\right)\mathbf{e}^{\frac{t}{2}}$$

Now that we have the solution, let's look at the long term behavior (*i.e.* $t \to \infty$) of the solution. The first two terms of the solution will remain finite for all values of t. It is the last term that will determine the behavior of the solution. The exponential will always go to infinity as $t \to \infty$, however depending on the sign of the coefficient c (yes we've already found it, but for ease of this discussion we'll continue to call it c). The following table gives the long term behavior of the solution for all values of c.

Range of c	Behavior of solution as $t \to \infty$
$c < 0$	$y(t) \to -\infty$
$c = 0$	$y(t)$ remains finite
$c > 0$	$y(t) \to \infty$

This behavior can also be seen in the following graph of several of the solutions.



Now, because we know how c relates to y_0 we can relate the behavior of the solution to y_0. The following table give the behavior of the solution in terms of y_0 instead of c.

Range of y_0	Behavior of solution as $t \to \infty$
$y_0 < -\dfrac{24}{37}$	$y(t) \to -\infty$
$y_0 = -\dfrac{24}{37}$	$y(t)$ remains finite
$y_0 > -\dfrac{24}{37}$	$y(t) \to \infty$

Note that for $y_0 = -\frac{24}{37}$ the solution will remain finite. That will not always happen.

Investigating the long term behavior of solutions is sometimes more important than the solution itself. Suppose that the solution above gave the temperature in a bar of metal. In this case we would want the solution(s) that remains finite in the long term. With this investigation we would now have the value of the initial condition that will give us that solution and more importantly values of the initial condition that we would need to avoid so that we didn't melt the bar.

2.2 Separable Equations

We are now going to start looking at nonlinear first order differential equations. The first type of nonlinear first order differential equations that we will look at is separable differential equations.

A separable differential equation is any differential equation that we can write in the following form.

$$N\left(y\right)\frac{dy}{dx} = M\left(x\right) \tag{2.12}$$

Note that in order for a differential equation to be separable all the y's in the differential equation must be multiplied by the derivative and all the x's in the differential equation must be on the other side of the equal sign.

To solve this differential equation we first integrate both sides with respect to x to get,

$$\int N\left(y\right)\frac{dy}{dx}\,dx = \int M\left(x\right)\,dx$$

Now, remember that y is really $y\left(x\right)$ and so we can use the following substitution,

$$u = y\left(x\right) \qquad du = y'\left(x\right)\,dx = \frac{dy}{dx}\,dx$$

Applying this substitution to the integral we get,

$$\int N\left(u\right)\,du = \int M\left(x\right)\,dx \tag{2.13}$$

At this point we can (hopefully) integrate both sides and then back substitute for the u on the left side. Note, that as implied in the previous sentence, it might not actually be possible to evaluate one or both of the integrals at this point. If that is the case, then there won't be a lot we can do to proceed using this method to solve the differential equation.

Now, the process above is the mathematically correct way of solving this differential equation. Note however, that if we "separate" the derivative as well we can write the differential equation as,

$$N\left(y\right)dy = M\left(x\right)dx$$

We obviously can't separate the derivative like that, but let's pretend we can for a bit and we'll see that we arrive at the answer with less work.

Now we integrate both sides of this to get,

$$\int N\left(y\right)dy = \int M\left(x\right)dx \tag{2.14}$$

So, if we compare Equation 2.13 and Equation 2.14 we can see that the only difference is on the left side and even then the only real difference is Equation 2.13 has the integral in terms of u and Equation 2.14 has the integral in terms of y. Outside of that there is no real difference. The integral on the left is exactly the same integral in each equation. The only difference is the letter used in the integral. If we integrate Equation 2.13 and then back substitute in for u we would arrive at the same thing as if we'd just integrated Equation 2.14 from the start.

Therefore, to make the work go a little easier, we'll just use Equation 2.14 to find the solution to the differential equation. Also, after doing the integrations, we will have an implicit solution that we can hopefully solve for the explicit solution, $y(x)$. Note that it won't always be possible to solve for an explicit solution.

Recall from the Definitions section that an implicit solution is a solution that is not in the form $y = y(x)$ while an explicit solution has been written in that form.

We will also have to worry about the interval of validity for many of these solutions. Recall that the interval of validity was the range of the independent variable, x in this case, on which the solution is valid. In other words, we need to avoid division by zero, complex numbers, logarithms of negative numbers or zero, *etc.* Most of the solutions that we will get from separable differential equations will not be valid for all values of x.

Let's start things off with a fairly simple example so we can see the process without getting lost in details of the other issues that often arise with these problems.

Example 1

Solve the following differential equation and determine the interval of validity for the solution.

$$\frac{dy}{dx} = 6y^2x \qquad y(1) = \frac{1}{25}$$

Solution

It is clear, hopefully, that this differential equation is separable. So, let's separate the differential equation and integrate both sides. As with the linear first order officially we will pick up a constant of integration on both sides from the integrals on each side of the equal sign. The two can be moved to the same side and absorbed into each other. We will use the convention that puts the single constant on the side with the x's given that we will eventually be solving for y and so the constant would end up on that side anyway.

$$\begin{aligned} y^{-2}dy &= 6x\,dx \\ \int y^{-2}dy &= \int 6x\,dx \\ -\frac{1}{y} &= 3x^2 + c \end{aligned}$$

So, we now have an implicit solution. This solution is easy enough to get an explicit solution, however before getting that it is usually easier to find the value of the constant at this point.

So apply the initial condition and find the value of c.

$$-\frac{1}{{}^{1}/_{25}} = 3(1)^2 + c \qquad c = -28$$

Plug this into the general solution and then solve to get an explicit solution.

$$-\frac{1}{y} = 3x^2 - 28$$
$$y(x) = \frac{1}{28 - 3x^2}$$

Now, as far as solutions go we've got the solution. We do need to start worrying about intervals of validity however.

Recall that there are two conditions that define an interval of validity. First, it must be a continuous interval with no breaks or holes in it. Second it must contain the value of the independent variable in the initial condition, $x = 1$ in this case.

So, for our case we've got to avoid two values of x. Namely, $x \neq \pm\sqrt{\frac{28}{3}} \approx \pm 3.05505$ since these will give us division by zero. This gives us three possible intervals of validity.

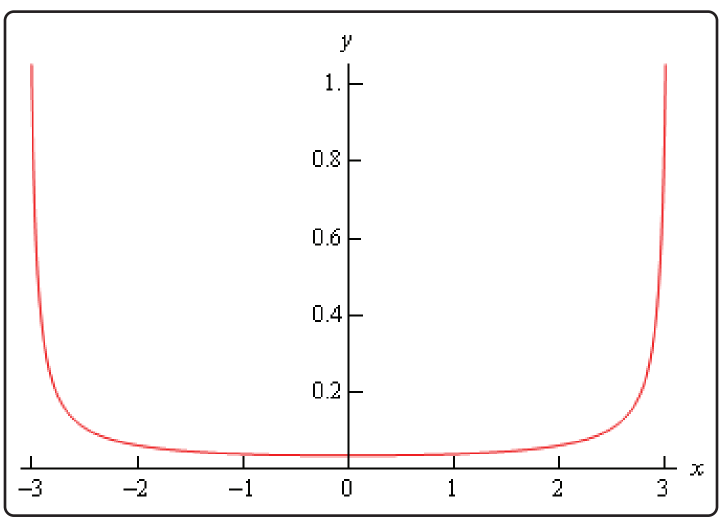
$$-\infty < x < -\sqrt{\frac{28}{3}} \qquad -\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}} \qquad \sqrt{\frac{28}{3}} < x < \infty$$

However, only one of these will contain the value of x from the initial condition and so we can see that

$$-\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}}$$

must be the interval of validity for this solution.

Here is a graph of the solution.



Note that this does not say that either of the other two intervals listed above can't be the interval of validity for any solution to the differential equation. With the proper initial condition either of these could have been the interval of validity.

We'll leave it to you to verify the details of the following claims. If we use an initial condition of

$$y(-4) = -\frac{1}{20}$$

we will get exactly the same solution however in this case the interval of validity would be the first one.

$$-\infty < x < -\sqrt{\frac{28}{3}}$$

Likewise, if we use

$$y(6) = -\frac{1}{80}$$

as the initial condition we again get exactly the same solution and, in this case, the third interval becomes the interval of validity.

$$\sqrt{\frac{28}{3}} < x < \infty$$

So, simply changing the initial condition a little can give any of the possible intervals.

