3 Second Order Differential Equations

In the previous chapter we looked at first order differential equations. In this chapter we will move on to second order differential equations and just as we did in the last chapter we will look at some special cases of second order differential equations that we can solve. Unlike the previous chapter however, we are going to have to be even more restrictive as to the kinds of differential equations that we'll look at. This will be required in order for us to actually be able to solve them. This is going to mean that, generally, we will only be looking at differential equations in which the coefficients of the function and it's derivatives are constants.

In addition we'll be looking at differential equations in which every term has either the function or it's derivative in it (homogeneous differential equation) and differential equations that allow terms that do not contain the function or it's derivative (nonhomogeneous differential equation). As we'll see we can only solve nonhomogeneous differential equations if we can first solve an associated homogeneous differential equation.

We'll close out this chapter with a quick look at mechanical vibrations an application of second order differential equations.

3.1 Basic Concepts

In this chapter we will be looking exclusively at linear second order differential equations. The most general linear second order differential equation is in the form.

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = g(t) \tag{3.1}$$

In fact, we will rarely look at non-constant coefficient linear second order differential equations. In the case where we assume constant coefficients we will use the following differential equation.

$$ay'' + by' + cy = g(t) \tag{3.2}$$

Where possible we will use Equation 3.1 just to make the point that certain facts, theorems, properties, and/or techniques can be used with the non-constant form. However, most of the time we will be using Equation 3.2 as it can be fairly difficult to solve second order non-constant coefficient differential equations.

Initially we will make our life easier by looking at differential equations with $g(t) = 0$. When $g(t) = 0$ we call the differential equation **homogeneous** and when $g(t) \neq 0$ we call the differential equation **nonhomogeneous**.

So, let's start thinking about how to go about solving a constant coefficient, homogeneous, linear, second order differential equation. Here is the general constant coefficient, homogeneous, linear, second order differential equation.

$$ay'' + by' + cy = 0$$

It's probably best to start off with an example. This example will lead us to a very important fact that we will use in every problem from this point on. The example will also give us clues into how to go about solving these in general.

Example 1

Determine some solutions to

$$y'' - 9y = 0$$

Solution

We can get some solutions here simply by inspection. We need functions whose second derivative is 9 times the original function. One of the first functions that I can think of that comes back to itself after two derivatives is an exponential function and with proper exponents the 9 will get taken care of as well.

So, it looks like the following two functions are solutions.

$$y(t) = \mathbf{e}^{3t} \qquad \text{and} \qquad y(t) = \mathbf{e}^{-3t}$$

We'll leave it to you to verify that these are in fact solutions.

These two functions are not the only solutions to the differential equation however. Any of the following are also solutions to the differential equation.

$$y(t) = -9\mathbf{e}^{3t} \qquad y(t) = 123\mathbf{e}^{3t}$$
$$y(t) = 56\mathbf{e}^{-3t} \qquad y(t) = \frac{14}{9}\mathbf{e}^{-3t}$$
$$y(t) = 7\mathbf{e}^{3t} - 6\mathbf{e}^{-3t} \qquad y(t) = -92\mathbf{e}^{3t} - 16\mathbf{e}^{-3t}$$

In fact if you think about it any function that is in the form

$$y(t) = c_1\mathbf{e}^{3t} + c_2\mathbf{e}^{-3t}$$

will be a solution to the differential equation.

This example leads us to a very important fact that we will use in practically every problem in this chapter.

Principle of Superposition

If $y_1(t)$ and $y_2(t)$ are two solutions to a linear, homogeneous differential equation then so is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3.3}$$

Note that we didn't include the restriction of constant coefficient or second order in this. This will work for any linear homogeneous differential equation.

If we further assume second order and one other condition (which we'll give in a second) we can go a step further.

If $y_1(t)$ and $y_2(t)$ are two solutions to a linear, second order homogeneous differential equation and they are "nice enough" then the general solution to the linear, second order homogeneous differential equation is given by Equation 3.3.

So, just what do we mean by "nice enough"? We'll hold off on that until a later section. At this point you'll hopefully believe it when we say that specific functions are "nice enough".

So, if we now make the assumption that we are dealing with a linear, second order homogeneous differential equation, we now know that Equation 3.3 will be its general solution. The next question that we can ask is how to find the constants c_1 and c_2. Since we have two constants it makes sense, hopefully, that we will need two equations, or conditions, to find them.

One way to do this is to specify the value of the solution at two distinct points, or,

$$y(t_0) = y_0 \qquad y(t_1) = y_1$$

These are typically called boundary values and are not really the focus of this course so we won't be working with them here. We do give a brief introduction to boundary values in a later chapter if you are interested in seeing how they work and some of the issues that arise when working with boundary values.

Another way to find the constants would be to specify the value of the solution and its derivative at a particular point. Or,

$$y(t_0) = y_0 \qquad y'(t_0) = y'_0$$

These are the two conditions that we'll be using here. As with the first order differential equations these will be called initial conditions.

Example 2

Solve the following IVP.

$$y'' - 9y = 0 \qquad y(0) = 2 \quad y'(0) = -1$$

Solution

First, the two functions

$$y(t) = \mathbf{e}^{3t} \qquad \text{and} \qquad y(t) = \mathbf{e}^{-3t}$$

are "nice enough" for us to form the general solution to the differential equation. At this point, please just believe this. You will be able to verify this for yourself in a couple of sections.

The general solution to our differential equation is then

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{3t}$$

Now all we need to do is apply the initial conditions. This means that we need the derivative of the solution.

$$y'(t) = -3c_1\mathbf{e}^{-3t} + 3c_2\mathbf{e}^{3t}$$

Plug in the initial conditions

$$\begin{aligned} 2 &= y(0) = c_1 + c_2 \\ -1 &= y'(0) = -3c_1 + 3c_2 \end{aligned}$$

This gives us a system of two equations and two unknowns that can be solved. Doing this yields

$$c_1 = \frac{7}{6} \qquad c_2 = \frac{5}{6}$$

The solution to the IVP is then,

$$y(t) = \frac{7}{6}\mathbf{e}^{-3t} + \frac{5}{6}\mathbf{e}^{3t}$$

Up to this point we've only looked at a single differential equation and we got its solution by inspection. For a rare few differential equations we can do this. However, for the vast majority of the second order differential equations out there we will be unable to do this.

So, we would like a method for arriving at the two solutions we will need in order to form a general solution that will work for any linear, constant coefficient, second order homogeneous differential equation. This is easier than it might initially look.

We will use the solutions we found in the first example as a guide. All of the solutions in this example were in the form

$$y(t) = \mathbf{e}^{r\,t}$$

Note, that we didn't include a constant in front of it since we can literally include any constant that we want and still get a solution. The important idea here is to get the exponential function. Once we have that we can add on constants to our hearts content.

