Chapter Five

Linear oscillations

and normal modes

KEY FEATURES

The key features of this chapter are the properties of **free undamped** oscillations, **free damped** oscillations, **driven** oscillations, and **coupled** oscillations.

Oscillations are a particularly important part of mechanics and indeed of physics as a whole. This is because of their widespread occurrence and the practical importance of oscillation problems. In this chapter we study the classical **linear theory** of oscillations, which is important for two reasons: (i) the linear theory usually gives a good approximation to the motion when the amplitude of the oscillations is small, and (ii) in the linear theory, most problems can be solved explicitly in closed form. The importance of this last fact should not be underestimated! We develop the theory in the context of the oscillations of a body attached to a spring, but the same equations apply to many different problems in mechanics and throughout physics.

In the course of this chapter we will need to solve linear second order ODEs with constant coefficients. For a description of the standard method of solution see Boyce & DiPrima [8].

5.1 BODY ON A SPRING

Suppose a body of mass m is attached to one end of a light spring. The other end of the spring is attached to a fixed point A on a smooth horizontal table, and the body slides on

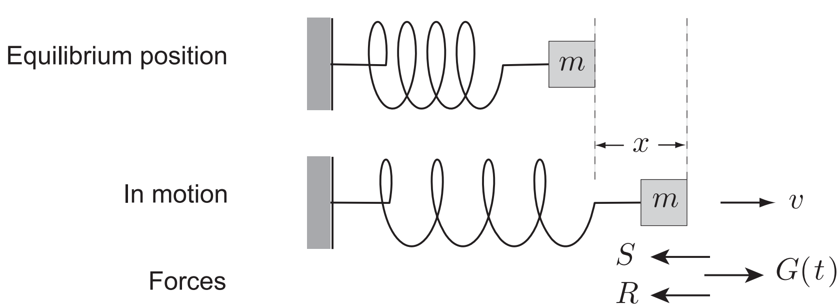

FIGURE 5.1 The body m is attached to one end of a light spring and moves in a straight line.

the table in a straight line through A. Let x be the displacement and v the velocity of the body at time t, as shown in Figure 5.1; note that x is measured from the *equilibrium position* of the body.

Consider now the forces acting on the body. When the spring is extended, it exerts a **restoring force** S in the opposite direction to the extension. Also, the body may encounter a **resistance force** R acting in the opposite direction to its velocity. Finally, there may be an external **driving force** $G(t)$ that is a specified function of the time. The equation of motion for the body is then

$$m\frac{dv}{dt} = -S - R + G(t). \tag{5.1}$$

The **restoring force** S is determined by the design of the spring and the extension x. For sufficiently *small strains*,* the relationship between S and x is approximately *linear*, that is,

$$S = \alpha\, x, \tag{5.2}$$

where α is a positive constant called the **spring constant** (or **strength**) of the spring. A powerful spring, such as those used in automobile suspensions, has a large value of α; the spring behind a doorbell has a small value of α. The formula (5.2) is called **Hooke's law**† and a spring that obeys Hooke's law exactly is called a **linear** spring.

The **resistance force** R depends on the physical process that is causing the resistance. For fluid resistance, the linear or quadratic resistance laws considered in Chapter 4 may be appropriate. However, neither of these laws represents the frictional force exerted by a rough table. In this chapter we assume the law of **linear resistance**

$$R = \beta\, v, \tag{5.3}$$

where β is a positive constant called the **resistance constant**; it is a measure of the strength of the resistance. There is no point in disguising the fact that our major reason for assuming linear resistance is that (together with Hooke's law) it leads to a linear equation of motion that can be solved explicitly. However, it does give insight into the general effect of all resistances, and actually is appropriate when the resistance arises from slow viscous flow (automobile shock absorbers, for instance); it is also appropriate in the electric circuit analogue, where it is equivalent to Ohm's law.

With **Hooke's law** and **linear resistance**, the equation of motion (5.1) for the body becomes

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \alpha\, x = G(t),$$

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* The strain is the extension of the spring divided by its natural length. If the strain is large, then the linear approximation will break down and a non-linear approximation, such as $S = a\,x + b\,x^3$ must be used instead.

† After Robert Hooke (1635–1703). Hooke was an excellent scientist, full of ideas and a first class experimenter, but he lacked the mathematical skills to develop his ideas. When other scientists (Newton in particular) did so, he accused them of stealing his work and this led to a succession of bitter disputes. So that his rivals could not immediately make use of his discovery, Hooke first published the law that bears his name as an anagram on the Latin phrase '*ut tensio, sic vis*' (as the extension, so the force).

where α is the spring constant, β is the resistance constant and $G(t)$ is the prescribed driving force. *This is a second order, linear ODE with constant coefficients for the unknown displacement $x(t)$*. We could go ahead with the solution of this equation as it stands, but the algebra is made much easier by introducing two new constants Ω and K (instead of α and β) defined by the relations

$$\alpha = m\Omega \qquad\qquad \beta = 2mK.$$

The equation of motion for the body then becomes

$$\boxed{\frac{d^2x}{dt^2} + 2K\,\frac{dx}{dt} + \Omega^2\,x = F(t)} \tag{5.4}$$

where $F(t) = G(t)/m$, the *driving force per unit mass*. This is the standard form of the **equation of motion** for the body. Any system that leads to an equation of this form is called a **damped*** **linear oscillator**. When the force $F(t)$ is absent, the oscillations are said to be **free**; when it is present, the oscillations are said to be **driven**.

5.2 CLASSICAL SIMPLE HARMONIC MOTION

A linear oscillator that is both **undamped** and **undriven** is called a **classical linear oscillator**. This is the simplest case, but arguably the most important system in physics! The equation (5.4) reduces to

$$\frac{d^2x}{dt^2} + \Omega^2 x = 0, \tag{5.5}$$

which, because of the solutions we are about to obtain, is called the **SHM equation**.

Solution procedure

Seek solutions of the form $x = e^{\lambda t}$. Then λ must satisfy the equation

$$\lambda^2 + \Omega^2 = 0,$$

which gives $\lambda = \pm i\Omega$. We have thus found the pair of complex solutions

$$x = e^{\pm i\Omega t},$$

which form a basis for the space of complex solutions. The real and imaginary parts of the first complex solution are

$$x = \begin{cases} \cos\Omega t \\ \sin\Omega t \end{cases}$$

* Damping is another term for resistance. Indeed, automobile shock absorbers are sometimes called dampers.

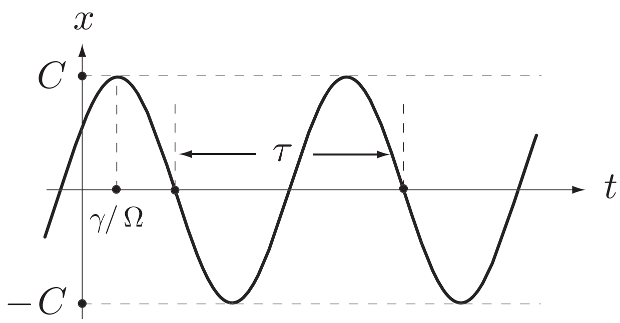

FIGURE 5.2 Classical simple harmonic motion $x = C\cos(\Omega t - \gamma)$.

and these functions form a basis for the space of real solutions. The **general real solution** of the SHM equation is therefore

$$x = A\cos\Omega t + B\sin\Omega t, \tag{5.6}$$

where A and B are real arbitrary constants. This general solution can be written in the alternative form*

$$x = C\cos(\Omega t - \gamma), \tag{5.7}$$

where C and γ are real arbitrary constants with $C > 0$.

