Summary of Lecture 8 – CONSERVATION OF ENERGY

1. Potential energy is, as the word suggests, the energy "locked up" up somewhere and which can do work. Potential energy *kam karnay ki salahiat hai*! Potential energy can be converted into kinetic energy, $\frac{1}{2}mv^2$. As I showed you earlier, this follows directly from Newton's Laws.

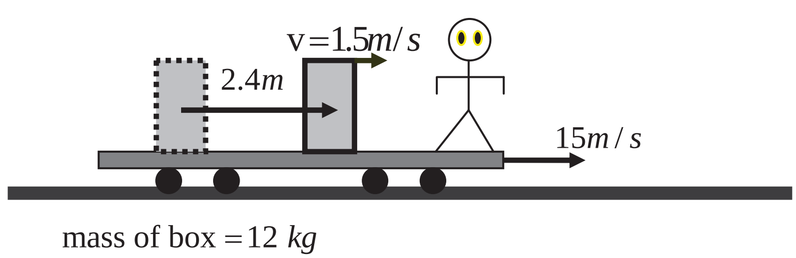
2. If you lift a stone of mass m from the ground up a distance x, you have to do work against gravity. The (constant) force is mg, and so $W = mgx$. By conservation of energy, the work done by you was transformed into gravitational potential energy whose values is exactly equal to mgx. Where is the energy stored? Answer: it is stored neither in the mass or in the earth - it is stored in the gravitational field of the combined system of stone+earth.

3. Suppose you pull on a spring and stretch it by an amount x away from its normal (equilibrium) position. How much energy is stored in the spring? Obviously, the spring gets harder and harder to pull as it becomes longer. When it is extended by length x and you pull it a further distance dx, the small amount of work done is $dW = Fdx = kxdx$. Adding up all the small bits of work gives the total work:

$$W = \int_0^x Fdx = \int_0^x kxdx = \frac{1}{2}kx^2$$

   This is the work you did. Maybe you got tired working so hard. What was the result of your working so hard? Answer: this work was transformed into energy stored in the spring. The spring contains energy exactly equal to $\frac{1}{2}kx^2$.

4. Kinetic energy obviously depends on the frame you choose to measure it in. If you are running with a ball, it has zero kinetic energy with respect to you. But someone who is standing will see that it has kinetic energy! Now consider the following situation: a box of mass 12kg is pushed with a constant force so that so that its speed goes from zero to 1.5m/sec (as measured by the person at rest on the cart) and it covers a distance of 2.4m. Assume there is no friction.



Let's first calculate the change in kinetic energy:

$$\Delta K = K_f - K_i = \frac{1}{2}(12kg)(1.5m/s)^2 - 0 = 13.5J$$

And then the (constant) acceleration:

$$a = \frac{v_f^2 - v_i^2}{2(x_f - x_i)} = \frac{(1.5m/s)^2 - 0}{2(2\cdot 4m)} = 0.469m/s^2$$

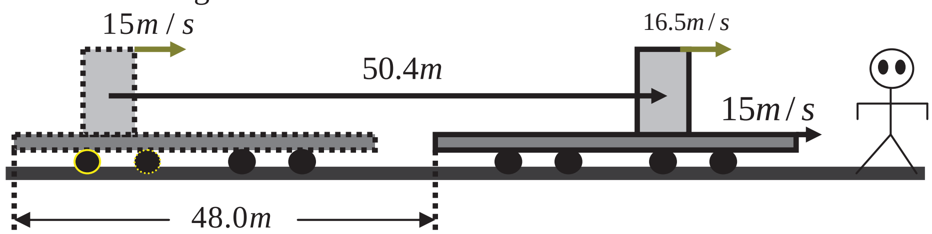
This acceleration results from a constant net force given by:

$$F = ma = (12kg)(0.469\, m/s^2) = 5.63N$$

From this, the work done on the crate is:

$$W = F\Delta x = (5.63N)(2.4m) = 13.5J \quad (\text{same as } \Delta K = 13.5J \text{ !})$$

5. Now suppose there is somebody standing on the ground, and that the trolley moves at 15 m/sec relative to the ground:



Let us repeat the same calculation:

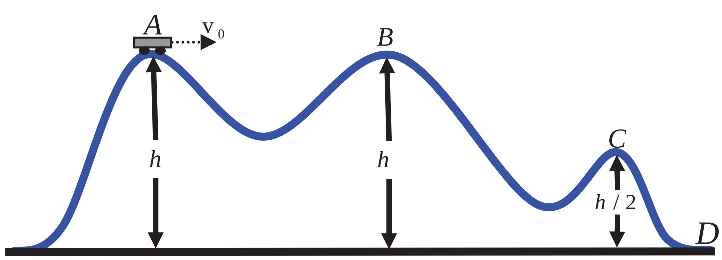
$$\Delta K' = K_f' - K_i' = \frac{1}{2}mv_f{}^2 - \frac{1}{2}mv_i{}^2$$

$$= \frac{1}{2}(12kg)(16.5m/s)^2 - \frac{1}{2}(12kg)(15.0m/s)^2 \; = 284J$$

This example clearly shows that work and energy have different values in different frames.