Example 2

Solve the following IVP and find the interval of validity for the solution.

$$y' = \frac{3x^2 + 4x - 4}{2y - 4} \qquad y(1) = 3$$

Solution

This differential equation is clearly separable, so let's put it in the proper form and then integrate both sides.

$$\begin{aligned}(2y-4)\,dy &= \left(3x^2 + 4x - 4\right)\,dx \\ \int (2y-4)\,dy &= \int \left(3x^2 + 4x - 4\right)\,dx \\ y^2 - 4y &= x^3 + 2x^2 - 4x + c\end{aligned}$$

We now have our implicit solution, so as with the first example let's apply the initial condition at this point to determine the value of c.

$$(3)^2 - 4(3) = (1)^3 + 2(1)^2 - 4(1) + c \qquad c = -2$$

The implicit solution is then

$$y^2 - 4y = x^3 + 2x^2 - 4x - 2$$

We now need to find the explicit solution. This is actually easier than it might look and you already know how to do it. First, we need to rewrite the solution a little

$$y^2 - 4y - \left(x^3 + 2x^2 - 4x - 2\right) = 0$$

To solve this all we need to recognize is that this is quadratic in y and so we can use the quadratic formula to solve it. However, unlike quadratics you are used to, at least some of the "constants" will not actually be constant but will in fact involve x's.

So, upon using the quadratic formula on this we get.

$$\begin{aligned} y(x) &= \frac{4 \pm \sqrt{16 - 4(1)\left(-\left(x^3 + 2x^2 - 4x - 2\right)\right)}}{2} \\ &= \frac{4 \pm \sqrt{16 + 4\left(x^3 + 2x^2 - 4x - 2\right)}}{2} \end{aligned}$$

Next, notice that we can factor a 4 out from under the square root (it will come out as a 2...) and then simplify a little.

$$\begin{aligned} y(x) &= \frac{4 \pm 2\sqrt{4 + \left(x^3 + 2x^2 - 4x - 2\right)}}{2} \\ &= 2 \pm \sqrt{x^3 + 2x^2 - 4x + 2} \end{aligned}$$

We are almost there. Notice that we've actually got two solutions here (the "$\pm$") and we only want a single solution. In fact, only one of the signs can be correct. So, to figure out which one is correct we can reapply the initial condition to this. Only one of the signs will give the correct value so we can use this to figure out which one of the signs is correct. Plugging $x = 1$ into the solution gives.

$$3 = y(1) = 2 \pm \sqrt{1 + 2 - 4 + 2} = 2 \pm 1 = 3,\ 1$$

In this case it looks like the "$+$" is the correct sign for our solution. Note that it is completely possible that the "$-$" could be the solution (*i.e.* using an initial condition of $y(1) = 1$) so don't always expect it to be one or the other.

The explicit solution for our differential equation is.

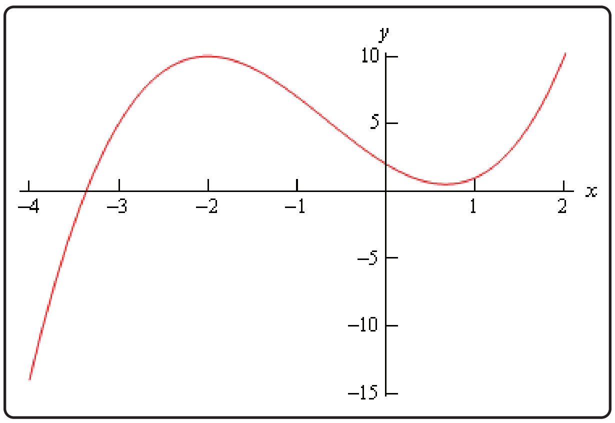
$$y(x) = 2 + \sqrt{x^3 + 2x^2 - 4x + 2}$$

To finish the example out we need to determine the interval of validity for the solution. If we were to put a large negative value of x in the solution we would end up with complex values in our solution and we want to avoid complex numbers in our solutions here. So, we will need to determine which values of x will give real solutions. To do this we will need to solve the following inequality.

$$x^3 + 2x^2 - 4x + 2 \geq 0$$

In other words, we need to make sure that the quantity under the radical stays positive.

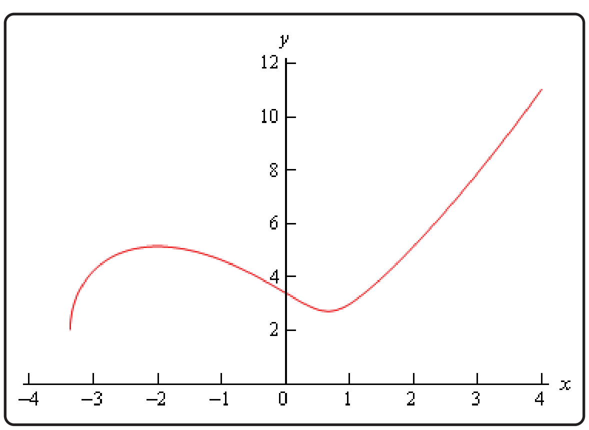
Using a computer algebra system like Maple or Mathematica we see that the left side is zero at $x = -3.36523$ as well as two complex values, but we can ignore complex values for interval of validity computations. Finally, a graph of the quantity under the radical is shown below.



So, in order to get real solutions we will need to require $x \geq -3.36523$ because this is the range of x's for which the quantity is positive. Notice as well that this interval also contains the value of x that is in the initial condition as it should.

Therefore, the interval of validity of the solution is $x \geq -3.36523$.

Here is graph of the solution.



Example 3

Solve the following IVP and find the interval of validity of the solution.

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$$

Solution

First separate and then integrate both sides.

$$\begin{aligned} y^{-3}dy &= x\left(1+x^2\right)^{-\frac{1}{2}}\,dx \\ \int y^{-3}dy &= \int x\left(1+x^2\right)^{-\frac{1}{2}}\,dx \\ -\frac{1}{2y^2} &= \sqrt{1+x^2}+c \end{aligned}$$

Apply the initial condition to get the value of c.

$$-\frac{1}{2} = \sqrt{1}+c \qquad c = -\frac{3}{2}$$

The implicit solution is then,

$$-\frac{1}{2y^2} = \sqrt{1+x^2}-\frac{3}{2}$$

Now let's solve for $y(x)$.

$$\begin{aligned} \frac{1}{y^2} &= 3-2\sqrt{1+x^2} \\ y^2 &= \frac{1}{3-2\sqrt{1+x^2}} \\ y(x) &= \pm\frac{1}{\sqrt{3-2\sqrt{1+x^2}}} \end{aligned}$$

Reapplying the initial condition shows us that the "–" is the correct sign. The explicit solution is then,

$$y(x) = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Let's get the interval of validity. That's easier than it might look for this problem. First, since $1+x^2 \geq 0$ the "inner" root will not be a problem. Therefore, all we need to worry about is division by zero and negatives under the "outer" root. We can take care of both by requiring

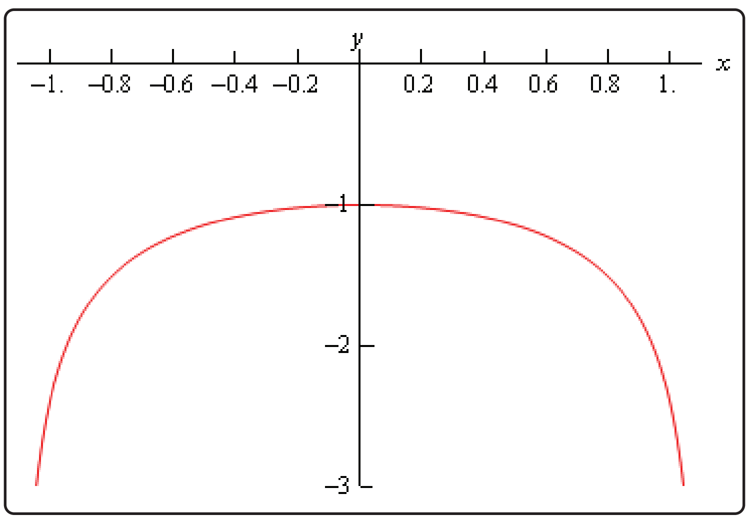
$$\begin{aligned}3-2\sqrt{1+x^2} &> 0\\ 3 &> 2\sqrt{1+x^2}\\ 9 &> 4\left(1+x^2\right)\\ \frac{9}{4} &> 1+x^2\\ \frac{5}{4} &> x^2\end{aligned}$$

Note that we were able to square both sides of the inequality because both sides of the inequality are guaranteed to be positive in this case. Finally solving for x we see that the only possible range of x's that will not give division by zero or square roots of negative numbers will be,

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

and nicely enough this also contains the initial condition $x = 0$. This interval is therefore our interval of validity.