General form of the motion

The general form of the motion is most easily deduced from the form (5.7) and is shown in Figure 5.2. This is called **simple harmonic motion (SHM)**. The body makes infinitely many oscillations of constant **amplitude** C; the constant γ is simply a 'phase factor' which shifts the whole graph by γ/Ω in the t-direction. Since the cosine function repeats itself when the argument Ωt increases by 2π, it follows that the **period** of the oscillations is given by

$$\tau = \frac{2\pi}{\Omega}. \tag{5.8}$$

The quantity Ω, which is related to the frequency ν by $\Omega = 2\pi\nu$, is called the **angular frequency** of the oscillations.

Example 5.1 *An initial value problem for classical SHM*

A body of mass m is suspended from a fixed point by a light spring and can move under uniform gravity. In equilibrium, the spring is found to be extended by a distance b. Find the period of vertical oscillations of the body about this equilibrium position. [Assume small strains.]

* This transformation is based on the result from trigonometry that $a\cos\theta + b\sin\theta$ can always be written in the form $c\cos(\theta - \gamma)$, where $c = (a^2 + b^2)^{1/2}$ and $\tan\gamma = b/a$.

The body is hanging in its equilibrium position when it receives a sudden blow which projects it upwards with speed u. Find the subsequent motion.

Solution

When the spring is subjected to a constant force of magnitude mg, the extension is b. Hence α, the strength of the spring, is given by $\alpha = mg/b$.

Let z be the *downwards* displacement of the body from its equilibrium position. Then the extension of the spring is $b + z$ and the restoring force is $\alpha(b + z) = g(b + z)/b$. The equation of motion for the body is therefore

$$m\frac{d^2z}{dt^2} = mg - \frac{mg(b+z)}{b}$$

that is

$$\frac{d^2z}{dt^2} + \left(\frac{g}{b}\right)z = 0.$$

This is the **SHM equation** with $\Omega^2 = g/b$. It follows that the **period** τ of vertical oscillations about the equilibrium position is given by

$$\tau = \frac{2\pi}{\Omega} = 2\pi\left(\frac{b}{g}\right)^{1/2}.$$

In the **initial value problem**, the subsequent motion must have the form

$$x = A\cos\Omega t + B\sin\Omega t,$$

where $\Omega = (g/b)^{1/2}$. The initial condition $x = 0$ when $t = 0$ shows that $A = 0$ and the initial condition $\dot{x} = -u$ when $t = 0$ then gives $\Omega B = -u$, that is, $B = -u/\Omega$. The subsequent motion is therefore

$$x = -\frac{u}{\Omega}\sin\Omega t,$$

where $\Omega = (g/b)^{1/2}$. ■

5.3 DAMPED SIMPLE HARMONIC MOTION

When **damping** is present but there is no external force, the general equation (5.4) reduces to

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = 0, \tag{5.9}$$

the **damped SHM equation**.

The solution procedure is the same as in the last section. Seek solutions of the form $x = e^{\lambda t}$. Then λ must satisfy the equation

$$\lambda^2 + 2K\lambda + \Omega^2 = 0,$$

that is

$$(\lambda + K)^2 = K^2 - \Omega^2.$$

We see that *different cases arise depending on whether* $K < \Omega$, $K = \Omega$ *or* $K > \Omega$. These cases give rise to different kinds of solution and must be treated separately.

Under-damping (sub-critical damping): $K < \Omega$

In this case, we write the equation for λ in the form

$$(\lambda + K)^2 = -\Omega_D^2,$$

where $\Omega_D = (\Omega^2 - K^2)^{1/2}$, a *positive real number*. The λ values are then $\lambda = -K \pm i\Omega_D$. We have thus found the pair of complex solutions

$$x = e^{-Kt} e^{\pm i\Omega_D t},$$

which form a basis for the space of complex solutions. The real and imaginary parts of the first complex solution are

$$x = \begin{cases} e^{-Kt} \cos \Omega_D t \\ e^{-Kt} \sin \Omega_D t \end{cases}$$

and these functions form a basis for the space of real solutions. The **general real solution** of the damped SHM equation in this case is therefore

$$x = e^{-Kt} \left(A \cos \Omega_D t + B \sin \Omega_D t \right), \tag{5.10}$$

where A and B are real arbitrary constants. This general solution can be written in the alternative form

$$x = C e^{-Kt} \cos(\Omega_D t - \gamma), \tag{5.11}$$

where C and γ are real arbitrary constants with $C > 0$.

General form of the motion

The general form of the motion is most easily deduced from the form (5.11) and is shown in Figure 5.3. This is called **under-damped SHM**. The body still executes infinitely many oscillations, but now they have *exponentially decaying amplitude* Ce^{-Kt}. Suppose the **period** τ of the oscillations is defined as shown in Figure 5.3.* The introduction of damping decreases the angular frequency of the oscillations from Ω to Ω_D, which *increases* the period of the oscillations from $2\pi/\Omega$ to

$$\tau = \frac{2\pi}{\Omega_D} = \frac{2\pi}{(\Omega^2 - K^2)^{1/2}}. \tag{5.12}$$

* The period might also be defined as the time interval between successive maxima of the function $x(t)$. Since these maxima do *not* occur at the points at which $x(t)$ touches the bounding curves, it is not obvious that this time interval is even a constant. However, it *is* a constant and has the same value as (5.12) (see Problem 5.5)

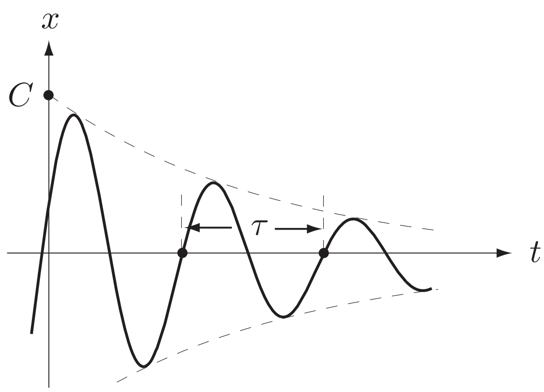

FIGURE 5.3 Under-damped simple harmonic motion $x = Ce^{-Kt}\cos(\Omega_D t - \gamma)$.

Over-damping (super-critical damping): $K > \Omega$

In this case, we write the equation for λ in the form

$$(\lambda + K)^2 = \delta^2,$$

where $\delta = (K^2 - \Omega^2)^{1/2}$, a *positive real number*. The λ values are then $\lambda = -k \pm \delta$, which are now real. We have thus found the pair of real solutions

$$x = e^{-Kt}e^{\pm\delta t},$$

which form a basis for the space of real solutions. The **general real solution** of the damped SHM equation in this case is therefore

$$x = e^{-Kt}\left(Ae^{\delta t} + Be^{-\delta t}\right), \tag{5.13}$$

where A and B are real arbitrary constants.

General form of the motion

Three typical forms for the motion are shown in Figure 5.4. This is called **over-damped SHM**. Somewhat surprisingly, *the body does not oscillate at all*. For example, if the body is released from rest, then it simply drifts back towards the equilibrium position. On the other hand, if the body is projected towards the equilibrium position with sufficient speed, then it passes the equilibrium position once and then drifts back towards it from the other side.

Critical damping: $K = \Omega$

The case of critical damping is solved in Problem 5.6. Qualitatively, the motions look like those in Figure 5.4.