So, let's assume that all solutions to

$$ay'' + by' + cy = 0 \tag{3.4}$$

will be of the form

$$y(t) = \mathbf{e}^{r\,t} \tag{3.5}$$

To see if we are correct all we need to do is plug this into the differential equation and see what happens. So, let's get some derivatives and then plug in.

$$y'(t) = r\mathbf{e}^{r\,t} \qquad y''(t) = r^2\mathbf{e}^{r\,t}$$

$$\begin{aligned} a\left(r^2\mathbf{e}^{r\,t}\right) + b\left(r\mathbf{e}^{r\,t}\right) + c\left(\mathbf{e}^{r\,t}\right) &= 0 \\ \mathbf{e}^{r\,t}\left(ar^2 + br + c\right) &= 0 \end{aligned}$$

So, if Equation 3.5 is to be a solution to Equation 3.4 then the following must be true

$$\mathbf{e}^{r\,t}\left(ar^2 + br + c\right) = 0$$

This can be reduced further by noting that exponentials are never zero. Therefore, Equation 3.5 will be a solution to Equation 3.4 provided r is a solution to

$$ar^2 + br + c = 0 \tag{3.6}$$

This equation is typically called the **characteristic equation** for Equation 3.4.

Okay, so how do we use this to find solutions to a linear, constant coefficient, second order homogeneous differential equation? First write down the characteristic equation, Equation 3.6, for the differential equation, Equation 3.4. This will be a quadratic equation and so we should expect two roots, r_1 and r_2. Once we have these two roots we have two solutions to the differential equation.

$$y_1(t) = \mathbf{e}^{r_1 t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{r_2 t} \tag{3.7}$$

Let's take a look at a quick example.

Example 3

Find two solutions to

$$y'' - 9y = 0$$

Solution

This is the same differential equation that we looked at in the first example. This time however, let's not just guess. Let's go through the process as outlined above to see the functions that we guess above are the same as the functions the process gives us.

First write down the characteristic equation for this differential equation and solve it.

$$r^2 - 9 = 0 \qquad \Rightarrow \qquad r = \pm 3$$

The two roots are 3 and -3. Therefore, two solutions are

$$y_1(t) = \mathbf{e}^{3t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{-3t}$$

These match up with the first guesses that we made in the first example.

You'll notice that we neglected to mention whether or not the two solutions listed in Equation 3.7 are in fact "nice enough" to form the general solution to Equation 3.4. This was intentional. We have three cases that we need to look at and this will be addressed differently in each of these cases.

So, what are the cases? As we previously noted the characteristic equation is quadratic and so will have two roots, r_1 and r_2. The roots will have three possible forms. These are

1. Real, distinct roots, $r_1 \neq r_2$.
2. Complex root, $r_{1,2} = \lambda \pm \mu\, i$.
3. Double roots, $r_1 = r_2 = r$.

The next three sections will look at each of these in some more depth, including giving forms for the solution that will be "nice enough" to get a general solution.

3.2 Real & Distinct Roots

It's time to start solving constant coefficient, homogeneous, linear, second order differential equations. So, let's recap how we do this from the last section. We start with the differential equation.

$$ay'' + by' + cy = 0$$

Write down the characteristic equation.

$$ar^2 + br + c = 0$$

Solve the characteristic equation for the two roots, r_1 and r_2. This gives the two solutions

$$y_1(t) = \mathbf{e}^{r_1 t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{r_2 t}$$

Now, if the two roots are real and distinct (*i.e.* $r_1 \neq r_2$) it will turn out that these two solutions are "nice enough" to form the general solution

$$y(t) = c_1\mathbf{e}^{r_1 t} + c_2\mathbf{e}^{r_2 t}$$

As with the last section, we'll ask that you believe us when we say that these are "nice enough". You will be able to prove this easily enough once we reach a later section.

With real, distinct roots there really isn't a whole lot to do other than work a couple of examples so let's do that.

Example 1

Solve the following IVP.

$$y'' + 11y' + 24y = 0 \qquad y(0) = 0 \qquad y'(0) = -7$$

Solution

The characteristic equation is

$$\begin{aligned} r^2 + 11r + 24 &= 0 \\ (r+8)(r+3) &= 0 \end{aligned}$$

Its roots are $r_1 = -8$ and $r_2 = -3$ and so the general solution and its derivative is.

$$\begin{aligned} y(t) &= c_1\mathbf{e}^{-8t} + c_2\mathbf{e}^{-3t} \\ y'(t) &= -8c_1\mathbf{e}^{-8t} - 3c_2\mathbf{e}^{-3t} \end{aligned}$$

Now, plug in the initial conditions to get the following system of equations.

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 \\ -7 &= y'(0) = -8c_1 - 3c_2 \end{aligned}$$

Solving this system gives $c_1 = \frac{7}{5}$ and $c_2 = -\frac{7}{5}$. The actual solution to the differential equation is then

$$y(t) = \frac{7}{5}\mathbf{e}^{-8t} - \frac{7}{5}\mathbf{e}^{-3t}$$

Example 2

Solve the following IVP

$$y'' + 3y' - 10y = 0 \qquad y(0) = 4 \qquad y'(0) = -2$$

Solution

The characteristic equation is

$$\begin{aligned} r^2 + 3r - 10 &= 0 \\ (r+5)(r-2) &= 0 \end{aligned}$$

Its roots are $r_1 = -5$ and $r_2 = 2$ and so the general solution and its derivative is.

$$\begin{aligned} y(t) &= c_1\mathbf{e}^{-5t} + c_2\mathbf{e}^{2t} \\ y'(t) &= -5c_1\mathbf{e}^{-5t} + 2c_2\mathbf{e}^{2t} \end{aligned}$$

Now, plug in the initial conditions to get the following system of equations.

$$\begin{aligned} 4 &= y(0) = c_1 + c_2 \\ -2 &= y'(0) = -5c_1 + 2c_2 \end{aligned}$$

Solving this system gives $c_1 = \frac{10}{7}$ and $c_2 = \frac{18}{7}$. The actual solution to the differential equation is then

$$y(t) = \frac{10}{7}\mathbf{e}^{-5t} + \frac{18}{7}\mathbf{e}^{2t}$$

Example 3

Solve the following IVP.

$$3y'' + 2y' - 8y = 0 \qquad y(0) = -6 \qquad y'(0) = -18$$

Solution

The characteristic equation is

$$\begin{aligned} 3r^2 + 2r - 8 &= 0 \\ (3r-4)(r+2) &= 0 \end{aligned}$$

Its roots are $r_1 = \frac{4}{3}$ and $r_2 = -2$ and so the general solution and its derivative is.

$$\begin{aligned} y(t) &= c_1 \mathbf{e}^{\frac{4t}{3}} + c_2 \mathbf{e}^{-2t} \\ y'(t) &= \frac{4}{3} c_1 \mathbf{e}^{\frac{4t}{3}} - 2c_2 \mathbf{e}^{-2t} \end{aligned}$$

Now, plug in the initial conditions to get the following system of equations.