Here is a graph of the solution.



Example 4

Solve the following IVP and find the interval of validity of the solution.

$$y' = \mathbf{e}^{-y}(2x-4) \qquad y(5) = 0$$

Solution

This differential equation is easy enough to separate, so let's do that and then integrate both sides.

$$\begin{aligned} \mathbf{e}^{y}\,dy &= (2x-4)\,dx \\ \int \mathbf{e}^{y}\,dy &= \int (2x-4)\,dx \\ \mathbf{e}^{y} &= x^2 - 4x + c \end{aligned}$$

Applying the initial condition gives

$$1 = 25 - 20 + c \qquad c = -4$$

This then gives an implicit solution of.

$$\mathbf{e}^{y} = x^2 - 4x - 4$$

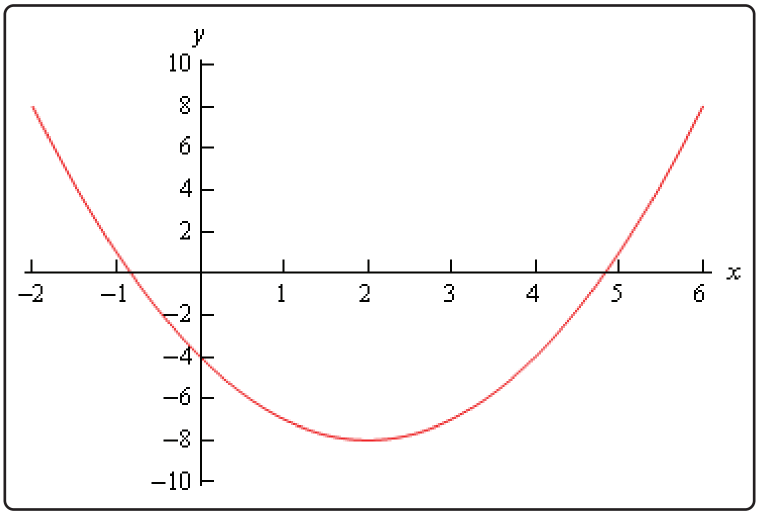
We can easily find the explicit solution to this differential equation by simply taking the natural log of both sides.

$$y(x) = \ln\left(x^2 - 4x - 4\right)$$

Finding the interval of validity is the last step that we need to take. Recall that we can't plug negative values or zero into a logarithm, so we need to solve the following inequality

$$x^2 - 4x - 4 > 0$$

The quadratic will be zero at the two points $x = 2 \pm 2\sqrt{2}$. A graph of the quadratic (shown below) shows that there are in fact two intervals in which we will get positive values of the polynomial and hence can be possible intervals of validity.



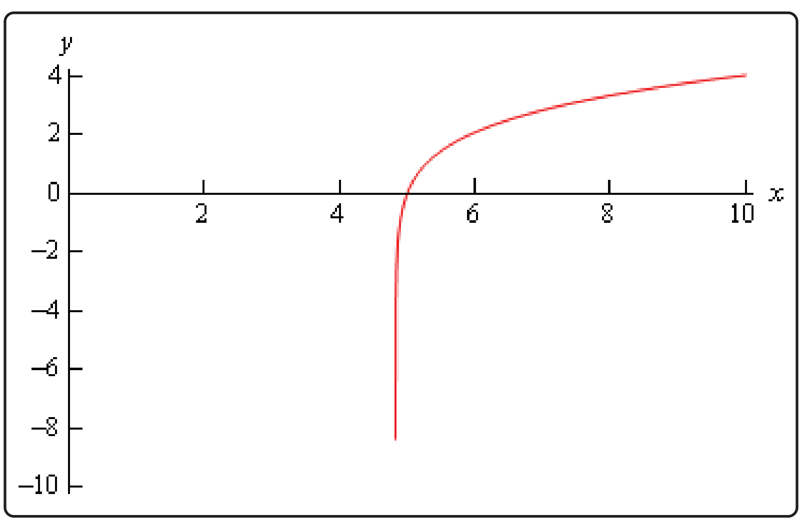
So, possible intervals of validity are

$$-\infty < x < 2 - 2\sqrt{2}$$
$$2 + 2\sqrt{2} < x < \infty$$

From the graph of the quadratic we can see that the second one contains $x = 5$, the value of the independent variable from the initial condition. Therefore, the interval of validity for this solution is.

$$2 + 2\sqrt{2} < x < \infty$$

Here is a graph of the solution.



Example 5

Solve the following IVP and find the interval of validity for the solution.

$$\frac{dr}{d\theta} = \frac{r^2}{\theta} \qquad r(1) = 2$$

Solution

This is actually a fairly simple differential equation to solve. We're doing this one mostly because of the interval of validity.

So, get things separated out and then integrate.

$$\begin{aligned} \frac{1}{r^2} dr &= \frac{1}{\theta} d\theta \\ \int \frac{1}{r^2} dr &= \int \frac{1}{\theta} d\theta \\ -\frac{1}{r} &= \ln|\theta| + c \end{aligned}$$

Now, apply the initial condition to find c.

$$-\frac{1}{2} = \ln(1) + c \qquad c = -\frac{1}{2}$$

So, the implicit solution is then,

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

Solving for r gets us our explicit solution.

$$r = \frac{1}{\frac{1}{2} - \ln|\theta|}$$

Now, there are two problems for our solution here. First, we need to avoid $\theta = 0$ because of the natural log. Notice that because of the absolute value on the θ we don't need to worry about θ being negative. We will also need to avoid division by zero. In other words, we need to avoid the following points.

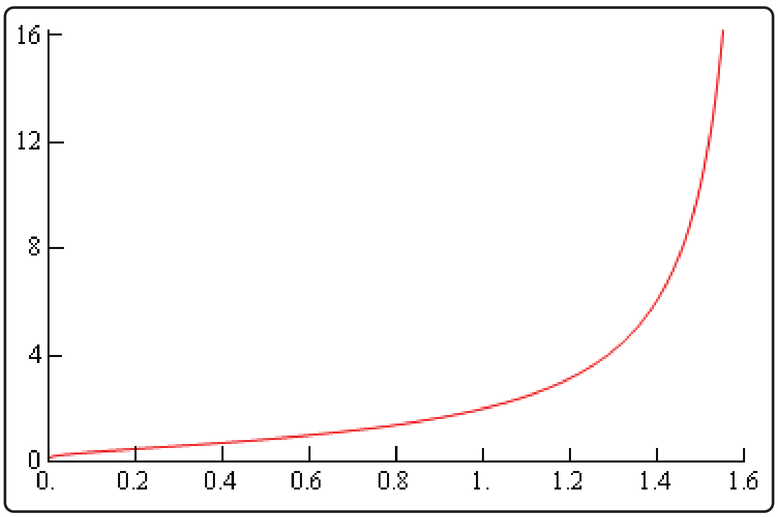
$$\begin{aligned} \frac{1}{2} - \ln|\theta| &= 0 \\ \ln|\theta| &= \frac{1}{2} \qquad \text{exponentiate both sides} \\ |\theta| &= \mathbf{e}^{\frac{1}{2}} \\ \theta &= \pm\sqrt{\mathbf{e}} \end{aligned}$$

So, these three points break the number line up into four portions, each of which could be an interval of validity.

$$-\infty < \theta < -\sqrt{\mathbf{e}}$$
$$-\sqrt{\mathbf{e}} < \theta < 0$$
$$0 < \theta < \sqrt{\mathbf{e}}$$
$$\sqrt{\mathbf{e}} < \theta < \infty$$

The interval that will be the actual interval of validity is the one that contains $\theta = 1$. Therefore, the interval of validity is $0 < \theta < \sqrt{\mathbf{e}}$.

Here is a graph of the solution.