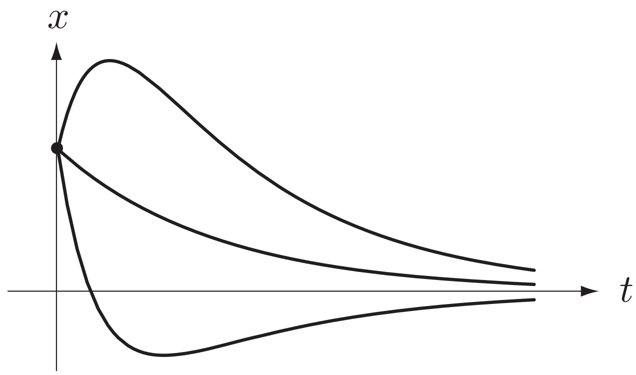

FIGURE 5.4 Three typical cases of over-damped simple harmonic motion.

5.4 DRIVEN (FORCED) MOTION

We now include the effect of an external **driving force** $G(t)$ which we suppose to be a *given* function of the time. In the case of a body suspended by a spring, we could apply such a force directly, but, in practice, the external 'force' often arises indirectly by virtue of the suspension point being made to oscillate in some prescribed way. The seismograph described in the next section is an instance of this.

Whatever the origin of the driving force, the **governing equation** for driven motion is (5.4), namely

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = F(t), \tag{5.14}$$

where $2mK$ is the damping constant, $m\Omega^2$ is the spring constant and $mF(t)$ is the driving force. Since this equation is linear and inhomogeneous, its general solution is the sum of (i) the general solution of the corresponding homogeneous equation (5.9) (the complementary function) and (ii) *any* particular solution of the inhomogeneous equation (5.14) (the particular integral). The complementary function has already been found in the last section, and it remains to find the particular integral for interesting choices of $F(t)$. Actually there is a (rather complicated) formula for a particular integral of this equation for *any choice* of the driving force $mF(t)$. However, the most important case by far is that of **time harmonic** forcing and, in this case, it is easier to find a particular integral directly. Time harmonic forcing is the case in which

$$F(t) = F_0 \cos pt, \tag{5.15}$$

where F_0 and p are positive constants; mF_0 is the amplitude of the applied force and p is its angular frequency.

Solution procedure

We first replace the forcing term $F_0 \cos pt$ by its complex counterpart $F_0 e^{ipt}$. This gives the complex equation

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = F_0 e^{ipt}. \tag{5.16}$$

We then seek a particular integral of this complex equation in the form

$$x = ce^{ipt}, \tag{5.17}$$

where c is a complex constant called the **complex amplitude**. On substituting (5.17) into equation (5.16) we find that

$$c = \frac{F_0}{\Omega^2 - p^2 + 2iKp}, \tag{5.18}$$

so that the complex function

$$\frac{F_0 e^{ipt}}{\Omega^2 - p^2 + 2iKp} \tag{5.19}$$

is a particular integral of the complex equation (5.16). A particular integral of the real equation (5.14) is then given by the real part of the complex expression (5.19). It follows that a **particular integral** of equation (5.14) is given by

$$x^D = a\cos(pt - \gamma),$$

where $a = |c|$ and $\gamma = -\arg c$. This particular integral, which is also time harmonic with the same frequency as the applied force, is called the **driven response** of the oscillator to the force $mF_0 \cos pt$; a is the **amplitude** of the driven response and γ $(0 < \gamma \le \pi)$ is the **phase angle** by which the response lags behind the force. From the expression (5.18) for c, it follows that

$$a = \frac{F_0}{\left((\Omega^2 - p^2)^2 + 4K^2p^2\right)^{1/2}}, \qquad \tan\gamma = \frac{2Kp}{\Omega^2 - p^2}. \tag{5.20}$$

The **general solution** of equation (5.14) therefore has the form

$$x = a\cos(pt - \gamma) + x^{CF}, \tag{5.21}$$

where x^{CF} is the complementary function, that is, the general solution of the corresponding *undriven* problem.

The undriven problem has already been solved in the last section. The solution took three different forms depending on whether the damping was supercritical, critical or subcritical. However, all these forms have one feature in common, that is, *they all decay to zero with increasing time*. For this reason, the complementary function for this equation is often called the **transient response** of the oscillator. Any solution of equation (5.21) is therefore the sum of the driven response x^D (which persists) and a transient response x^{CF} (which dies away). Thus, *no matter what the initial conditions, after a sufficently long time we are left with just the driven response*. In many problems, the transient response can be disregarded, but it must be included if inital conditions are to be satisfied.

Example 5.2 *An initial value problem for driven motion*

The equation of motion of a certain driven damped oscillator is

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 10\cos t$$

and initially the particle is at rest at the origin. Find the subsequent motion.

Solution

First we find the **driven response** x^D. The complex counterpart of the equation of motion is

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 10e^{it}$$

and we seek a solution of this equation of the form $x = ce^{it}$. On substituting in, we find that

$$c = \frac{10}{1+3i} = 1 - 3i.$$

It follows that the **driven response** x^D is given by

$$x^D = \Re\left[(1-3i)e^{it}\right] = \cos t + 3\sin t.$$

Now for the **complementary function** x^{CF}. This is the general solution of the corresponding undriven equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0,$$

which is easily found to be

$$x = Ae^{-t} + Be^{-2t},$$

where A and B are arbitrary constants. The **general solution** of the equation of motion is therefore

$$x = \cos t + 3\sin t + Ae^{-t} + Be^{-2t}.$$

It now remains to choose A and B so that the **initial conditions** are satisfied. The condition $x = 0$ when $t = 0$ implies that

$$0 = 1 + A + B,$$

and the condition $\dot{x} = 0$ when $t = 0$ implies that

$$0 = 3 - A - 2B.$$

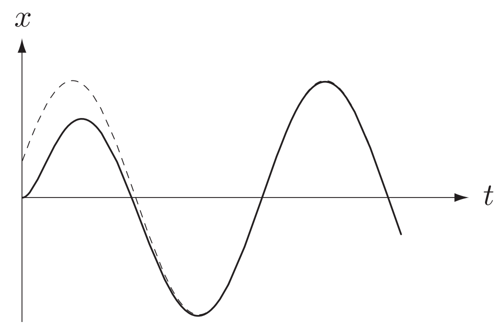

FIGURE 5.5 The solid curve is the actual response and the dashed curve the driven response only.

Solving these simultaneous equations gives $A = -5$ and $B = 4$. The **subsequent motion** of the oscillator is therefore given by

$$x = \underbrace{\cos t + 3\sin t}_{\text{driven response}} \; \underbrace{-5e^{-t} + 4e^{-2t}}_{\text{transient response}}.$$

This solution is shown in Figure 5.5 together with the driven response only. In this case, the transient response is insignificant after less than one cycle of the driving force. The **amplitude** of the driven reponse is $(1^2 + 3^2)^{1/2} = \sqrt{10}$ and the **phase lag** is $\tan^{-1}(3/1) \approx 72°$. ■

Resonance of an oscillating system

Consider the general formula

$$a = \frac{F_0}{\left((\Omega^2 - p^2)^2 + 4K^2p^2\right)^{1/2}}$$

for the amplitude a of the driven response to the force $mF_0 \cos pt$ (see equation (5.20)). Suppose that the amplitude of the applied force, the spring constant, and the resistance constant are held fixed and that the *angular frequency* p of the applied force is varied. Then a is a function of p only. Which value of p produces the largest driven response? Let

$$f(q) = (\Omega^2 - q)^2 + 4K^2q.$$

Then, since $a = F_0/\sqrt{f(p^2)}$, we need only find the minimum point of the function $f(q)$ lying in $q > 0$. Now

$$f'(q) = -2(\Omega^2 - q) + 4K^2 = 2\left(q - (\Omega^2 - 2K^2)\right)$$

so that $f(q)$ decreases for $q < \Omega^2 - 2K^2$ and increases for $q > \Omega^2 - 2K^2$. Hence $f(q)$ has a unique minimum point at $q = \Omega^2 - 2K^2$. Two cases arise depending on whether this value is positive or not.