$$\begin{aligned} -6 &= y(0) = c_1 + c_2 \\ -18 &= y'(0) = \frac{4}{3} c_1 - 2c_2 \end{aligned}$$

Solving this system gives $c_1 = -9$ and $c_2 = 3$. The actual solution to the differential equation is then.

$$y(t) = -9\mathbf{e}^{\frac{4t}{3}} + 3\mathbf{e}^{-2t}$$

Example 4

Solve the following IVP

$$4y'' - 5y' = 0 \qquad y(-2) = 0 \qquad y'(-2) = 7$$

Solution

The characteristic equation is

$$\begin{aligned} 4r^2 - 5r &= 0 \\ r(4r-5) &= 0 \end{aligned}$$

The roots of this equation are $r_1 = 0$ and $r_2 = \frac{5}{4}$. Here is the general solution as well as its derivative.

$$y(t) = c_1\mathbf{e}^0 + c_2\mathbf{e}^{\frac{5t}{4}} = c_1 + c_2\mathbf{e}^{\frac{5t}{4}}$$
$$y'(t) = \frac{5}{4}c_2\mathbf{e}^{\frac{5t}{4}}$$

Up to this point all of the initial conditions have been at $t = 0$ and this one isn't. Don't get too locked into initial conditions always being at $t = 0$ and you just automatically use that instead of the actual value for a given problem.

So, plugging in the initial conditions gives the following system of equations to solve.

$$0 = y(-2) = c_1 + c_2\mathbf{e}^{-\frac{5}{2}}$$
$$7 = y'(-2) = \frac{5}{4}c_2\mathbf{e}^{-\frac{5}{2}}$$

Solving this gives.

$$c_1 = -\frac{28}{5} \qquad c_2 = \frac{28}{5}\mathbf{e}^{\frac{5}{2}}$$

The solution to the differential equation is then.

$$y(t) = -\frac{28}{5} + \frac{28}{5}\mathbf{e}^{\frac{5}{2}}\mathbf{e}^{\frac{5t}{4}} = -\frac{28}{5} + \frac{28}{5}\mathbf{e}^{\frac{5t}{4}+\frac{5}{2}}$$

In a differential equations class most instructors (including me....) tend to use initial conditions at $t = 0$ because it makes the work a little easier for the students as they are trying to learn the subject. However, there is no reason to always expect that this will be the case, so do not start to always expect initial conditions at $t = 0$!

Let's do one final example to make another point that you need to be made aware of.

Example 5

Find the general solution to the following differential equation.

$$y'' - 6y' - 2y = 0$$

Solution

The characteristic equation is.

$$r^2 - 6r - 2 = 0$$

The roots of this equation are.

$$r_{1,2} = 3 \pm \sqrt{11}$$

Now, do NOT get excited about these roots they are just two real numbers.

$$r_1 = 3 + \sqrt{11} \qquad \text{and} \qquad r_2 = 3 - \sqrt{11}$$

Admittedly they are not as nice looking as we may be used to, but they are just real numbers. Therefore, the general solution is

$$y(t) = c_1 \mathbf{e}^{\left(3+\sqrt{11}\right)t} + c_2 \mathbf{e}^{\left(3-\sqrt{11}\right)t}$$

If we had initial conditions we could proceed as we did in the previous two examples although the work would be somewhat messy and so we aren't going to do that for this example.

The point of the last example is make sure that you don't get to used to "nice", simple roots. In practice roots of the characteristic equation will generally not be nice, simple integers or fractions so don't get too used to them!

3.3 Complex Roots

In this section we will be looking at solutions to the differential equation

$$ay'' + by' + cy = 0$$

in which roots of the characteristic equation,

$$ar^2 + br + c = 0$$

are complex roots in the form $r_{1,2} = \lambda \pm \mu\, i$.

Now, recall that we arrived at the characteristic equation by assuming that all solutions to the differential equation will be of the form

$$y(t) = \mathbf{e}^{rt}$$

Plugging our two roots into the general form of the solution gives the following solutions to the differential equation.

$$y_1(t) = \mathbf{e}^{(\lambda + \mu\, i)\, t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{(\lambda - \mu\, i)\, t}$$

Now, these two functions are "nice enough" (there's those words again... we'll get around to defining them eventually) to form the general solution. We do have a problem however. Since we started with only real numbers in our differential equation we would like our solution to only involve real numbers. The two solutions above are complex and so we would like to get our hands on a couple of solutions ("nice enough" of course...) that are real.

To do this we'll need Euler's Formula.

$$\mathbf{e}^{i\theta} = \cos(\theta) + i\sin(\theta)$$

A nice variant of Euler's Formula that we'll need is.

$$\mathbf{e}^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta)$$

Now, split up our two solutions into exponentials that only have real exponents and exponentials that only have imaginary exponents. Then use Euler's formula, or its variant, to rewrite the second exponential.

$$y_1(t) = \mathbf{e}^{\lambda\, t}\mathbf{e}^{i\mu\, t} = \mathbf{e}^{\lambda\, t}\left(\cos(\mu t) + i\sin(\mu\, t)\right)$$
$$y_2(t) = \mathbf{e}^{\lambda\, t}\mathbf{e}^{-i\mu\, t} = \mathbf{e}^{\lambda\, t}\left(\cos(\mu t) - i\sin(\mu\, t)\right)$$

This doesn't eliminate the complex nature of the solutions, but it does put the two solutions into a form that we can eliminate the complex parts.

Recall from the basics section that if two solutions are "nice enough" then any solution can be written as a combination of the two solutions. In other words,

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

will also be a solution.

Using this let's notice that if we add the two solutions together we will arrive at.

$$y_1(t) + y_2(t) = 2\mathbf{e}^{\lambda t}\cos(\mu t)$$

This is a real solution and just to eliminate the extraneous 2 let's divide everything by a 2. This gives the first real solution that we're after.

$$u(t) = \frac{1}{2}y_1(t) + \frac{1}{2}y_2(t) = \mathbf{e}^{\lambda t}\cos(\mu t)$$

Note that this is just equivalent to taking

$$c_1 = c_2 = \frac{1}{2}$$

Now, we can arrive at a second solution in a similar manner. This time let's subtract the two original solutions to arrive at.

$$y_1(t) - y_2(t) = 2i\,\mathbf{e}^{\lambda t}\sin(\mu t)$$

On the surface this doesn't appear to fix the problem as the solution is still complex. However, upon learning that the two constants, c_1 and c_2 can be complex numbers we can arrive at a real solution by dividing this by $2i$. This is equivalent to taking

$$c_1 = \frac{1}{2i} \qquad \text{and} \qquad c_2 = -\frac{1}{2i}$$

Our second solution will then be

$$v(t) = \frac{1}{2i}y_1(t) - \frac{1}{2i}y_2(t) = \mathbf{e}^{\lambda t}\sin(\mu t)$$

We now have two solutions (we'll leave it to you to check that they are in fact solutions) to the differential equation.

$$u(t) = \mathbf{e}^{\lambda t}\cos(\mu t) \qquad \text{and} \qquad v(t) = \mathbf{e}^{\lambda t}\sin(\mu t)$$

It also turns out that these two solutions are "nice enough" to form a general solution.