Example 6

Solve the following IVP.

$$\frac{dy}{dt} = \mathbf{e}^{y-t}\sec(y)\left(1+t^2\right) \qquad y(0) = 0$$

Solution

This problem will require a little work to get it separated and in a form that we can integrate, so let's do that first.

$$\frac{dy}{dt} = \frac{\mathbf{e}^{y}\mathbf{e}^{-t}}{\cos(y)}\left(1+t^2\right)$$
$$\mathbf{e}^{-y}\cos(y)\;dy = \mathbf{e}^{-t}\left(1+t^2\right)dt$$

Now, with a little integration by parts on both sides we can get an implicit solution.

$$\int \mathbf{e}^{-y}\cos(y)\,dy = \int \mathbf{e}^{-t}\left(1+t^2\right)dt$$
$$\frac{\mathbf{e}^{-y}}{2}\left(\sin(y)-\cos(y)\right) = -\mathbf{e}^{-t}\left(t^2+2t+3\right)+c$$

Applying the initial condition gives.

$$\frac{1}{2}(-1) = -(3)+c \qquad c=\frac{5}{2}$$

Therefore, the implicit solution is.

$$\frac{\mathbf{e}^{-y}}{2}\left(\sin(y)-\cos(y)\right) = -\mathbf{e}^{-t}\left(t^2+2t+3\right)+\frac{5}{2}$$

It is not possible to find an explicit solution for this problem and so we will have to leave the solution in its implicit form. Finding intervals of validity from implicit solutions can often be very difficult so we will also not bother with that for this problem.

As this last example showed it is not always possible to find explicit solutions so be on the lookout for those cases.

2.3 Exact Equations

The next type of first order differential equations that we'll be looking at is exact differential equations. Before we get into the full details behind solving exact differential equations it's probably best to work an example that will help to show us just what an exact differential equation is. It will also show some of the behind the scenes details that we usually don't bother with in the solution process.

The vast majority of the following example will not be done in any of the remaining examples and the work that we will put into the remaining examples will not be shown in this example. The whole point behind this example is to show you just what an exact differential equation is, how we use this fact to arrive at a solution and why the process works as it does. The majority of the actual solution details will be shown in a later example.

Example 1

Solve the following differential equation.

$$2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0$$

Solution

Let's start off by supposing that somewhere out there in the world is a function $\Psi\left(x, y\right)$ that we can find. For this example the function that we need is

$$\Psi\left(x, y\right) = y^2 + \left(x^2 + 1\right)y - 3x^3$$

Do not worry at this point about where this function came from and how we found it. Finding the function, $\Psi\left(x, y\right)$, that is needed for any particular differential equation is where the vast majority of the work for these problems lies. As stated earlier however, the point of this example is to show you why the solution process works rather than showing you the actual solution process. We will see how to find this function in the next example, so at this point do not worry about how to find it, simply accept that it can be found and that we've done that for this particular differential equation.

Now, take some partial derivatives of the function.

$$\Psi_x = 2xy - 9x^2$$
$$\Psi_y = 2y + x^2 + 1$$

Now, compare these partial derivatives to the differential equation and you'll notice that with these we can now write the differential equation as.

$$\Psi_x + \Psi_y\frac{dy}{dx} = 0 \qquad (2.15)$$

Now, recall from your multi-variable calculus class (probably Calculus III), that Equation 2.15 is nothing more than the following derivative (you'll need the multi-variable chain rule for this...).

$$\frac{d}{dx}\left(\Psi\left(x, y\left(x\right)\right)\right)$$

So, the differential equation can now be written as

$$\frac{d}{dx}\left(\Psi\left(x, y\left(x\right)\right)\right) = 0$$

Now, if the ordinary (not partial...) derivative of something is zero, that something must have been a constant to start with. In other words, we've got to have $\Psi\left(x, y\right) = c$. Or,

$$y^2 + \left(x^2 + 1\right) y - 3x^3 = c$$

This then is an implicit solution for our differential equation! If we had an initial condition we could solve for c. We could also find an explicit solution if we wanted to, but we'll hold off on that until the next example.

Okay, so what did we learn from the last example? Let's look at things a little more generally. Suppose that we have the following differential equation.

$$M\left(x, y\right) + N\left(x, y\right)\frac{dy}{dx} = 0 \tag{2.16}$$

Note that it's important that it must be in this form! There must be an "$= 0$" on one side and the sign separating the two terms must be a "$+$". Now, if there is a function somewhere out there in the world, $\Psi\left(x, y\right)$, so that,

$$\Psi_x = M\left(x, y\right) \qquad \text{and} \qquad \Psi_y = N\left(x, y\right)$$

then we call the differential equation **exact**. In these cases we can write the differential equation as

$$\Psi_x + \Psi_y\frac{dy}{dx} = 0 \tag{2.17}$$

Then using the chain rule from your Multivariable Calculus class we can further reduce the differential equation to the following derivative,

$$\frac{d}{dx}\left(\Psi\left(x, y\left(x\right)\right)\right) = 0$$

The (implicit) solution to an exact differential equation is then

$$\Psi\left(x, y\right) = c \tag{2.18}$$

Well, it's the solution provided we can find $\Psi(x, y)$ anyway. Therefore, once we have the function we can always just jump straight to Equation 2.18 to get an implicit solution to our differential equation.

Finding the function $\Psi(x, y)$ is clearly the central task in determining if a differential equation is exact and in finding its solution. As we will see, finding $\Psi(x, y)$ can be a somewhat lengthy process in which there is the chance of mistakes. Therefore, it would be nice if there was some simple test that we could use before even starting to see if a differential equation is exact or not. This will be especially useful if it turns out that the differential equation is not exact, since in this case $\Psi(x, y)$ will not exist. It would be a waste of time to try and find a nonexistent function!

So, let's see if we can find a test for exact differential equations. Let's start with Equation 2.16 and assume that the differential equation is in fact exact. Since its exact we know that somewhere out there is a function $\Psi(x, y)$ that satisfies

$$\begin{aligned}\Psi_x &= M \\ \Psi_y &= N\end{aligned}$$

Now, provided $\Psi(x, y)$ is continuous and its first order derivatives are also continuous we know that

$$\Psi_{x\,y} = \Psi_{y\,x}$$

However, we also have the following.

$$\begin{aligned}\Psi_{x\,y} &= (\Psi_x)_y = (M)_y = M_y \\ \Psi_{y\,x} &= (\Psi_y)_x = (N)_x = N_x\end{aligned}$$

Therefore, if a differential equation is exact and $\Psi(x, y)$ meets all of its continuity conditions we must have.

$$M_y = N_x \tag{2.19}$$

Likewise, if Equation 2.19 is not true there is no way for the differential equation to be exact.

Therefore, we will use Equation 2.19 as a test for exact differential equations. If Equation 2.19 is true we will assume that the differential equation is exact and that $\Psi(x, y)$ meets all of its continuity conditions and proceed with finding it. Note that for all the examples here the continuity conditions will be met and so this won't be an issue.

Okay, let's go back and rework the first example. This time we will use the example to show how to find $\Psi(x, y)$. We'll also add in an initial condition to the problem.

Example 2

Solve the following IVP and find the interval of validity for the solution.

$$2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0, \qquad y(0) = -3$$

Solution

First identify M and N and check that the differential equation is exact.

$$\begin{aligned} M &= 2xy - 9x^2 \qquad & M_y &= 2x \\ N &= 2y + x^2 + 1 \qquad & N_x &= 2x \end{aligned}$$

So, the differential equation is exact according to the test. However, we already knew that as we have given you $\Psi(x, y)$. It's not a bad thing to verify it however and to run through the test at least once however.

Now, how do we actually find $\Psi(x, y)$? Well recall that

$$\begin{aligned} \Psi_x &= M \\ \Psi_y &= N \end{aligned}$$

We can use either of these to get a start on finding $\Psi(x, y)$ by integrating as follows.

$$\Psi = \int M\,dx \qquad \text{OR} \qquad \Psi = \int N\,dy$$

However, we will need to be careful as this won't give us the exact function that we need. Often it doesn't matter which one you choose to work with while in other problems one will be significantly easier than the other. In this case it doesn't matter which one we use as either will be just as easy.

So, we'll use the first one.

$$\Psi(x, y) = \int 2xy - 9x^2\,dx = x^2 y - 3x^3 + h(y)$$

Note that in this case the "constant" of integration is not really a constant at all, but instead it will be a function of the remaining variable(s), y in this case.

Recall that in integration we are asking what function we differentiated to get the function we are integrating. Since we are working with two variables here and talking about partial differentiation with respect to x, this means that any term that contained only constants or y's would have differentiated away to zero, therefore we need to acknowledge that fact by adding on a function of y instead of the standard c.