Case **1**. When $\Omega^2 > 2K^2$, the minimum point $q = \Omega^2 - 2K^2$ is positive and a has its maximum value when $p = p^R$, where

$$p^R = (\Omega^2 - 2K^2)^{1/2}.$$

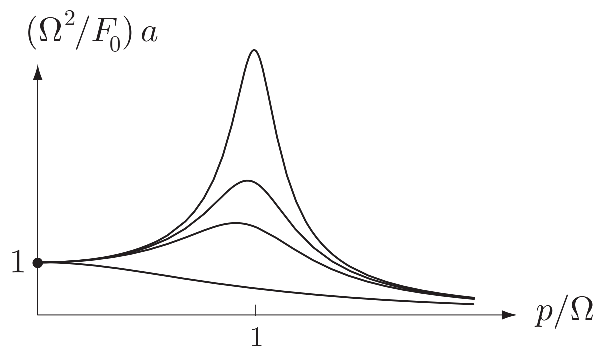

FIGURE 5.6 The dimensionless amplitude $(F_0/\Omega^2)a$ against the dimensionless driving frequency p/Ω for (from the top) $K/\Omega = 0.1, 0.2, 0.3, 1$.

The angular frequency p^R is called the **resonant frequency** of the oscillator. The value of a at the resonant frequency is

$$a_{max} = \frac{F_0}{2K(\Omega^2 - K^2)^{1/2}}.$$

Case **2**. When $\Omega^2 \le 2K^2$, a is a decreasing funcion of p for $p > 0$ so that a has *no maximum point*.

These results are illustrated in Figure 5.6. They are an example of the general physical phenomenon known as **resonance**, which can be loosely stated as follows:

The phenomenon of resonance

Suppose that, in the absence of damping, a physical system can perform free oscillations with angular frequency Ω. Then a driving force with angular frequency p will induce a large response in the system when p is close to Ω, providing that the damping is not too large.

This principle does not just apply to the mechanical systems we study here. It is a general physical principle that also applies, for example, to the oscillations of electric currents in circuits and to the quantum mechanical oscillations of atoms.

Note that the resonant frequency p^R is always less than Ω, but is *close* to Ω when K/Ω is small. The height of the resonance peak, a_{max}, is given approximately by

$$a_{max} \sim \frac{F_0}{2\Omega^2}\left(\frac{\Omega}{K}\right)^{-1}$$

in the limit in which K/Ω is small; a_{max} therefore tends to infinity in this limit. In the same limit, the width of the resonance peak is directly proportional to K/Ω and consequently tends to zero.

General periodic driving force

The method we have developed for the time harmonic driving force can be extended to any periodic driving force $mF(t)$. A function $f(t)$ is said to be **periodic** with period τ if the values taken by f in any interval of length τ are then repeated in the next interval of length τ. An example is the 'square wave' function shown in Figure 5.7. The solution method requires that $F(t)$ be expanded as a **Fourier series**.* A textbook on mechanics is not the place to develop the theory of Fourier series. Instead we will simply quote the essential results and then give an example of how the method works. To keep the algebra as simple as possible, we will suppose that the driving force has period 2π.†

Fourier's Theorem

Fourier's theorem states that any function $f(t)$ that is periodic with period 2π can be expanded as a **Fourier series** in the form

$$f(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt, \tag{5.22}$$

where the **Fourier coefficients** $\{a_n\}$ and $\{b_n\}$ are given by the formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt. \tag{5.23}$$

What this means is that *any* function $f(t)$ with period 2π can be expressed as a sum of *time harmonic* terms, each of which has period 2π. In order to find the driven response of the oscillator when the force $mF(t)$ is applied, we first expand $F(t)$ in a Fourier series. We then find the driven response that would be induced by each of the terms of this Fourier series applied separately, and then simply add these responses together. The method depends on the equation of motion being *linear*.

Example 5.3 *Periodic non-harmonic driving force*

Find the driven response of the damped linear oscillator

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = F(t)$$

for the case in which $F(t)$ is periodic with period 2π and takes the values

$$F(t) = \begin{cases} F_0 & (0 < t < \pi), \\ -F_0 & (\pi < t < 2\pi), \end{cases}$$

* After Jean Baptiste Joseph Fourier 1768–1830. The memoir in which he developed the theory of trigonometric series '*On the Propagation of Heat in Solid Bodies*' was submitted for the mathematics prize of the Paris Institute in 1811; the judges included such luminaries as Lagrange, Laplace and Legendre. They awarded Fourier the prize but griped about his lack of mathematical rigour.

† The general case can be reduced to this one by a scaling of the unit of time.

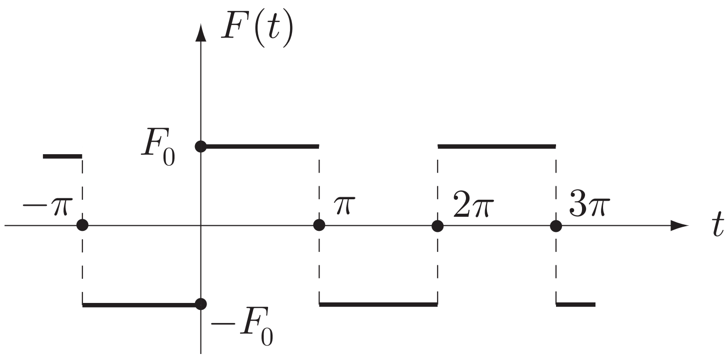

FIGURE 5.7 The 'square wave' input function $F(t)$ is periodic with period 2π. Its value alternates between $\pm F_0$.

in the interval $0 < t < 2\pi$. This function* is shown in Figure 5.7.

Solution

The first step is to find the Fourier series of the function $F(t)$. From the formula (5.23), the coefficient a_n is given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0}(-F_0)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}(+F_0)\cos nt\,dt \\ &= 0, \end{aligned}$$

since both integrals are zero for $n \geq 1$ and are equal and opposite when $n = 0$. In the same way,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0}(-F_0)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}(+F_0)\sin nt\,dt \\ &= \frac{2F_0}{\pi}\int_{0}^{\pi}\sin nt\,dt, \end{aligned}$$

since this time the two integrals are equal. Hence

$$\begin{aligned} b_n &= \frac{2F_0}{\pi}\left[\frac{-\cos nt}{n}\right]_0^{\pi} = \frac{2F_0}{\pi}\left(\frac{1-\cos n\pi}{n}\right) \\ &= \frac{2F_0}{\pi}\left(\frac{1-(-1)^n}{n}\right). \end{aligned}$$

Hence the **Fourier series** of the function $F(t)$ is

$$F(t) = \sum_{n=1}^{\infty}\frac{2F_0}{\pi}\left(\frac{1-(-1)^n}{n}\right)\sin nt.$$

* This function is the mechanical equivalent of a 'square wave input' in electric circuit theory.