So, if the roots of the characteristic equation happen to be $r_{1,2} = \lambda \pm \mu\, i$ the general solution to the differential equation is.

$$y(t) = c_1\mathbf{e}^{\lambda t}\cos(\mu t) + c_2\mathbf{e}^{\lambda t}\sin(\mu t)$$

Let's take a look at a couple of examples now.

Example 1

Solve the following IVP.

$$y'' - 4y' + 9y = 0 \qquad y(0) = 0 \qquad y'(0) = -8$$

Solution

The characteristic equation for this differential equation is.

$$r^2 - 4r + 9 = 0$$

The roots of this equation are $r_{1,2} = 2 \pm \sqrt{5}\,i$. The general solution to the differential equation is then.

$$y(t) = c_1 \mathbf{e}^{2t} \cos\left(\sqrt{5}t\right) + c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$

Now, you'll note that we didn't differentiate this right away as we did in the last section. The reason for this is simple. While the differentiation is not terribly difficult, it can get a little messy. So, first looking at the initial conditions we can see from the first one that if we just applied it we would get the following.

$$0 = y(0) = c_1$$

In other words, the first term will drop out in order to meet the first condition. This makes the solution, along with its derivative

$$\begin{aligned} y(t) &= c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right) \\ y'(t) &= 2c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right) + \sqrt{5}c_2 \mathbf{e}^{2t} \cos\left(\sqrt{5}t\right) \end{aligned}$$

A much nicer derivative than if we'd done the original solution. Now, apply the second initial condition to the derivative to get.

$$-8 = y'(0) = \sqrt{5}c_2 \qquad \Rightarrow \qquad c_2 = -\frac{8}{\sqrt{5}}$$

The actual solution is then.

$$y(t) = -\frac{8}{\sqrt{5}} \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$

Example 2

Solve the following IVP.

$$y'' - 8y' + 17y = 0 \qquad y(0) = -4 \qquad y'(0) = -1$$

Solution

The characteristic equation this time is.

$$r^2 - 8r + 17 = 0$$

The roots of this are $r_{1,2} = 4 \pm i$. The general solution as well as its derivative is

$$\begin{aligned} y(t) &= c_1 \mathbf{e}^{4t} \cos(t) + c_2 \mathbf{e}^{4t} \sin(t) \\ y'(t) &= 4c_1 \mathbf{e}^{4t} \cos(t) - c_1 \mathbf{e}^{4t} \sin(t) + 4c_2 \mathbf{e}^{4t} \sin(t) + c_2 \mathbf{e}^{4t} \cos(t) \end{aligned}$$

Notice that this time we will need the derivative from the start as we won't be having one of the terms drop out. Applying the initial conditions gives the following system.

$$\begin{aligned} -4 &= y(0) = c_1 \\ -1 &= y'(0) = 4c_1 + c_2 \end{aligned}$$

Solving this system gives $c_1 = -4$ and $c_2 = 15$. The actual solution to the IVP is then.

$$y(t) = -4\mathbf{e}^{4t} \cos(t) + 15\mathbf{e}^{4t} \sin(t)$$

Example 3

Solve the following IVP.

$$4y'' + 24y' + 37y = 0 \qquad y(\pi) = 1 \qquad y'(\pi) = 0$$

Solution

The characteristic equation this time is.

$$4r^2 + 24r + 37 = 0$$

The roots of this are $r_{1,2} = -3 \pm \frac{1}{2}\,i$. The general solution as well as its derivative is

$$y\left(t\right) = c_1\mathbf{e}^{-3t}\cos\left(\frac{t}{2}\right) + c_2\mathbf{e}^{-3t}\sin\left(\frac{t}{2}\right)$$
$$y'\left(t\right) = -3c_1\mathbf{e}^{-3t}\cos\left(\frac{t}{2}\right) - \frac{c_1}{2}\mathbf{e}^{-3t}\sin\left(\frac{t}{2}\right) - 3c_2\mathbf{e}^{-3t}\sin\left(\frac{t}{2}\right) + \frac{c_2}{2}\mathbf{e}^{-3t}\cos\left(\frac{t}{2}\right)$$

Applying the initial conditions gives the following system.

$$1 = y\left(\pi\right) = c_1\mathbf{e}^{-3\pi}\cos\left(\frac{\pi}{2}\right) + c_2\mathbf{e}^{-3\pi}\sin\left(\frac{\pi}{2}\right) = c_2\mathbf{e}^{-3\pi}$$
$$0 = y'\left(\pi\right) = -\frac{c_1}{2}\mathbf{e}^{-3\pi} - 3c_2\mathbf{e}^{-3\pi}$$

Do not forget to plug the $t = \pi$ into the exponential! This is one of the more common mistakes that students make on these problems. Also, make sure that you evaluate the trig functions as much as possible in these cases. It will only make your life simpler. Solving this system gives

$$c_1 = -6\mathbf{e}^{3\pi} \qquad c_2 = \mathbf{e}^{3\pi}$$

The actual solution to the IVP is then.

$$y\left(t\right) = -6\mathbf{e}^{3\pi}\mathbf{e}^{-3t}\cos\left(\frac{t}{2}\right) + \mathbf{e}^{3\pi}\mathbf{e}^{-3t}\sin\left(\frac{t}{2}\right)$$
$$y\left(t\right) = -6\mathbf{e}^{-3(t-\pi)}\cos\left(\frac{t}{2}\right) + \mathbf{e}^{-3(t-\pi)}\sin\left(\frac{t}{2}\right)$$

Let's do one final example before moving on to the next topic.

Example 4

Solve the following IVP.

$$y'' + 16y = 0 \qquad y\left(\frac{\pi}{2}\right) = -10 \qquad y'\left(\frac{\pi}{2}\right) = 3$$

Solution

The characteristic equation for this differential equation and its roots are.

$$r^2 + 16 = 0 \qquad \Rightarrow \qquad r = \pm 4\,i$$

Be careful with this characteristic polynomial. One of the biggest mistakes students make here is to write it as,

$$r^2 + 16r = 0$$

The problem is that the second term will only have an r if the second term in the differential equation has a y' in it and this one clearly does not. Students however, tend to just start at r^2 and write times down until they run out of terms in the differential equation. That can, and often does mean, they write down the wrong characteristic polynomial so be careful.

Okay, back to the problem.