Okay, we've got most of $\Psi(x,y)$ we just need to determine $h(y)$ and we'll be done. This is actually easy to do. We used $\Psi_x = M$ to find most of $\Psi(x,y)$ so we'll use $\Psi_y = N$ to find $h(y)$. Differentiate our $\Psi(x,y)$ with respect to y and set this equal to N (since they must be equal after all). Don't forget to "differentiate" $h(y)$! Doing this gives,

$$\Psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N$$

From this we can see that

$$h'(y) = 2y + 1$$

Note that at this stage $h(y)$ must be only a function of y and so if there are any x's in the equation at this stage we have made a mistake somewhere and it's time to go look for it.

We can now find $h(y)$ by integrating.

$$h(y) = \int 2y + 1\, dy = y^2 + y + k$$

You'll note that we included the constant of integration, k, here. It will turn out however that this will end up getting absorbed into another constant so we can drop it in general.

So, we can now write down $\Psi(x,y)$.

$$\Psi(x,y) = x^2 y - 3x^3 + y^2 + y + k = y^2 + \left(x^2 + 1\right) y - 3x^3 + k$$

With the exception of the k this is identical to the function that we used in the first example. We can now go straight to the implicit solution using Equation 2.18.

$$y^2 + \left(x^2 + 1\right) y - 3x^3 + k = c$$

We'll now take care of the k. Since both k and c are unknown constants all we need to do is subtract one from both sides and combine and we still have an unknown constant.

$$\begin{aligned} y^2 + \left(x^2 + 1\right) y - 3x^3 &= c - k \\ y^2 + \left(x^2 + 1\right) y - 3x^3 &= c \end{aligned}$$

Therefore, we'll not include the k in anymore problems.

This is where we left off in the first example. Let's now apply the initial condition to find c.

$$(-3)^2 + (0 + 1)(-3) - 3(0)^3 = c \qquad \Rightarrow \qquad c = 6$$

The implicit solution is then.

$$y^2 + \left(x^2 + 1\right) y - 3x^3 - 6 = 0$$

Now, as we saw in the separable differential equation section, this is quadratic in y and so we can solve for $y(x)$ by using the quadratic formula.

$$\begin{aligned} y(x) &= \frac{-\left(x^2+1\right) \pm \sqrt{\left(x^2+1\right)^2 - 4(1)\left(-3x^3-6\right)}}{2(1)} \\ &= \frac{-\left(x^2+1\right) \pm \sqrt{x^4+12x^3+2x^2+25}}{2} \end{aligned}$$

Now, reapply the initial condition to figure out which of the two signs in the $\pm$ that we need.

$$-3 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = -3,\ 2$$

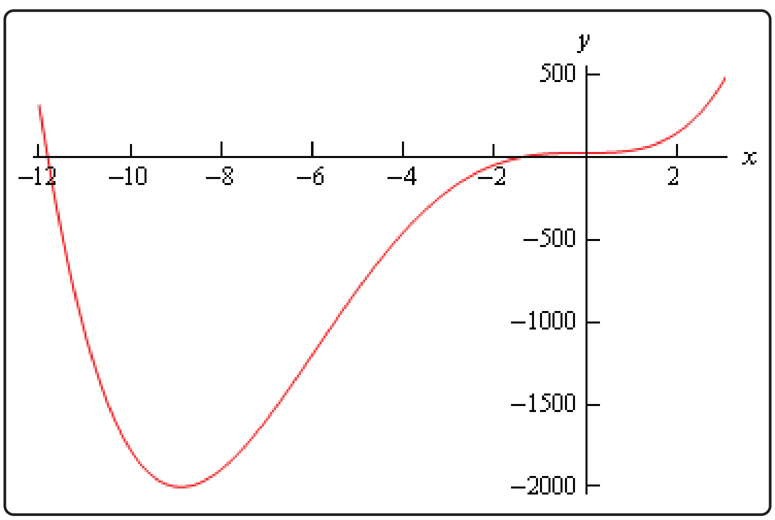
So, it looks like the "$-$" is the one that we need. The explicit solution is then.

$$y(x) = \frac{-\left(x^2+1\right) - \sqrt{x^4+12x^3+2x^2+25}}{2}$$

Now, for the interval of validity. It looks like we might well have problems with square roots of negative numbers. So, we need to solve

$$x^4+12x^3+2x^2+25 = 0$$

Upon solving this equation is zero at $x = -11.81557624$ and $x = -1.396911133$. Note that you'll need to use some form of computational aid in solving this equation. Here is a graph of the polynomial under the radical.



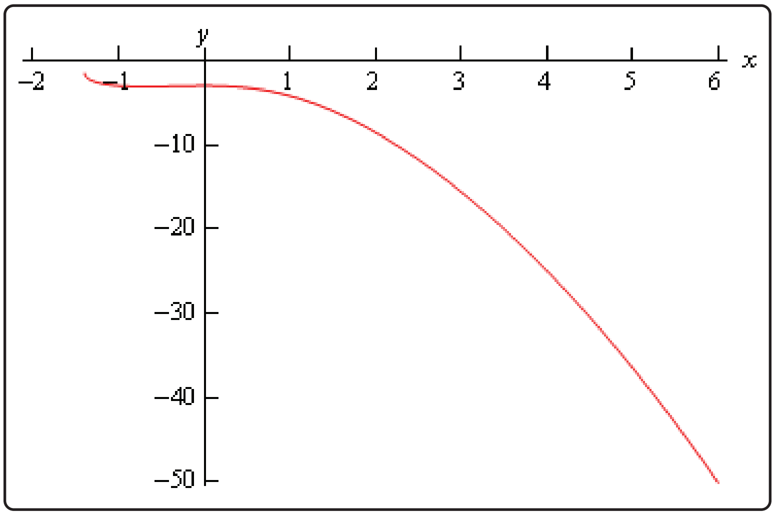
So, it looks like there are two intervals where the polynomial will be positive.

$$\begin{gathered} -\infty < x \le -11.81557624 \\ -1.396911133 \le x < \infty \end{gathered}$$

However, recall that intervals of validity need to be continuous intervals and contain the value of x that is used in the initial condition. Therefore, the interval of validity must be.

$$-1.396911133 \le x < \infty$$

Here is a quick graph of the solution.



That was a long example, but mostly because of the initial explanation of how to find $\Psi(x, y)$. The remaining examples will not be as long.

Example 3

Find the solution and interval of validity for the following IVP.

$$2xy^2 + 4 = 2\left(3 - x^2 y\right) y' \qquad y(-1) = 8$$

Solution

Here, we first need to put the differential equation into proper form before proceeding. Recall that it needs to be "$= 0$" and the sign separating the two terms must be a plus!

$$\begin{aligned} 2xy^2 + 4 - 2\left(3 - x^2 y\right) y' &= 0 \\ 2xy^2 + 4 + 2\left(x^2 y - 3\right) y' &= 0 \end{aligned}$$

So, we have the following

$$\begin{aligned} M &= 2xy^2 + 4 & M_y &= 4xy \\ N &= 2x^2 y - 6 & N_x &= 4xy \end{aligned}$$

and so the differential equation is exact. We can either integrate M with respect to x or integrate N with respect to y. In this case either would be just as easy so we'll integrate N this time so we can say that we've got an example of both down here.

$$\Psi(x,y) = \int 2x^2y - 6\,dy = x^2y^2 - 6y + h(x)$$

This time, as opposed to the previous example, our "constant" of integration must be a function of x since we integrated with respect to y. Now differentiate with respect to x and compare this to M.

$$\Psi_x = 2xy^2 + h'(x) = 2xy^2 + 4 = M$$

So, it looks like

$$h'(x) = 4 \qquad \Rightarrow \qquad h(x) = 4x$$

Again, we'll drop the constant of integration that technically should be present in $h(x)$ since it will just get absorbed into the constant we pick up in the next step. Also note that, $h(x)$ should only involve x's at this point. If there are any y's left at this point a mistake has been made so go back and look for it.