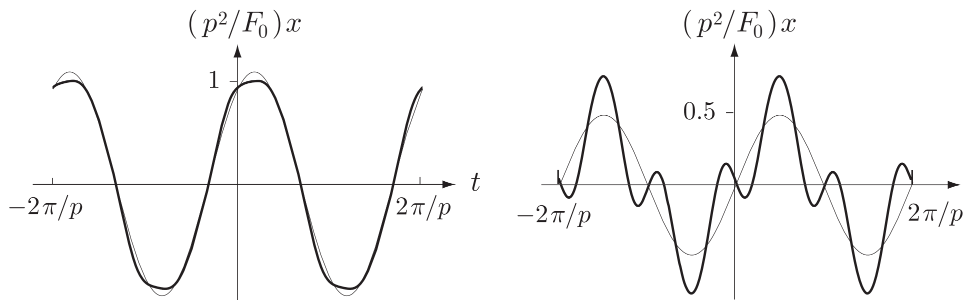

FIGURE 5.8 Driven response of a damped oscillator to the alternating constant force $\pm mF_0$ with angular frequency p : **Left** $\Omega/p = 1.5$, $K/p = 1$. **Right** $\Omega/p = 2.5$, $K/p = 0.1$. The light graphs show the first term of the exapansion series.

The next step is to find the driven response of the oscillator to the force $m(b_n \sin nt)$, that is, the particular integral of the equation

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = b_n \sin nt. \tag{5.24}$$

The complex counterpart of this equaton is

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = b_n e^{int}$$

for which the particular integral is ce^{int}, where the complex amplitude c is given by

$$c = \frac{b_n}{\Omega^2 - n^2 + 2iK}.$$

The particular integral of the real equation (5.24) is then given by

$$\Im\left(\frac{b_n e^{int}}{\Omega^2 - n^2 + 2iKn}\right) = b_n\left(\frac{(\Omega^2 - n^2)\sin nt + 2Kn\cos nt}{(\Omega^2 - n^2)^2 + 4K^2n^2}\right).$$

Finally we add together these separate responses to find the **driven response** of the oscillator to the force $mF(t)$. On inserting the value of the coefficient b_n, this gives

$$x = \frac{2F_0}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)\left(\frac{(\Omega^2 - n^2)\sin nt + 2Kn\cos nt}{(\Omega^2 - n^2)^2 + 4K^2n^2}\right). \tag{5.25}$$

In order to deduce anything from this complicated formula, we must either sum the series numerically or approximate the formula in some way. When Ω and K are both small compared to the forcing frequency p, the series (5.25) converges quite quickly and can be approximated (to within a few percent) by the first term. Even when $\Omega/p = 1.5$ and $K/p = 1$, this is still a reasonable approximation (see Figure 5.8 (left)). However, for larger values of Ω/p, the higher harmonics in the Fourier expansion of $F(t)$ that have frequencies close to Ω produce large contributions (see Figure 5.8 (right)). In this case, the series (5.25) must be summed numerically. ■

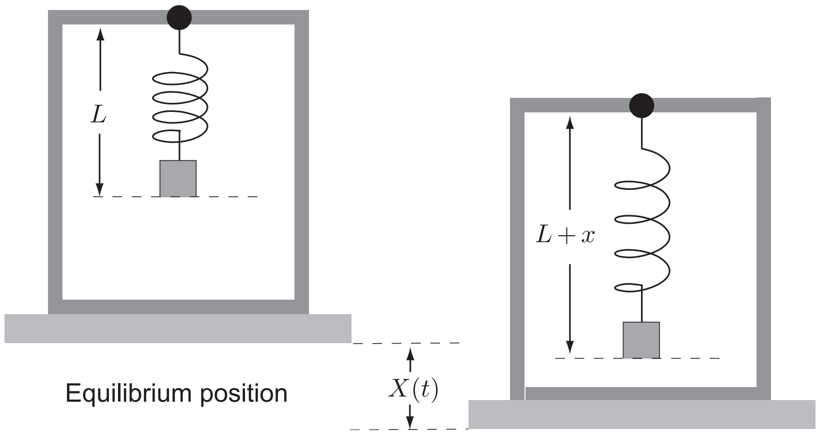

FIGURE 5.9 A simple seismograph for measuring vertical ground motion.

5.5 A SIMPLE SEISMOGRAPH

The seismograph is an instrument that measures the motion of the ground on which it stands. In real earthquakes, the ground motion will generally have both vertical and horizontal components, but, for simplicity, we describe here a device for measuring **vertical motion** only.

Our simple seismograph (see Figure 5.9) consists of a mass which is suspended from a rigid support by a spring; the motion of the mass relative to the support is resisted by a damper. The support is attached to the ground so that the suspension point has the same motion as the ground below it. This motion sets the suspended mass moving and the resulting spring extension is measured as a function of the time. Can we deduce what the ground motion was?

Suppose the ground (and therefore the support) has downward displacement $X(t)$ at time t and that the extension $x(t)$ of the spring is measured from its equilibrium length. Then the *displacement* of the mass is $x + X$, relative to an inertial frame. The equation of motion (5.9) is therefore modified to become

$$m\frac{d^2(x+X)}{dt^2} = -(2mK)\frac{dx}{dt} - (m\Omega^2)x,$$

that is,

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = -\frac{d^2X}{dt^2}.$$

This means *that the motion of the body relative to the moving support is the same as if the support were fixed and the external driving force* $-m(d^2X/dt^2)$ *were applied to the body*.

First consider the driven response of our seismograph to a train of harmonic waves with amplitude A and angular frequency p, that is,

$$X = A\cos pt.$$

The equation of motion for the spring extension x is then

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = Ap^2\cos pt.$$

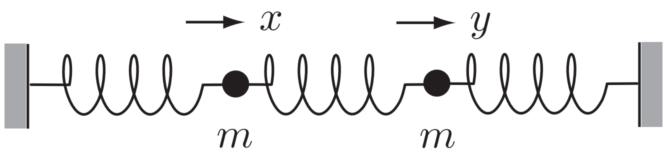

FIGURE 5.10 Two particles are connected between three springs and perform longitudinal oscillations.

The complex amplitude of the driven motion is

$$c = \frac{p^2 A}{-p^2 + 2iKp + \Omega^2},$$

and the real driven motion is

$$x = a\cos(pt - \gamma),$$

where

$$a = |c| = \frac{A}{\left|-1 + 2i(K/p) + (\Omega/p)^2\right|}. \tag{5.26}$$

Thus, providing that the spring and resistance constants are accurately known, the angular frequency p and amplitude A of the incident wave train can be deduced.

In practice, things may not be so simple. In particular, the incident wave train may be a mixture of harmonic waves with different amplitudes and frequencies, and these are not easily disentangled. However, if K and Ω are chosen so that K/p and Ω/p are small compared with unity (for all likely values of p), then $c = -A$ and $X = -x$ approximately. Thus, in this case, the record for $x(t)$ is simply the negative of the ground motion $X(t)$.* Since this result is independent of the incident frequency, it should also apply to complicated inputs such as a pulse of waves.

5.6 COUPLED OSCILLATIONS AND NORMAL MODES

Interesting new effects occur when two or more oscillators are coupled together. Figure 5.10 shows a typical case in which two bodies are connected between three springs and the motion takes place in a straight line. We restrict ourselves here to the classical theory in which the *restoring forces are linear and damping is absent*. If the springs are non-linear, then the displacements of the particles must be small enough so that the linear approximation is adequate.