The general solution to this differential equation and its derivative is.

$$\begin{aligned} y(t) &= c_1 \cos(4t) + c_2 \sin(4t) \\ y'(t) &= -4c_1 \sin(4t) + 4c_2 \cos(4t) \end{aligned}$$

Plugging in the initial conditions gives the following system.

$$\begin{aligned} -10 = y\left(\frac{\pi}{2}\right) &= c_1 \qquad c_1 = -10 \\ 3 = y'\left(\frac{\pi}{2}\right) &= 4c_2 \qquad c_2 = \frac{3}{4} \end{aligned}$$

So, the constants drop right out with this system and the actual solution is.

$$y(t) = -10\cos(4t) + \frac{3}{4}\sin(4t)$$

3.4 Repeated Roots

In this section we will be looking at the last case for the constant coefficient, linear, homogeneous second order differential equations. In this case we want solutions to

$$ay'' + by' + cy = 0$$

where solutions to the characteristic equation

$$ar^2 + br + c = 0$$

are double roots $r_1 = r_2 = r$.

This leads to a problem however. Recall that the solutions are

$$y_1(t) = \mathbf{e}^{r_1 t} = \mathbf{e}^{r t} \qquad y_2(t) = \mathbf{e}^{r_2 t} = \mathbf{e}^{r t}$$

These are the same solution and will NOT be "nice enough" to form a general solution. We do promise that we'll define "nice enough" eventually! So, we can use the first solution, but we're going to need a second solution.

Before finding this second solution let's take a little side trip. The reason for the side trip will be clear eventually. From the quadratic formula we know that the roots to the characteristic equation are,

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, since we have double roots we must have

$$b^2 - 4ac = 0$$

This is the only way that we can get double roots and in this case the roots will be

$$r_{1,2} = \frac{-b}{2a}$$

So, the one solution that we've got is

$$y_1(t) = \mathbf{e}^{-\frac{b\,t}{2a}}$$

To find a second solution we will use the fact that a constant times a solution to a linear homogeneous differential equation is also a solution. If this is true then *maybe* we'll get lucky and the following will also be a solution

$$y_2(t) = v(t)\, y_1(t) = v(t)\, \mathbf{e}^{-\frac{b\,t}{2a}} \tag{3.8}$$

with a proper choice of $v(t)$. To determine if this in fact can be done, let's plug this back into the differential equation and see what we get. We'll first need a couple of derivatives.

$$\begin{aligned} y'_2(t) &= v'\,\mathbf{e}^{-\frac{b\,t}{2a}} - \frac{b}{2a}v\,\mathbf{e}^{-\frac{b\,t}{2a}} \\ y''_2(t) &= v''\,\mathbf{e}^{-\frac{b\,t}{2a}} - \frac{b}{2a}v'\,\mathbf{e}^{-\frac{b\,t}{2a}} - \frac{b}{2a}v'\,\mathbf{e}^{-\frac{b\,t}{2a}} + \frac{b^2}{4a^2}v\,\mathbf{e}^{-\frac{b\,t}{2a}} \\ &= v''\,\mathbf{e}^{-\frac{b\,t}{2a}} - \frac{b}{a}v'\,\mathbf{e}^{-\frac{b\,t}{2a}} + \frac{b^2}{4a^2}v\,\mathbf{e}^{-\frac{b\,t}{2a}} \end{aligned}$$

We dropped the (t) part on the v to simplify things a little for the writing out of the derivatives. Now, plug these into the differential equation.

$$a\left(v''\,\mathbf{e}^{-\frac{b\,t}{2a}} - \frac{b}{a}v'\,\mathbf{e}^{-\frac{b\,t}{2a}} + \frac{b^2}{4a^2}v\,\mathbf{e}^{-\frac{b\,t}{2a}}\right) + b\left(v'\,\mathbf{e}^{-\frac{b\,t}{2a}} - \frac{b}{2a}v\,\mathbf{e}^{-\frac{b\,t}{2a}}\right) + c\left(v\,\mathbf{e}^{-\frac{b\,t}{2a}}\right) = 0$$

We can factor an exponential out of all the terms so let's do that. We'll also collect all the coefficients of v and its derivatives.

$$\begin{aligned} \mathbf{e}^{-\frac{b\,t}{2a}}\left(av'' + (-b+b)\,v' + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v\right) &= 0 \\ \mathbf{e}^{-\frac{b\,t}{2a}}\left(av'' + \left(-\frac{b^2}{4a} + c\right)v\right) &= 0 \\ \mathbf{e}^{-\frac{b\,t}{2a}}\left(av'' - \frac{1}{4a}\left(b^2 - 4ac\right)v\right) &= 0 \end{aligned}$$

Now, because we are working with a double root we know that that the second term will be zero. Also exponentials are never zero. Therefore, Equation 3.8 will be a solution to the differential equation provided $v(t)$ is a function that satisfies the following differential equation.

$$av'' = 0 \qquad \text{OR} \qquad v'' = 0$$

We can drop the a because we know that it can't be zero. If it were we wouldn't have a second order differential equation! So, we can now determine the most general possible form that is allowable for $v(t)$.

$$v' = \int v''\,dt = c \qquad v(t) = \int v'\,dt = ct + k$$

This is actually more complicated than we need and in fact we can drop both of the constants from this. To see why this is let's go ahead and use this to get the second solution. The two solutions are then

$$y_1(t) = \mathbf{e}^{-\frac{b\,t}{2a}} \qquad y_2(t) = (ct+k)\,\mathbf{e}^{-\frac{b\,t}{2a}}$$

Eventually you will be able to show that these two solutions are "nice enough" to form a general solution. The general solution would then be the following.

$$\begin{aligned} y(t) &= c_1\mathbf{e}^{-\frac{b\,t}{2a}} + c_2(ct+k)\,\mathbf{e}^{-\frac{b\,t}{2a}} \\ &= c_1\mathbf{e}^{-\frac{b\,t}{2a}} + (c_2ct + c_2k)\,\mathbf{e}^{-\frac{b\,t}{2a}} \\ &= (c_1 + c_2k)\,\mathbf{e}^{-\frac{b\,t}{2a}} + c_2c\,t\,\mathbf{e}^{-\frac{b\,t}{2a}} \end{aligned}$$

Notice that we rearranged things a little. Now, c, k, c_1, and c_2 are all unknown constants so any combination of them will also be unknown constants. In particular, $c_1 + c_2 k$ and $c_2 c$ are unknown constants so we'll just rewrite them as follows.

$$y(t) = c_1 \mathbf{e}^{-\frac{bt}{2a}} + c_2 t \, \mathbf{e}^{-\frac{bt}{2a}}$$

So, if we go back to the most general form for $v(t)$ we can take $c = 1$ and $k = 0$ and we will arrive at the same general solution.

Let's recap. If the roots of the characteristic equation are $r_1 = r_2 = r$, then the general solution is then

$$y(t) = c_1 \mathbf{e}^{rt} + c_2 t \mathbf{e}^{rt}$$

Now, let's work a couple of examples.