Writing everything down gives us the following for $\Psi(x,y)$.

$$\Psi(x,y) = x^2y^2 - 6y + 4x$$

So, the implicit solution to the differential equation is

$$x^2y^2 - 6y + 4x = c$$

Applying the initial condition gives,

$$64 - 48 - 4 = c \qquad \rightarrow \qquad c = 12$$

The solution is then

$$x^2y^2 - 6y + 4x - 12 = 0$$

Using the quadratic formula gives us

$$\begin{aligned} y(x) &= \frac{6 \pm \sqrt{36 - 4x^2(4x - 12)}}{2x^2} \\ &= \frac{6 \pm \sqrt{36 + 48x^2 - 16x^3}}{2x^2} \\ &= \frac{6 \pm 2\sqrt{9 + 12x^2 - 4x^3}}{2x^2} \\ &= \frac{3 \pm \sqrt{9 + 12x^2 - 4x^3}}{x^2} \end{aligned}$$

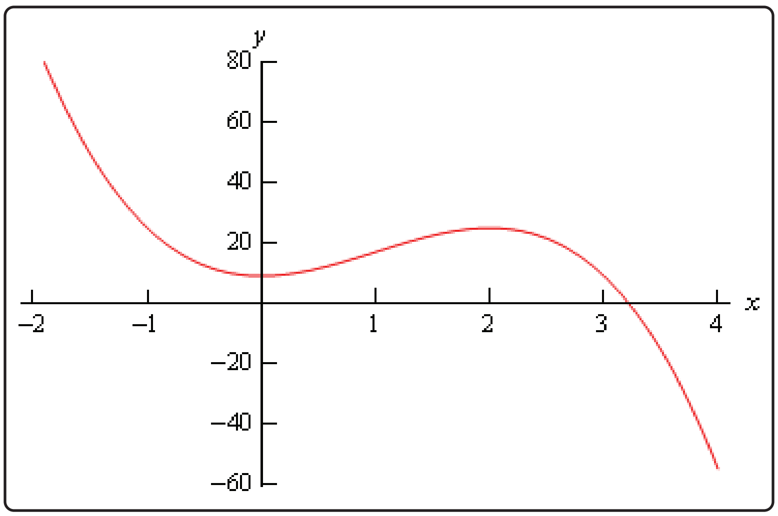
Reapplying the initial condition shows that this time we need the "$+$" (we'll leave those details to you to check). Therefore, the explicit solution is

$$y(x) = \frac{3 + \sqrt{9 + 12x^2 - 4x^3}}{x^2}$$

Now let's find the interval of validity. We'll need to avoid $x = 0$ so we don't get division by zero. We'll also have to watch out for square roots of negative numbers so solve the following equation.

$$-4x^3 + 12x^2 + 9 = 0$$

The only real solution here is $x = 3.217361577$. Below is a graph of the polynomial.



So, it looks like the polynomial will be positive, and hence okay under the square root on

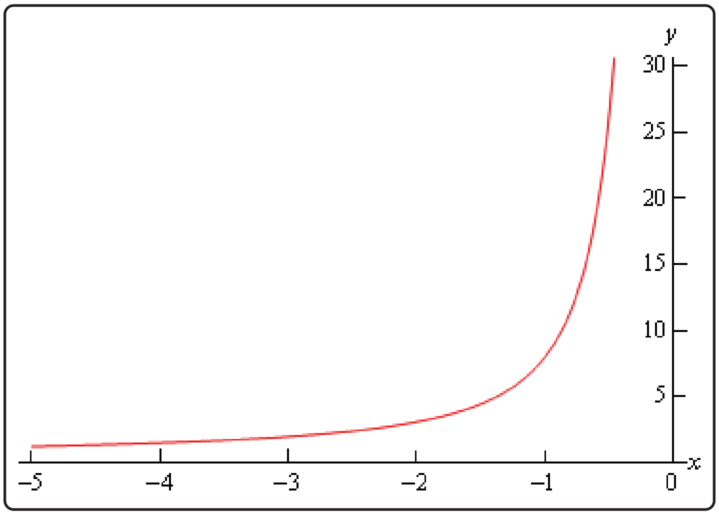
$$-\infty < x < 3.217361577$$

Now, this interval can't be the interval of validity because it contains $x = 0$ and we need to avoid that point. Therefore, this interval actually breaks up into two different possible intervals of validity.

$$-\infty < x < 0$$
$$0 < x < 3.217361577$$

The first one contains $x = -1$, the x value from the initial condition. Therefore, the interval of validity for this problem is $-\infty < x < 0$.

Here is a graph of the solution.



Example 4

Find the solution and interval of validity to the following IVP.

$$\frac{2ty}{t^2+1} - 2t - \left(2 - \ln\left(t^2+1\right)\right) y' = 0 \qquad y\left(5\right) = 0$$

Solution

So, first deal with that minus sign separating the two terms.

$$\frac{2ty}{t^2+1} - 2t + \left(\ln\left(t^2+1\right) - 2\right) y' = 0$$

Now, find M and N and check that it's exact.

$$M = \frac{2ty}{t^2+1} - 2t \qquad M_y = \frac{2t}{t^2+1}$$

$$N = \ln\left(t^2+1\right) - 2 \qquad N_t = \frac{2t}{t^2+1}$$

So, it's exact. We'll integrate the first one in this case.

$$\Psi\left(t,y\right) = \int \frac{2ty}{t^2+1} - 2t\,dt = y\ln\left(t^2+1\right) - t^2 + h\left(y\right)$$

Differentiate with respect to y and compare to N.

$$\Psi_y = \ln\left(t^2+1\right) + h'\left(y\right) = \ln\left(t^2+1\right) - 2 = N$$

So, it looks like we've got.

$$h'(y) = -2 \qquad \Rightarrow \qquad h(y) = -2y$$

This gives us

$$\Psi(t,y) = y\ln\left(t^2+1\right) - t^2 - 2y$$

The implicit solution is then,

$$y\ln\left(t^2+1\right) - t^2 - 2y = c$$

Applying the initial condition gives,

$$-25 = c$$

The implicit solution is now,

$$y\left(\ln\left(t^2+1\right) - 2\right) - t^2 = -25$$

This solution is much easier to solve than the previous ones. No quadratic formula is needed this time, all we need to do is solve for y. Here's what we get for an explicit solution.

$$y(t) = \frac{t^2 - 25}{\ln\left(t^2+1\right) - 2}$$

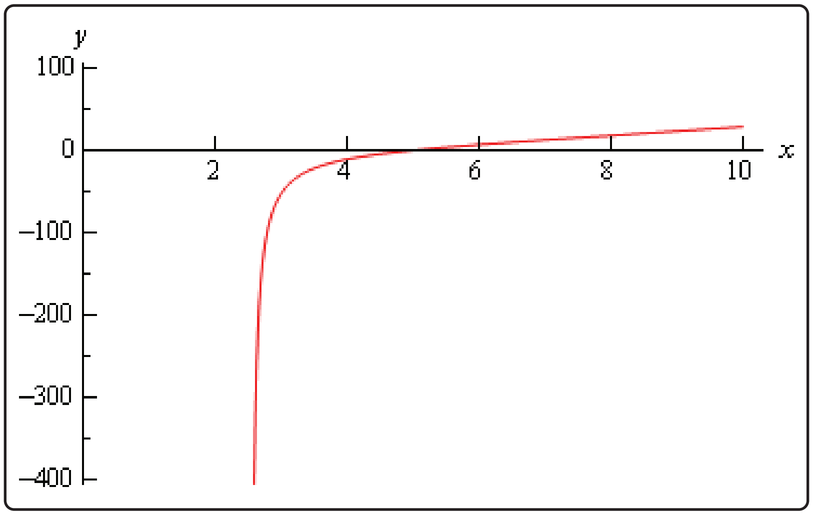
Alright, let's get the interval of validity. The term in the logarithm is always positive so we don't need to worry about negative numbers in that. We do need to worry about division by zero however. We will need to avoid the following point(s).

$$\begin{aligned} \ln\left(t^2+1\right) - 2 &= 0 \\ \ln\left(t^2+1\right) &= 2 \\ t^2 + 1 &= \mathbf{e}^2 \\ t &= \pm\sqrt{\mathbf{e}^2 - 1} \end{aligned}$$

We now have three possible intervals of validity.

$$\begin{gathered} -\infty < t < -\sqrt{\mathbf{e}^2-1} \\ -\sqrt{\mathbf{e}^2-1} < t < \sqrt{\mathbf{e}^2-1} \\ \sqrt{\mathbf{e}^2-1} < t < \infty \end{gathered}$$

The last one contains $t = 5$ and so is the interval of validity for this problem is $\sqrt{\mathbf{e}^2-1} < t < \infty$. Here's a graph of the solution.