Let x and y be the displacements of the two bodies from their respective equilibrium positions at time t; because two coordinates are needed to specify the configuration, the system is said to have two *degrees of freedom*. Then, at time t, the extensions of the three springs are x, $y - x$ and $-y$ respectively. Suppose that the strengths of the three springs are α, 2α and

* What is actually happening is that the mass is hardly moving at all (relative to an inertial frame).

4α respectively. Then the three restoring forces are αx, $2\alpha(y-x)$, $-4\alpha y$ and the equations of motion for the two bodies are

$$m\ddot{x} = -\alpha x + 2\alpha(y-x),$$
$$m\ddot{y} = -2\alpha(y-x) - 4\alpha y,$$

which can be written in the form

$$\begin{aligned} \ddot{x} + 3n^2 x - 2n^2 y &= 0, \\ \ddot{y} - 2n^2 x + 6n^2 y &= 0, \end{aligned} \tag{5.27}$$

where the positive constant n is defined by $n^2 = \alpha/m$. These are the **governing equations** for the motion. They are a *pair of simultaneous second order homogeneous linear ODEs* with constant coefficients. The equations are **coupled** in the sense that both unknown functions appear in each equation; thus neither equation can be solved on its own.

The solution procedure: normal modes

The solution procedure is simply an extension of the usual method for finding the complementary function for a single homogeneous linear ODE with constant coefficients. However, rather than seek solutions in exponential form, it is simpler to seek solutions directly in the trigonometric form

$$\begin{aligned} x &= A\cos(\omega t - \gamma), \\ y &= B\cos(\omega t - \gamma), \end{aligned} \tag{5.28}$$

where A, B, ω and γ are constants. A solution of the governing equations (5.27) that has the form (5.28) is called a **normal mode** of the oscillating system. In a normal mode, all the coordinates that specify the configuration of the system vary harmonically in time with the *same frequency* and the *same phase*; however, they generally have *different amplitudes*. On substituting the normal mode form (5.28) into the governing equations (5.27), we obtain

$$-\omega^2 A\cos(\omega t - \gamma) + 3n^2 A\cos(\omega t - \gamma) - 2n^2 B\cos(\omega t - \gamma) = 0,$$
$$-\omega^2 B\cos(\omega t - \gamma) - 2n^2 A\cos(\omega t - \gamma) + 6n^2 B\cos(\omega t - \gamma) = 0,$$

which simplifies to give

$$\begin{aligned} (3n^2 - \omega^2)A - 2n^2 B \quad &= 0, \\ -2n^2 A + (6n^2 - \omega^2)B &= 0, \end{aligned} \tag{5.29}$$

a pair of *simultaneous linear algebraic equations* for the amplitudes A and B. Thus a normal mode will exist if we can find constants A, B and ω so that the equations (5.29) are satisfied. Since the equations are homogeneous, they always have the *trivial solution* $A = B = 0$, whatever the value of ω. However, the trivial solution corresponds to the *equilibrium solution* $x = y = 0$ of the governing equations (5.27), which is not a motion at all. We therefore require the equations (5.29) to have a **non-trivial solution** for A, B. There is a simple condition that this should be so, namely that the determinant of the system of equations should be zero, that is,

$$\det\begin{pmatrix} 3n^2-\omega^2 & -2n^2 \\ -2n^2 & 6n^2-\omega^2 \end{pmatrix} = 0. \tag{5.30}$$

On simplification, this gives the condition

$$\omega^4 - 9n^2\omega^2 + 14n^4 = 0, \tag{5.31}$$

a quadratic equation in the variable ω^2. If this equation has *real positive* roots ω_1^2, ω_2^2, then, for *each* of these values, the linear equations (5.29) will have a non-trivial solution for the amplitudes A, B. In the present case, the equation (5.31) factorises and the roots are found to be

$$\omega_1^2 = 2n^2, \qquad\qquad \omega_2^2 = 7n^2. \tag{5.32}$$

Hence there are **two normal modes** with (angular) frequencies $\sqrt{2}n$ and $\sqrt{7}n$ respectively. These frequencies are known as the **normal frequencies** of the oscillating system.

Slow mode: In the slow mode we have $\omega^2 = 2n^2$ so that the linear equations (5.29) become

$$\begin{aligned} n^2A - 2n^2B &= 0, \\ -2n^2A + 4n^2B &= 0. \end{aligned}$$

These two equations are each equivalent to the single equation $A = 2B$. This is to be expected since, if the equations were linearly independent, then there would be no non-trivial solution for A and B. We have thus found a family of non-trivial solutions $A = 2\delta$, $B = \delta$, where δ can take any (non-zero) value. Thus the *amplitude of the normal mode is not uniquely determined*; this happens because the governing ODEs are linear and homogeneous. The **slow normal mode** therefore has the form

$$\begin{aligned} x &= 2\delta\cos(\sqrt{2}\,nt - \gamma), \\ y &= \ \delta\cos(\sqrt{2}\,nt - \gamma), \end{aligned} \tag{5.33}$$

where the amplitude factor δ and phase factor γ can take any values. We see that, in the slow mode, the two bodies always move in the *same* direction with the body on the left having twice the amplitude as the body on the right.

Fast mode: In the fast mode we have $\omega^2 = 7n^2$ and, by following the same procedure, we find that the form of the **fast normal mode** is

$$\begin{aligned} x &= \ \delta\cos(\sqrt{7}\,nt - \gamma), \\ y &= -2\delta\cos(\sqrt{7}\,nt - \gamma), \end{aligned} \tag{5.34}$$

where the amplitude factor δ and phase factor γ can take any values. We see that, in the fast mode, the two bodies always move in *opposite* directions with the body on the right having twice the amplitude as the body on the left.

The general motion

Since the governing equations (5.27) are linear and homogeneous, a sum of normal mode solutions is also a solution. Indeed, the general solution can be written as a sum of normal

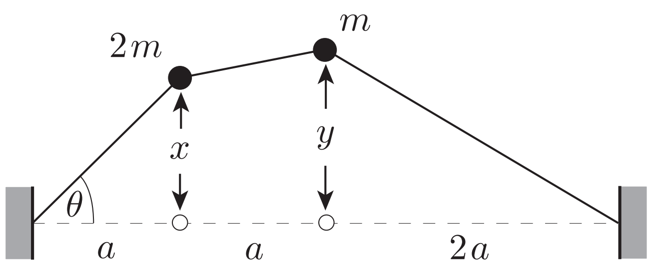

FIGURE 5.11 The two particles are attached to a light stretched string and perform *small* transverse oscillations. The displacements are shown to be large for clarity.

modes. Consider the expression

$$\begin{aligned} x &= 2\delta_1 \cos(\sqrt{2}\,nt - \gamma_1) + \delta_2 \cos(\sqrt{7}\,nt - \gamma_2), \\ y &= \delta_1 \cos(\sqrt{2}\,nt - \gamma_1) - 2\delta_2 \cos(\sqrt{7}\,nt - \gamma_2). \end{aligned} \tag{5.35}$$

This is simply a sum of the first normal mode (with amplitude factor δ_1 and phase factor γ_1) and the second normal mode (with amplitude factor δ_2 and phase factor γ_2). Since it is possible to choose these *four* arbitrary constants so that x, y, $\dot{x}$, $\dot{y}$ take any set of assigned values when $t = 0$, this must be the **general solution** of the governing equations (5.27).

Question ***Periodicity of the general motion***

Is the general motion periodic?