Example 1

Solve the following IVP.

$$y'' - 4y' + 4y = 0 \qquad y(0) = 12 \qquad y'(0) = -3$$

Solution

The characteristic equation and its roots are.

$$r^2 - 4r + 4 = (r-2)^2 = 0 \qquad r_{1,2} = 2$$

The general solution and its derivative are

$$y(t) = c_1 \mathbf{e}^{2t} + c_2 t \mathbf{e}^{2t}$$
$$y'(t) = 2c_1 \mathbf{e}^{2t} + c_2 \mathbf{e}^{2t} + 2c_2 t \mathbf{e}^{2t}$$

Don't forget to product rule the second term! Plugging in the initial conditions gives the following system.

$$12 = y(0) = c_1$$
$$-3 = y'(0) = 2c_1 + c_2$$

This system is easily solved to get $c_1 = 12$ and $c_2 = -27$. The actual solution to the IVP is then.

$$y(t) = 12\mathbf{e}^{2t} - 27t\mathbf{e}^{2t}$$

Example 2

Solve the following IVP.

$$16y'' - 40y' + 25y = 0 \qquad y(0) = 3 \qquad y'(0) = -\frac{9}{4}$$

Solution

The characteristic equation and its roots are.

$$16r^2 - 40r + 25 = (4r-5)^2 = 0 \qquad r_{1,2} = \frac{5}{4}$$

The general solution and its derivative are

$$\begin{aligned} y(t) &= c_1 \mathbf{e}^{\frac{5t}{4}} + c_2 t \mathbf{e}^{\frac{5t}{4}} \\ y'(t) &= \frac{5}{4} c_1 \mathbf{e}^{\frac{5t}{4}} + c_2 \mathbf{e}^{\frac{5t}{4}} + \frac{5}{4} c_2 t \mathbf{e}^{\frac{5t}{4}} \end{aligned}$$

Don't forget to product rule the second term! Plugging in the initial conditions gives the following system.

$$\begin{aligned} 3 &= y(0) = c_1 \\ -\frac{9}{4} &= y'(0) = \frac{5}{4} c_1 + c_2 \end{aligned}$$

This system is easily solve to get $c_1 = 3$ and $c_2 = -6$. The actual solution to the IVP is then.

$$y(t) = 3\mathbf{e}^{\frac{5t}{4}} - 6t\mathbf{e}^{\frac{5t}{4}}$$

Example 3

Solve the following IVP

$$y'' + 14y' + 49y = 0 \qquad y(-4) = -1 \qquad y'(-4) = 5$$

Solution

The characteristic equation and its roots are.

$$r^2 + 14r + 49 = (r+7)^2 = 0 \qquad r_{1,2} = -7$$

The general solution and its derivative are

$$\begin{aligned} y(t) &= c_1\mathbf{e}^{-7t} + c_2 t\mathbf{e}^{-7t} \\ y'(t) &= -7c_1\mathbf{e}^{-7t} + c_2\mathbf{e}^{-7t} - 7c_2 t\mathbf{e}^{-7t} \end{aligned}$$

Plugging in the initial conditions gives the following system of equations.

$$\begin{aligned} -1 &= y(-4) = c_1\mathbf{e}^{28} - 4c_2\mathbf{e}^{28} \\ 5 &= y'(-4) = -7c_1\mathbf{e}^{28} + c_2\mathbf{e}^{28} + 28c_2\mathbf{e}^{28} = -7c_1\mathbf{e}^{28} + 29c_2\mathbf{e}^{28} \end{aligned}$$

Solving this system gives the following constants.

$$c_1 = -9\mathbf{e}^{-28} \qquad c_2 = -2\mathbf{e}^{-28}$$

The actual solution to the IVP is then.

$$\begin{aligned} y(t) &= -9\mathbf{e}^{-28}\mathbf{e}^{-7t} - 2t\mathbf{e}^{-28}\mathbf{e}^{-7t} \\ y(t) &= -9\mathbf{e}^{-7(t+4)} - 2t\mathbf{e}^{-7(t+4)} \end{aligned}$$

3.5 Reduction of Order

We're now going to take a brief detour and look at solutions to non-constant coefficient, second order differential equations of the form.

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = 0$$

In general, finding solutions to these kinds of differential equations can be much more difficult than finding solutions to constant coefficient differential equations. However, if we already know one solution to the differential equation we can use the method that we used in the last section to find a second solution. This method is called **reduction of order**.

Let's take a quick look at an example to see how this is done.

Example 1

Find the general solution to

$$2t^2y'' + ty' - 3y = 0, \quad t > 0$$

given that $y_1(t) = t^{-1}$ is a solution.

Solution

Reduction of order requires that a solution already be known. Without this known solution we won't be able to do reduction of order.

Once we have this first solution we will then assume that a second solution will have the form

$$y_2(t) = v(t)\,y_1(t) \tag{3.9}$$

for a proper choice of $v(t)$. To determine the proper choice, we plug the guess into the differential equation and get a new differential equation that can be solved for $v(t)$.

So, let's do that for this problem. Here is the form of the second solution as well as the derivatives that we'll need.

$$y_2(t) = t^{-1}v \qquad y_2'(t) = -t^{-2}v + t^{-1}v' \qquad y_2''(t) = 2t^{-3}v - 2t^{-2}v' + t^{-1}v''$$

Plugging these into the differential equation gives

$$2t^2\left(2t^{-3}v - 2t^{-2}v' + t^{-1}v''\right) + t\left(-t^{-2}v + t^{-1}v'\right) - 3\left(t^{-1}v\right) = 0$$

Rearranging and simplifying gives

$$\begin{aligned} 2tv'' + (-4+1)\,v' + \left(4t^{-1} - t^{-1} - 3t^{-1}\right)v &= 0 \\ 2tv'' - 3v' &= 0 \end{aligned}$$

Note that upon simplifying the only terms remaining are those involving the derivatives of v. The term involving v drops out. If you've done all of your work correctly this should always happen. Sometimes, as in the repeated roots case, the first derivative term will also drop out.

So, in order for Equation 3.9 to be a solution then v must satisfy

$$2tv'' - 3v' = 0 \tag{3.10}$$

This appears to be a problem. In order to find a solution to a second order non-constant coefficient differential equation we need to solve a different second order non-constant coefficient differential equation.

However, this isn't the problem that it appears to be. Because the term involving the v drops out we can actually solve Equation 3.10 and we can do it with the knowledge that we already have at this point. We will solve this by making the following **change of variable**.

$$w = v' \qquad \Rightarrow \qquad w' = v''$$

With this change of variable Equation 3.10 becomes

$$2tw' - 3w = 0$$

and this is a linear, first order differential equation that we can solve. This also explains the name of this method. We've managed to reduce a second order differential equation down to a first order differential equation.