Example 5

Find the solution and interval of validity for the following IVP.

$$3y^3\mathbf{e}^{3xy} - 1 + \left(2y\mathbf{e}^{3xy} + 3xy^2\mathbf{e}^{3xy}\right)y' = 0 \qquad y(0) = 1$$

Solution

Let's identify M and N and check that it's exact.

$$\begin{aligned} M &= 3y^3\mathbf{e}^{3xy} - 1 & \qquad M_y &= 9y^2\mathbf{e}^{3xy} + 9xy^3\mathbf{e}^{3xy} \\ N &= 2y\mathbf{e}^{3xy} + 3xy^2\mathbf{e}^{3xy} & \qquad N_x &= 9y^2\mathbf{e}^{3xy} + 9xy^3\mathbf{e}^{3xy} \end{aligned}$$

So, it's exact. With the proper simplification integrating the second one isn't too bad.

However, the first is already set up for easy integration so let's do that one.

$$\Psi(x,y) = \int 3y^3\mathbf{e}^{3xy} - 1\,dx = y^2\mathbf{e}^{3xy} - x + h(y)$$

Differentiate with respect to y and compare to N.

$$\Psi_y = 2y\mathbf{e}^{3xy} + 3xy^2\mathbf{e}^{3xy} + h'(y) = 2y\mathbf{e}^{3xy} + 3xy^2\mathbf{e}^{3xy} = N$$

So, it looks like we've got

$$h'(y) = 0 \qquad \Rightarrow \qquad h(y) = 0$$

Recall that actually $h(y) = k$, but we drop the k because it will get absorbed in the next step. That gives us $h(y) = 0$. Therefore, we get.

$$\Psi(x, y) = y^2 \mathbf{e}^{3xy} - x$$

The implicit solution is then

$$y^2 \mathbf{e}^{3xy} - x = c$$

Apply the initial condition.

$$1 = c$$

The implicit solution is then

$$y^2 \mathbf{e}^{3xy} - x = 1$$

This is as far as we can go. There is no way to solve this for y and get an explicit solution.

2.4 Bernoulli Differential Equations

In this section we are going to take a look at differential equations in the form,

$$y' + p(x)\,y = q(x)\,y^n$$

where $p(x)$ and $q(x)$ are continuous functions on the interval we're working on and n is a real number. Differential equations in this form are called **Bernoulli Equations**.

First notice that if $n = 0$ or $n = 1$ then the equation is linear and we already know how to solve it in these cases. Therefore, in this section we're going to be looking at solutions for values of n other than these two.

In order to solve these we'll first divide the differential equation by y^n to get,

$$y^{-n}\,y' + p(x)\,y^{1-n} = q(x)$$

We are now going to use the substitution $v = y^{1-n}$ to convert this into a differential equation in terms of v. As we'll see this will lead to a differential equation that we can solve.

We are going to have to be careful with this however when it comes to dealing with the derivative, y'. We need to determine just what y' is in terms of our substitution. This is easier to do than it might at first look to be. All that we need to do is differentiate both sides of our substitution with respect to x. Remember that both v and y are functions of x and so we'll need to use the chain rule on the right side. If you remember your Calculus I you'll recall this is just implicit differentiation. So, taking the derivative gives us,

$$v' = (1-n)\,y^{-n}y'$$

Now, plugging this as well as our substitution into the differential equation gives,

$$\frac{1}{1-n}v' + p(x)\,v = q(x)$$

This is a linear differential equation that we can solve for v and once we have this in hand we can also get the solution to the original differential equation by plugging v back into our substitution and solving for y.

Let's take a look at an example.

Example 1

Solve the following IVP and find the interval of validity for the solution.

$$y' + \frac{4}{x}y = x^3y^2 \qquad y(2) = -1, \qquad x > 0$$

Solution

So, the first thing that we need to do is get this into the "proper" form and that means dividing everything by y^2. Doing this gives,

$$y^{-2}y' + \frac{4}{x}y^{-1} = x^3$$

The substitution and derivative that we'll need here is,

$$v = y^{-1} \qquad v' = -y^{-2}y'$$

With this substitution the differential equation becomes,

$$-v' + \frac{4}{x}v = x^3$$

So, as noted above this is a linear differential equation that we know how to solve. We'll do the details on this one and then for the rest of the examples in this section we'll leave the details for you to fill in. If you need a refresher on solving linear differential equations then go back to that section for a quick review.

Here's the solution to this differential equation.

$$v' - \frac{4}{x}v = -x^3 \qquad \Rightarrow \qquad \mu(x) = \mathbf{e}^{\int -\frac{4}{x}\,dx} = \mathbf{e}^{-4\ln|x|} = x^{-4}$$

$$\int \left(x^{-4}v\right)'\,dx = \int -x^{-1}\,dx$$

$$x^{-4}v = -\ln|x| + c \qquad \Rightarrow \qquad v(x) = cx^4 - x^4\ln(x)$$

Note that we dropped the absolute value bars on the x in the logarithm because of the assumption that $x > 0$.

Now we need to determine the constant of integration. This can be done in one of two ways. We can can convert the solution above into a solution in terms of y and then use the original initial condition or we can convert the initial condition to an initial condition in terms of v and use that. Because we'll need to convert the solution to y's eventually anyway and it won't add that much work in we'll do it that way.

So, to get the solution in terms of y all we need to do is plug the substitution back in. Doing this gives,

$$y^{-1} = x^4 \left(c - \ln(x)\right)$$

At this point we can solve for y and then apply the initial condition or apply the initial condition and then solve for y. We'll generally do this with the later approach so let's apply the initial condition to get,

$$(-1)^{-1} = c2^4 - 2^4 \ln 2 \qquad \Rightarrow \qquad c = \ln 2 - \frac{1}{16}$$

Plugging in for c and solving for y gives,

$$y(x) = \frac{1}{x^4 \left(\ln 2 - \frac{1}{16} - \ln(x)\right)} = \frac{-16}{x^4 \left(1 + 16\ln(x) - 16\ln 2\right)} = \frac{-16}{x^4 \left(1 + 16\ln \frac{x}{2}\right)}$$

Note that we did a little simplification in the solution. This will help with finding the interval of validity.

Before finding the interval of validity however, we mentioned above that we could convert the original initial condition into an initial condition for v. Let's briefly talk about how to do that. To do that all we need to do is plug $x = 2$ into the substitution and then use the original initial condition. Doing this gives,

$$v(2) = y^{-1}(2) = (-1)^{-1} = -1$$

So, in this case we got the same value for v that we had for y. Don't expect that to happen in general if you chose to do the problems in this manner.

Okay, let's now find the interval of validity for the solution. First, we already know that $x > 0$ and that means we'll avoid the problems of having logarithms of negative numbers and division by zero at $x = 0$. So, all that we need to worry about then is division by zero in the second term and this will happen where,

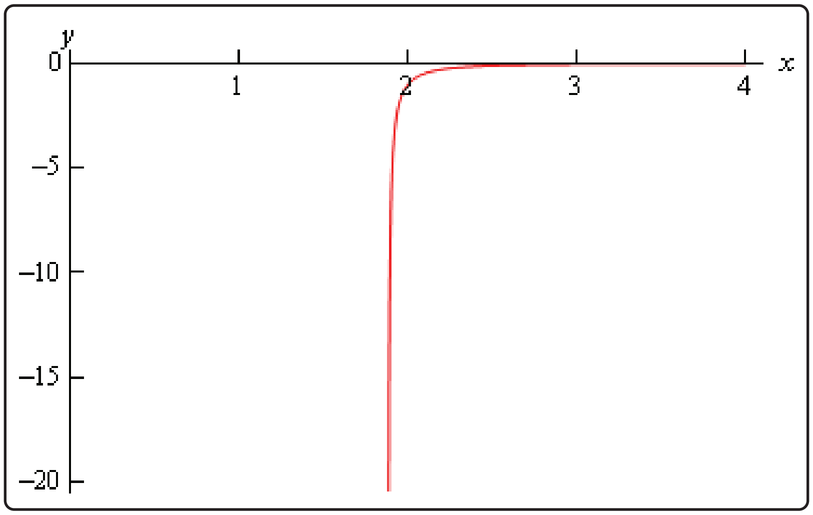
$$\begin{aligned} 1 + 16\ln\frac{x}{2} &= 0 \\ \ln\frac{x}{2} &= -\frac{1}{16} \\ \frac{x}{2} &= \mathbf{e}^{-\frac{1}{16}} \qquad \Rightarrow \qquad x = 2\mathbf{e}^{-\frac{1}{16}} \approx 1.8788 \end{aligned}$$

The two possible intervals of validity are then,

$$0 < x < 2\mathbf{e}^{-\frac{1}{16}} \qquad 2\mathbf{e}^{-\frac{1}{16}} < x < \infty$$

and since the second one contains the initial condition we know that the interval of validity is then $2\mathbf{e}^{-\frac{1}{16}} < x < \infty$.