Answer

The general motion is a sum of normal mode motions with periods τ_1, τ_2 respectively. This sum will be periodic with period τ if (and only if) τ is an integer multiple of both τ_1 and τ_2, that is, if τ_1/τ_2 is a *rational* number. (In this case, the periods are said to be *commensurate*.) This in turn requires that ω_1/ω_2 is a rational number. In the present case, $\omega_1/\omega_2 = (2/7)^{1/2}$, which is irrational. The general motion is therefore **not periodic** in this case. ■

We conclude by solving another typical normal mode problem.

Example 5.4 *Small transverse oscillations*

Two particles P and Q, of masses $2m$ and m, are secured to a light string that is stretched to tension T_0 between two fixed supports, as shown in Figure 5.11. The particles undergo *small transverse oscillations* perpendicular to the equilibrium line of the string. Find the normal frequencies, the forms of the normal modes, and the general motion of this system. Is the general motion periodic?

Solution

First we need to make some simplifying assumptions.* We will assume that the transverse displacements x, y of the two particles are small compared with a; the three sections of the string then make small angles with the equilibrium line. We will also neglect any change in the tensions of the three sections of string.

The left section of string then has constant tension T_0. When the particle P is displaced, this tension force has the transverse component $-T_0 \sin\theta$, which acts as a restoring force on P; since θ is small, this component is approximately $-T_0 x/a$. Similar remarks apply to the other sections of string. The **equations of transverse motion** for P and Q are therefore

$$\begin{aligned} 2m\ddot{x} &= -\frac{T_0 x}{a} + \frac{T_0(y-x)}{a}, \\ m\ddot{y} &= -\frac{T_0(y-x)}{a} - \frac{T_0 y}{2a}. \end{aligned}$$

which can be written in the form

$$2\ddot{x} + 2n^2 x - n^2 y = 0, \tag{5.36}$$

$$2\ddot{y} - 2n^2 x + 3n^2 y = 0, \tag{5.37}$$

where the positive constant n is defined by $n^2 = T_0/ma$.

These equations will have **normal mode** solutions of the form

$$\begin{aligned} x &= A\cos(\omega t - \gamma), \\ y &= B\cos(\omega t - \gamma), \end{aligned}$$

when the simultaneous linear equations

$$\begin{aligned} (2n^2 - 2\omega^2)A - n^2 B \quad &= 0, \\ -2n^2 A + (3n^2 - 2\omega^2)B &= 0, \end{aligned} \tag{5.38}$$

have a non-trivial solution for the amplitudes A, B. The condition for this is

$$\det \begin{pmatrix} 2n^2 - 2\omega^2 & -n^2 \\ -n^2 & 3n^2 - 2\omega^2 \end{pmatrix} = 0. \tag{5.39}$$

On simplification, this gives

$$2\omega^4 - 5n^2\omega^2 + 2n^4 = 0, \tag{5.40}$$

a quadratic equation in the variable ω^2. This equation factorises and the roots are found to be

$$\omega_1^2 = \tfrac{1}{2}n^2, \qquad\qquad \omega_2^2 = 2n^2. \tag{5.41}$$

Hence there are **two normal modes** with **normal frequencies** $n/\sqrt{2}$ and $\sqrt{2}n$ respectively.

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* These assumptions are consistent with the more complete treatment given in Chapter 15.

Slow mode: In the slow mode we have $\omega^2 = n^2/2$ so that the linear equations (5.38) become

$$n^2 A - n^2 B = 0,$$
$$-2n^2 A + 2n^2 B = 0.$$

These two equations are each equivalent to the single equation $A = B$ so that we have the family of non-trivial solutions $A = \delta$, $B = \delta$, where δ can take any (non-zero) value. The **slow normal mode** therefore has the form

$$\begin{aligned} x &= \delta \cos(nt/\sqrt{2} - \gamma), \\ y &= \delta \cos(nt/\sqrt{2} - \gamma), \end{aligned} \tag{5.42}$$

where the amplitude factor δ and phase factor γ can take any values. We see that, in the slow mode, the two particles always have the *same displacement*.

Fast mode: In the fast mode we have $\omega^2 = 2n^2$ and, by following the same procedure, we find that the form of the **fast normal mode** is

$$\begin{aligned} x &= \delta \cos(\sqrt{2}\, nt - \gamma), \\ y &= -2\delta \cos(\sqrt{2}\, nt - \gamma), \end{aligned} \tag{5.43}$$

where the amplitude factor δ and phase factor γ can take any values. We see that, in the fast mode, the two particles always move in *opposite* directions with Q having twice the amplitude of P.

The **general motion** is now the sum of the first normal mode (with amplitude factor δ_1 and phase factor γ_1) and the second normal mode (with amplitude factor δ_2 and phase factor γ_2). This gives

$$\begin{aligned} x &= \delta_1 \cos(nt/\sqrt{2} - \gamma_1) + \delta_2 \cos(\sqrt{2}\, nt - \gamma_2), \\ y &= \delta_1 \cos(nt/\sqrt{2} - \gamma_1) - 2\delta_2 \cos(\sqrt{2}\, nt - \gamma_2). \end{aligned} \tag{5.44}$$

For this system $\tau_1/\tau_2 = \omega_2/\omega_1 = 2$ so that the general motion is **periodic** with period $\tau_1 = 2\sqrt{2}\pi/n$. ■

Problems on Chapter 5

Answers and comments are at the end of the book.
Harder problems carry a star (∗).

Free linear oscillations

5.1 A certain oscillator satisfies the equation

$$\ddot{x} + 4x = 0.$$

Initially the particle is at the point $x = \sqrt{3}$ when it is projected towards the origin with speed 2. Show that, in the subsequent motion,

$$x = \sqrt{3}\cos 2t - \sin 2t.$$

Deduce the amplitude of the oscillations. How long does it take for the particle to first reach the origin?

5.2 When a body is suspended from a fixed point by a certain linear spring, the angular frequency of its vertical oscillations is found to be Ω_1. When a different linear spring is used, the oscillations have angular frequency Ω_2. Find the angular frequency of vertical oscillations when the two springs are used together (i) in parallel, and (ii) in series. Show that the first of these frequencies is at least twice the second.

5.3 A particle of mass m moves along the x-axis and is acted upon by the restoring force $-m(n^2 + k^2)x$ and the resistance force $-2mk\dot{x}$, where n, k are positive constants. If the particle is released from rest at $x = a$, show that, in the subsequent motion,

$$x = \frac{a}{n}e^{-kt}(n \cos nt + k \sin nt).$$

Find how far the particle travels before it next comes to rest.

5.4 An overdamped harmonic oscillator satisfies the equation

$$\ddot{x} + 10\dot{x} + 16x = 0.$$

At time $t = 0$ the particle is projected from the point $x = 1$ towards the origin with speed u. Find x in the subsequent motion.

Show that the particle will reach the origin at some later time t if

$$\frac{u-2}{u-8} = e^{6t}.$$

How large must u be so that the particle will pass through the origin?

5.5 A damped oscillator satisfies the equation

$$\ddot{x} + 2K\dot{x} + \Omega^2 x = 0$$

where K and Ω are positive constants with $K < \Omega$ (under-damping). At time $t = 0$ the particle is released from rest at the point $x = a$. Show that the subsequent motion is given by

$$x = ae^{-Kt}\left(\cos \Omega_D t + \frac{K}{\Omega_D}\sin \Omega_D t\right),$$

where $\Omega_D = (\Omega^2 - K^2)^{1/2}$.

Find all the turning points of the function $x(t)$ and show that the ratio of successive maximum values of x is $e^{-2\pi K/\Omega_D}$.

A certain damped oscillator has mass 10 kg, period 5 s and successive maximum values of its displacement are in the ratio 3 : 1. Find the values of the spring and damping constants α and β.