This is a fairly simple first order differential equation so I'll leave the details of the solving to you. If you need a refresher on solving linear, first order differential equations go back to the second chapter and check out that section. The solution to this differential equation is

$$w(t) = c\,t^{\frac{3}{2}}$$

Now, this is not quite what we were after. We are after a solution to Equation 3.10. However, we can now find this. Recall our change of variable.

$$v' = w$$

With this we can easily solve for $v(t)$.

$$v(t) = \int w\,dt = \int ct^{\frac{3}{2}}\,dt = \frac{2}{5}ct^{\frac{5}{2}} + k$$

This is the most general possible $v(t)$ that we can use to get a second solution. So, just as we did in the repeated roots section, we can choose the constants to be anything we want so choose them to clear out all the extraneous constants. In this case we can use

$$c = \frac{5}{2} \qquad k = 0$$

Using these gives the following for $v(t)$ and for the second solution.

$$v\left(t\right) = t^{\frac{5}{2}} \qquad \Rightarrow \qquad y_2\left(t\right) = t^{-1}\left(t^{\frac{5}{2}}\right) = t^{\frac{3}{2}}$$

Then general solution will then be,

$$y\left(t\right) = c_1 t^{-1} + c_2 t^{\frac{3}{2}}$$

If we had been given initial conditions we could then differentiate, apply the initial conditions and solve for the constants.

Reduction of order, the method used in the previous example can be used to find second solutions to differential equations. However, this does require that we already have a solution and often finding that first solution is a very difficult task and often in the process of finding the first solution you will also get the second solution without needing to resort to reduction of order. So, for those cases when we do have a first solution this is a nice method for getting a second solution.

Let's do one more example.

Example 2

Find the general solution to

$$t^2 y'' + 2ty' - 2y = 0$$

given that $y_1\left(t\right) = t$ is a solution.

Solution

The form for the second solution as well as its derivatives are,

$$y_2\left(t\right) = tv \qquad y_2'\left(t\right) = v + tv' \qquad y_2''\left(t\right) = 2v' + tv''$$

Plugging these into the differential equation gives,

$$t^2\left(2v' + tv''\right) + 2t\left(v + tv'\right) - 2\left(tv\right) = 0 = 0$$

Rearranging and simplifying gives the differential equation that we'll need to solve in order to determine the correct v that we'll need for the second solution.

$$t^3 v'' + 4t^2 v' = 0$$

Next use the variable transformation as we did in the previous example.

$$w = v' \qquad \Rightarrow \qquad w' = v''$$

With this change of variable the differential equation becomes

$$t^3 w' + 4t^2 w = 0$$

and this is a linear, first order differential equation that we can solve. We'll leave the details of the solution process to you.

$$w(t) = c\,t^{-4}$$

Now solve for $v(t)$.

$$v(t) = \int w\,dt = \int ct^{-4}\,dt = -\frac{1}{3}ct^{-3} + k$$

As with the first example we'll drop the constants and use the following $v(t)$

$$v(t) = t^{-3} \qquad \Rightarrow \qquad y_2(t) = t\left(t^{-3}\right) = t^{-2}$$

Then general solution will then be,

$$y(t) = c_1 t + \frac{c_2}{t^2}$$

On a side note, both of the differential equations in this section were of the form,

$$t^2 y'' + \alpha\, t\, y' + \beta y = 0$$

These are called Euler differential equations and are fairly simple to solve directly for both solutions. To see how to solve these directly take a look at the Euler Differential Equation section.

3.6 Fundamental Sets of Solutions

The time has finally come to define "nice enough". We've been using this term throughout the last few sections to describe those solutions that could be used to form a general solution and it is now time to officially define it.

First, because everything that we're going to be doing here only requires linear and homogeneous we won't require constant coefficients in our differential equation. So, let's start with the following IVP.

$$\begin{aligned} p(t)\,y'' + q(t)\,y' + r(t)\,y = 0 \\ y(t_0) = y_0 \qquad y'(t_0) = {y'}_0 \end{aligned} \tag{3.11}$$

Let's also suppose that we have already found two solutions to this differential equation, $y_1(t)$ and $y_2(t)$. We know from the Principle of Superposition that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3.12}$$

will also be a solution to the differential equation. What we want to know is whether or not it will be a general solution. In order for Equation 3.12 to be considered a general solution it must satisfy the general initial conditions in Equation 3.11.

$$y(t_0) = y_0 \qquad y'(t_0) = y_0'$$

This will also imply that any solution to the differential equation can be written in this form.

So, let's see if we can find constants that will satisfy these conditions. First differentiate Equation 3.12 and plug in the initial conditions.

$$\begin{aligned} y_0 = y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) \\ {y'}_0 = y'(t_0) = c_1 {y'}_1(t_0) + c_2 {y'}_2(t_0) \end{aligned} \tag{3.13}$$

Since we are assuming that we've already got the two solutions everything in this system is technically known and so this is a system that can be solved for c_1 and c_2. This can be done in general using Cramer's Rule. Using Cramer's Rule gives the following solution.

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ {y'}_0 & {y'}_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ {y'}_1(t_0) & {y'}_2(t_0) \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ {y'}_1(t_0) & {y'}_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ {y'}_1(t_0) & {y'}_2(t_0) \end{vmatrix}} \tag{3.14}$$

where,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

is the determinant of a 2x2 matrix. If you don't know about determinants that is okay, just use the formula that we've provided above.

Now, Equation 3.14 will give the solution to the system Equation 3.13. Note that in practice we generally don't use Cramer's Rule to solve systems, we just proceed in a straightforward manner and solve the system using basic algebra techniques. So, why did we use Cramer's Rule here then?

We used Cramer's Rule because we can use Equation 3.14 to develop a condition that will allow us to determine when we can solve for the constants. All three (yes three, the denominators are the same!) of the quantities in Equation 3.14 are just numbers and the only thing that will prevent us from actually getting a solution will be when the denominator is zero.

The quantity in the denominator is called the **Wronskian** and is denoted as

$$W(f,g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)\,g'(t) - g(t)\,f'(t)$$

When it is clear what the functions and/or t are we often just denote the Wronskian by W.

Let's recall what we were after here. We wanted to determine when two solutions to Equation 3.11 would be nice enough to form a general solution. The two solutions will form a general solution to Equation 3.11 if they satisfy the general initial conditions given in Equation 3.11 and we can see from Cramer's Rule that they will satisfy the initial conditions provided the Wronskian isn't zero. Or,

$$W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix} = y_1(t_0)\,y'_2(t_0) - y_2(t_0)\,y'_1(t_0) \neq 0$$

So, suppose that $y_1(t)$ and $y_2(t)$ are two solutions to Equation 3.11 and that $W(y_1,y_2)(t) \neq 0$. Then the two solutions are called a **fundamental set of solutions** and the general solution to Equation 3.11 is

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

We know now what "nice enough" means. Two solutions are "nice enough" if they are a fundamental set of solutions.

So, let's check one of the claims that we made in a previous section. We'll leave the other two to you to check if you'd like to.

Example 1

Back in the complex root section we made the claim that

$$y_1(t) = \mathbf{e}^{\lambda t}\cos(\mu\,t) \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{\lambda t}\sin(\mu\,t)$$

were a fundamental set of solutions. Prove that they in fact are.