Here is a graph of the solution.



Let's do a couple more examples and as noted above we're going to leave it to you to solve the linear differential equation when we get to that stage.

Example 2

Solve the following IVP and find the interval of validity for the solution.

$$y' = 5y + \mathbf{e}^{-2x} y^{-2} \qquad y(0) = 2$$

Solution

The first thing we'll need to do here is multiply through by y^2 and we'll also do a little rearranging to get things into the form we'll need for the linear differential equation. This gives,

$$y^2 y' - 5y^3 = \mathbf{e}^{-2x}$$

The substitution here and its derivative is,

$$v = y^3 \qquad v' = 3y^2 y'$$

Plugging the substitution into the differential equation gives,

$$\frac{1}{3}v' - 5v = \mathbf{e}^{-2x} \qquad \Rightarrow \qquad v' - 15v = 3\mathbf{e}^{-2x} \qquad \mu(x) = \mathbf{e}^{-15x}$$

We rearranged a little and gave the integrating factor for the linear differential equation solution. Upon solving we get,

$$v(x) = c\mathbf{e}^{15x} - \frac{3}{17}\mathbf{e}^{-2x}$$

Now go back to y's.

$$y^3 = c\mathbf{e}^{15x} - \frac{3}{17}\mathbf{e}^{-2x}$$

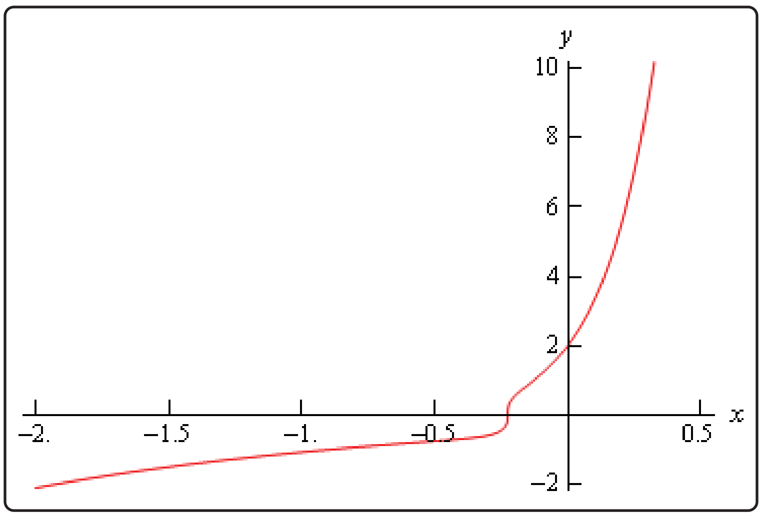
Applying the initial condition and solving for c gives,

$$8 = c - \frac{3}{17} \qquad \Rightarrow \qquad c = \frac{139}{17}$$

Plugging in c and solving for y gives,

$$y(x) = \left(\frac{139\mathbf{e}^{15x} - 3\mathbf{e}^{-2x}}{17}\right)^{\frac{1}{3}}$$

There are no problem values of x for this solution and so the interval of validity is all real numbers. Here's a graph of the solution.



Example 3

Solve the following IVP and find the interval of validity for the solution.

$$6y' - 2y = x\,y^4 \qquad y(0) = -2$$

Solution

First get the differential equation in the proper form and then write down the substitution.

$$6y^{-4}y' - 2y^{-3} = x \qquad \Rightarrow \qquad v = y^{-3} \qquad v' = -3y^{-4}y'$$

Plugging the substitution into the differential equation gives,

$$-2v' - 2v = x \qquad \Rightarrow \qquad v' + v = -\frac{1}{2}x \qquad \mu(x) = \mathbf{e}^{x}$$

Again, we've rearranged a little and given the integrating factor needed to solve the linear differential equation. Upon solving the linear differential equation we have,

$$v(x) = -\frac{1}{2}(x-1) + c\mathbf{e}^{-x}$$

Now back substitute to get back into y's.

$$y^{-3} = -\frac{1}{2}(x-1) + c\mathbf{e}^{-x}$$

Now we need to apply the initial condition and solve for c.

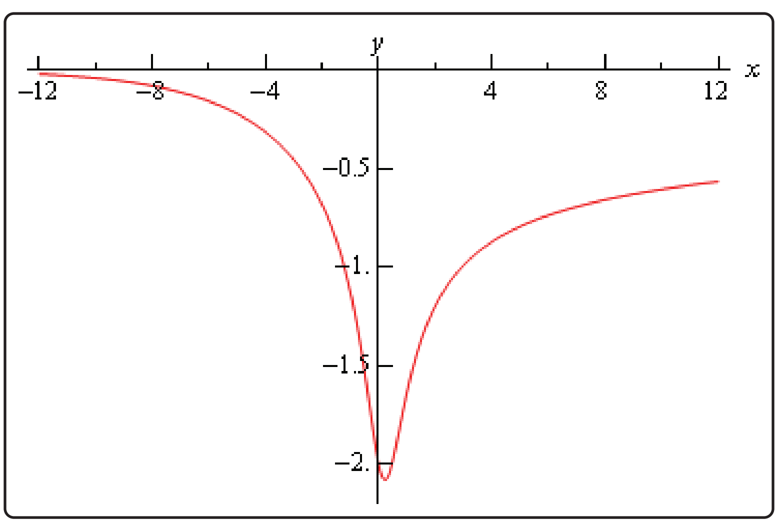
$$-\frac{1}{8} = \frac{1}{2} + c \qquad \Rightarrow \qquad c = -\frac{5}{8}$$

Plugging in c and solving for y gives,

$$y(x) = -\frac{2}{\left(4x - 4 + 5\mathbf{e}^{-x}\right)^{\frac{1}{3}}}$$

Next, we need to think about the interval of validity. In this case all we need to worry about it is division by zero issues and using some form of computational aid (such as Maple or Mathematica) we will see that the denominator of our solution is never zero and so this solution will be valid for all real numbers.

Here is a graph of the solution.



To this point we've only worked examples in which n was an integer (positive and negative) and so we should work a quick example where n is not an integer.

Example 4

Solve the following IVP and find the interval of validity for the solution.

$$y' + \frac{y}{x} - \sqrt{y} = 0 \qquad y(1) = 0$$

Solution

Let's first get the differential equation into proper form.

$$y' + \frac{1}{x}y = y^{\frac{1}{2}} \qquad \Rightarrow \qquad y^{-\frac{1}{2}}y' + \frac{1}{x}y^{\frac{1}{2}} = 1$$

The substitution is then,

$$v = y^{\frac{1}{2}} \qquad v' = \frac{1}{2}y^{-\frac{1}{2}}y'$$

Now plug the substitution into the differential equation to get,

$$2v' + \frac{1}{x}v = 1 \qquad \Rightarrow \qquad v' + \frac{1}{2x}v = \frac{1}{2} \qquad \mu(x) = x^{\frac{1}{2}}$$

As we've done with the previous examples we've done some rearranging and given the integrating factor needed for solving the linear differential equation. Solving this gives us,

$$v(x) = \frac{1}{3}x + cx^{-\frac{1}{2}}$$

In terms of y this is,

$$y^{\frac{1}{2}} = \frac{1}{3}x + cx^{-\frac{1}{2}}$$

Applying the initial condition and solving for c gives,

$$0 = \frac{1}{3} + c \qquad \Rightarrow \qquad c = -\frac{1}{3}$$

Plugging in for c and solving for y gives us the solution.

$$y(x) = \left(\frac{1}{3}x - \frac{1}{3}x^{-\frac{1}{2}}\right)^2 = \frac{x^3 - 2x^{\frac{3}{2}} + 1}{9x}$$

Note that we multiplied everything out and converted all the negative exponents to positive exponents to make the interval of validity clear here. Because of the root (in the second term in the numerator) and the x in the denominator we can see that we need to require $x > 0$ in order for the solution to exist and it will exist for all positive x's and so this is also the interval of validity.

Here is the graph of the solution.