5.6 ***Critical damping*** Find the general solution of the damped SHM equation (5.9) for the special case of critical damping, that is, when $K = \Omega$. Show that, if the particle is initially released from rest at $x = a$, then the subsequent motion is given by

$$x = ae^{-\Omega t}(1 + \Omega t).$$

Sketch the graph of x against t.

5.7* ***Fastest decay*** The oscillations of a galvanometer satisfy the equation

$$\ddot{x} + 2K\dot{x} + \Omega^2 x = 0.$$

The galvanometer is released from rest with $x = a$ and we wish to bring the reading permanently within the interval $-\epsilon a \le x \le \epsilon a$ as quickly as possible, where ϵ is a small positive constant. What value of K should be chosen? One possibility is to choose a sub-critical value of K such that the first minimum point of $x(t)$ occurs when $x = -\epsilon a$. [Sketch the graph of $x(t)$ in this case.] Show that this can be acheived by setting the value of K to be

$$K = \Omega\left[1 + \left(\frac{\pi}{\ln(1/\epsilon)}\right)^2\right]^{-1/2}.$$

If K has this value, show that the time taken for x to reach its first minimum is approximately $\Omega^{-1}\ln(1/\epsilon)$ when ϵ is small.

5.8 A block of mass M is connected to a second block of mass m by a linear spring of natural length $8a$. When the system is in equilibrium with the first block on the floor, and with the spring and second block vertically above it, the length of the spring is $7a$. The upper block is then pressed down until the spring has half its natural length and is then resleased from rest. Show that the lower block will leave the floor if $M < 2m$. For the case in which $M = 3m/2$, find when the lower block leaves the floor.

Driven linear oscillations

5.9 A block of mass 2 kg is suspended from a fixed support by a spring of strength $2000\ \mathrm{N\,m^{-1}}$. The block is subject to the vertical driving force $36\cos pt$ N. Given that the spring will yield if its extension exceeds 4 cm, find the range of frequencies that can safely be applied.

5.10 A driven oscillator satisfies the equation

$$\ddot{x} + \Omega^2 x = F_0\cos[\Omega(1+\epsilon)t],$$

where ϵ is a positive constant. Show that the solution that satisfies the initial conditions $x = 0$ and $\dot{x} = 0$ when $t = 0$ is

$$x = \frac{F_0}{\epsilon(1+\frac{1}{2}\epsilon)\Omega^2}\sin\tfrac{1}{2}\epsilon\Omega t\,\sin\Omega(1+\tfrac{1}{2}\epsilon)t.$$

Sketch the graph of this solution for the case in which ϵ is small.

5.11 Figure 5.12 shows a simple model of a car moving with constant speed c along a gently undulating road with profile $h(x)$, where $h'(x)$ is small. The car is represented by a chassis

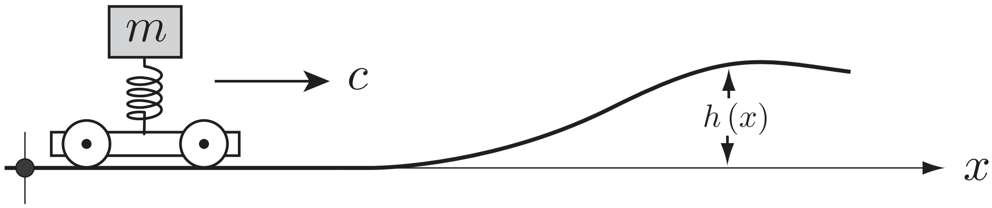

FIGURE 5.12 The car moves along a gently undulating road.

which keeps contact with the road, connected to an upper mass m by a spring and a damper. At time t the upper mass has displacement $y(t)$ above its equilibrium level. Show that, under suitable assumptions, y satisfies a differential equation of the form

$$\ddot{y} + 2K\dot{y} + \Omega^2 y = 2K\,ch'(ct) + \Omega^2 h(ct)$$

where K and Ω are positive constants.

Suppose that the profile of the road surface is given by $h(x) = h_0 \cos(px/c)$, where h_0 and p are positive constants. Find the amplitude a of the *driven* oscillations of the upper mass.

The vehicle designer adjusts the damper so that $K = \Omega$. Show that

$$a \leq \frac{2}{\sqrt{3}}\,h_0,$$

whatever the values of the consants Ω and p.

5.12 ***Solution by Fourier series*** A driven oscillator satisfies the equation

$$\ddot{x} + 2K\dot{x} + \Omega^2 x = F(t),$$

where K and Ω are positive constants. Find the driven response of the oscillator to the saw tooth' input, that is, when $F(t)$ is given by

$$F(t) = F_0\,t \qquad\qquad (-\pi < t < \pi)$$

and $F(t)$ is periodic with period 2π. [It is a good idea to sketch the graph of the function $F(t)$.]

Non-linear oscillations that are piecewise linear

5.13 A particle of mass m is connected to a fixed point O on a smooth horizontal table by a linear elastic string of natural length $2a$ and strength $m\Omega^2$. Initially the particle is released from rest at a point on the table whose distance from O is $3a$. Find the period of the resulting oscillations.

5.14 ***Coulomb friction*** The displacement x of a spring mounted mass under the action of Coulomb friction satisfies the equation

$$\ddot{x} + \Omega^2 x = \begin{cases} -F_0 & \dot{x} > 0 \\ F_0 & \dot{x} < 0 \end{cases}$$

where Ω and F_0 are positive constants. If $|x| > F_0/\Omega^2$ when $\dot{x} = 0$, then the motion continues; if $|x| \leq F_0/\Omega^2$ when $\dot{x} = 0$, then the motion ceases. Initially the body is released from rest with $x = 9F_0/2\Omega^2$. Find where it finally comes to rest. How long was the body in motion?

5.15 A partially damped oscillator satisfies the equation

$$\ddot{x} + 2\kappa\,\dot{x} + \Omega^2 x = 0,$$

where Ω is a positive constant and κ is given by

$$\kappa = \begin{cases} 0 & x < 0 \\ K & x > 0 \end{cases}$$

where K is a positive constant such that $K < \Omega$. Find the period of the oscillator and the ratio of successive maximum values of x.

Normal modes

5.16 A particle P of mass $3m$ is suspended fron a fixed point O by a light linear spring with strength α. A second particle Q of mass $2m$ is in turn suspended from P by a second spring of the same strength. The system moves in the vertical straight line through O. Find the normal frequencies and the form of the normal modes for this system. Write down the form of the general motion.

5.17 Two particles P and Q, each of mass m, are secured at the points of trisection of a light string that is stretched to tension T_0 between two fixed supports a distance $3a$ apart. The particles undergo small *transverse* oscillations perpendicular to the equlilibrium line of the string. Find the normal frequencies, the forms of the normal modes, and the general motion of this system. [Note that the forms of the modes could have been deduced from the symmetry of the system.] Is the general motion periodic?

5.18 A particle P of mass $3m$ is suspended from a fixed point O by a light inextensible string of length a. A second particle Q of mass m is in turn suspended from P by a second string of length a. The system moves in a vertical plane through O. Show that the linearised equations of motion for *small* oscillations near the downward vertical are

$$4\ddot{\theta} + \ddot{\phi} + 4n^2\theta = 0,$$

$$\ddot{\theta} + \ddot{\phi} + n^2\phi = 0,$$

where θ and ϕ are the angles that the two strings make with the downward vertical, and $n^2 = g/a$. Find the normal frequencies and the forms of the normal modes for this system.