Solution

So, to prove this we will need to take the Wronskian for these two solutions and show that it isn't zero.

$$\begin{aligned} W &= \begin{vmatrix} \mathbf{e}^{\lambda t}\cos(\mu t) & \mathbf{e}^{\lambda t}\sin(\mu t) \\ \lambda\mathbf{e}^{\lambda t}\cos(\mu t) - \mu\mathbf{e}^{\lambda t}\sin(\mu t) & \lambda\mathbf{e}^{\lambda t}\sin(\mu t) + \mu\mathbf{e}^{\lambda t}\cos(\mu t) \end{vmatrix} \\ &= \mathbf{e}^{\lambda t}\cos(\mu t)\left(\lambda\mathbf{e}^{\lambda t}\sin(\mu t) + \mu\mathbf{e}^{\lambda t}\cos(\mu t)\right) - \\ &\qquad \mathbf{e}^{\lambda t}\sin(\mu t)\left(\lambda\mathbf{e}^{\lambda t}\cos(\mu t) - \mu\mathbf{e}^{\lambda t}\sin(\mu t)\right) \\ &= \mu\mathbf{e}^{2\lambda t}\cos^2(\mu t) + \mu\mathbf{e}^{2\lambda t}\sin^2(\mu t) \\ &= \mu\mathbf{e}^{2\lambda t}\left(\cos^2(\mu t) + \sin^2(\mu t)\right) \\ &= \mu\mathbf{e}^{2\lambda t} \end{aligned}$$

Now, the exponential will never be zero and $\mu \neq 0$ (if it were we wouldn't have complex roots!) and so $W \neq 0$. Therefore, these two solutions are in fact a fundamental set of solutions and so the general solution in this case is.

$$y(t) = c_1\mathbf{e}^{\lambda t}\cos(\mu t) + c_2\mathbf{e}^{\lambda t}\sin(\mu t)$$

Example 2

In the first example that we worked in the Reduction of Order section we found a second solution to

$$2t^2y'' + ty' - 3y = 0$$

Show that this second solution, along with the given solution, form a fundamental set of solutions for the differential equation.

Solution

The two solutions from that example are

$$y_1(t) = t^{-1} \qquad\qquad y_2(t) = t^{\frac{3}{2}}$$

Let's compute the Wronskian of these two solutions.

$$W = \begin{vmatrix} t^{-1} & t^{\frac{3}{2}} \\ -t^{-2} & \frac{3}{2}t^{\frac{1}{2}} \end{vmatrix} = \frac{3}{2}t^{-\frac{1}{2}} - \left(-t^{-\frac{1}{2}}\right) = \frac{5}{2}t^{-\frac{1}{2}} = \frac{5}{2\sqrt{t}}$$

So, the Wronskian will never be zero. Note that we can't plug $t = 0$ into the Wronskian. This would be a problem in finding the constants in the general solution, except that we also can't plug $t = 0$ into the solution either and so this isn't the problem that it might appear to be.

So, since the Wronskian isn't zero for any t the two solutions form a fundamental set of solutions and the general solution is

$$y(t) = c_1 t^{-1} + c_2 t^{\frac{3}{2}}$$

as we claimed in that example.

To this point we've found a set of solutions then we've claimed that they are in fact a fundamental set of solutions. Of course, you can now verify all those claims that we've made, however this does bring up a question. How do we know that for a given differential equation a set of fundamental solutions will exist? The following theorem answers this question.

Theorem

Consider the differential equation

$$y'' + p(t)\,y' + q(t)\,y = 0$$

where $p(t)$ and $q(t)$ are continuous functions on some interval I. Choose t_0 to be any point in the interval I. Let $y_1(t)$ be a solution to the differential equation that satisfies the initial conditions.

$$y(t_0) = 1 \qquad y'(t_0) = 0$$

Let $y_2(t)$ be a solution to the differential equation that satisfies the initial conditions.

$$y(t_0) = 0 \qquad y'(t_0) = 1$$

Then $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions for the differential equation.

It is easy enough to show that these two solutions form a fundamental set of solutions. Just compute the Wronskian.

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ {y'}_1(t_0) & {y'}_2(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$$

So, fundamental sets of solutions will exist provided we can solve the two IVP's given in the theorem.

Example 3

Use the theorem to find a fundamental set of solutions for

$$y'' + 4y' + 3y = 0$$

using $t_0 = 0$.

Solution

Using the techniques from the first part of this chapter we can find the two solutions that we've been using to this point.

$$y(t) = \mathbf{e}^{-3t} \qquad\qquad y(t) = \mathbf{e}^{-t}$$

These do form a fundamental set of solutions as we can easily verify. However, they are NOT the set that will be given by the theorem. Neither of these solutions will satisfy either of the two sets of initial conditions given in the theorem. We will have to use these to find the fundamental set of solutions that is given by the theorem.

We know that the following is also a solution to the differential equation.

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t}$$

So, let's apply the first set of initial conditions and see if we can find constants that will work.

$$y(0) = 1 \qquad y'(0) = 0$$

We'll leave it to you to verify that we get the following solution upon doing this.

$$y_1(t) = -\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t}$$

Likewise, if we apply the second set of initial conditions,

$$y(0) = 0 \qquad y'(0) = 1$$

we will get

$$y_2(t) = -\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t}$$

According to the theorem these should form a fundament set of solutions. This is easy enough to check.

$$\begin{aligned} W &= \begin{vmatrix} -\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t} & -\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t} \\ \frac{3}{2}\mathbf{e}^{-3t} - \frac{3}{2}\mathbf{e}^{-t} & \frac{3}{2}\mathbf{e}^{-3t} - \frac{1}{2}\mathbf{e}^{-t} \end{vmatrix} \\ &= \left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t}\right)\left(\frac{3}{2}\mathbf{e}^{-3t} - \frac{1}{2}\mathbf{e}^{-t}\right) - \left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t}\right)\left(\frac{3}{2}\mathbf{e}^{-3t} - \frac{3}{2}\mathbf{e}^{-t}\right) \\ &= \mathbf{e}^{-4t} \neq 0 \end{aligned}$$

So, we got a completely different set of fundamental solutions from the theorem than what we've been using up to this point. This is not a problem. There are an infinite number of pairs of functions that we could use as a fundamental set of solutions for this problem.

So, which set of fundamental solutions should we use? Well, if we use the ones that we originally found, the general solution would be,

$$y(t) = c_1 \mathbf{e}^{-3t} + c_2 \mathbf{e}^{-t}$$

Whereas, if we used the set from the theorem the general solution would be,

$$y(t) = c_1 \left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t} \right) + c_2 \left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t} \right)$$

This would not be very fun to work with when it came to determining the coefficients to satisfy a general set of initial conditions.

So, which set of fundamental solutions should we use? We should always try to use the set that is the most convenient to use for a given problem.