6. The total mechanical energy is: $E_{mech} = KE + PE$. If there is no friction then E_{mech} is conserved. This means that the sum does not change with time. For example: a ball is thrown upwards at speed v_0. How high will it go before it stops? The loss of potential energy is equal to the gain of potential energy. Hence, $\frac{1}{2}mv_0^2 = mgh \Rightarrow h = \frac{v_0^2}{2g}$.

Now look at the smooth, frictionless motion of a car over the hills below:



Even though the motion is complicated, we can use the fact that the total energy is a constant to get the speeds at the points B,C,D:

At point A: $\frac{1}{2}m\mathrm{v}_A{}^2 + mgh = \frac{1}{2}m\mathrm{v}_B{}^2 + mgh \quad \Rightarrow \quad \mathrm{v}_B = \mathrm{v}_C$

At point C: $\frac{1}{2}m\mathrm{v}_A{}^2 + mgh = \frac{1}{2}m\mathrm{v}_C{}^2 + mg\frac{h}{2} \quad \Rightarrow \quad \mathrm{v}_\mathrm{C} = \sqrt{\mathrm{v}_\mathrm{A}{}^2 + gh}$

At point D: $\frac{1}{2}m\mathrm{v}_A{}^2 + mgh = \frac{1}{2}m\mathrm{v}_\mathrm{D}{}^2 \quad \Rightarrow \quad \mathrm{v}_\mathrm{D} = \sqrt{\mathrm{v}_\mathrm{A}{}^2 + 2gh}$

7. Remember that potential energy has meaning only for a force that is conservative. A conservative force is that for which the work done in going from point A to point B is independent of the path chosen. Friction is an example of a non-conservative force and a potential energy cannot be defined. For a conservative force, $\mathrm{F} = -\frac{dV}{dx}$. So, for a spring, $V = \frac{1}{2}kx^2$ and so $F = -kx$.

8. Derivation of $\mathrm{F} = -\frac{dV}{dx}$: If the particle moves distance Δx in a potential V, then change in PE is ΔV where, $\Delta V = -F\ \Delta x$. From this, $F = -\frac{\Delta V}{\Delta x}$. Now let $\Delta x \rightarrow 0$. Hence, $\mathrm{F} = -\lim_{\Delta x\to 0}\frac{\Delta \mathrm{V}}{\Delta x} = -\frac{dV}{dx}$.

QUESTIONS AND EXERCISES – 8

1. If energy is conserved, then why do you get tired simply standing in one place?

2. Suppose that the motion of the earth around the sun suddenly stopped. Using the conservation of energy, discuss what will happen to the earth after that.

3. In point 5 above, find the work done by the force (as seen by the ground observer) and show that it is equal to 284 J. So, even though work and energy are different in different frames, the law of conservation of energy holds in every frame.

4. The potential between 2 atoms as a function of distance is:

$$V(x) = -\frac{4}{x^6} + \frac{7}{x^{12}}$$

a) Find the force as a function of distance. Where is the force zero?

b) Calculate $\frac{d^2V}{dx^2}$ at the value of x where the force is zero. Is it positive or negative?

c) Note that the above expression for $V(x)$ is not dimensionally correct if x is actually a length. How can it be made correct?

Summary of Lecture 9 – MOMENTUM

1. Momentum is the "quantity of motion" possessed by a body. More precisely, it is defined as:

Mass of the body × Velocity of the body.

The dimensions of momentum are MLT^{-1} and the units of momentum are kg-m/s.

2. Momentum is a vector quantity and has both magnitude and direction, $p = m\vec{v}$. We can easily see that Newton's Second Law can be reexpressed in terms of momentum. When I wrote it down originally, it was in the form $m\vec{a} = \vec{\mathbf{F}}$. But since $\frac{d\vec{v}}{dt} = \vec{a}$, this can also be written as $\frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}}$ (new form). In words, the rate of change of momentum of a body equals the total force acting upon it. Of course, the old and new are exactly the same,

$$\frac{d\vec{\mathbf{p}}}{dt} = \frac{d(m\vec{\mathbf{v}})}{dt} = m\frac{d\vec{\mathbf{v}}}{dt} = m\vec{\mathbf{a}} = \vec{\mathbf{F}}.$$

3. When there are many particles, then the total momentum $\vec{\mathbf{P}}$ is,

$$\vec{\mathbf{P}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \cdots \vec{\mathbf{p}}_N$$

$$\frac{d}{dt}\vec{\mathbf{P}} = \frac{d\vec{\mathbf{p}}_1}{dt} + \frac{d\vec{\mathbf{p}}_2}{dt} + \cdots \frac{d\vec{\mathbf{p}}_N}{dt}$$

$$= \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots \vec{\mathbf{F}}_N = \vec{\mathbf{F}}$$

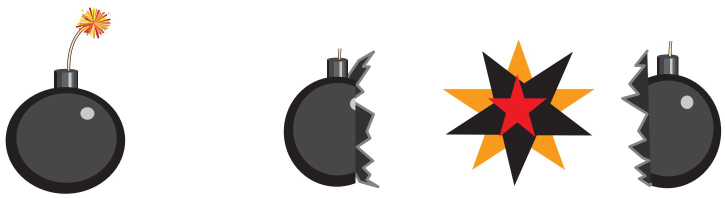
This shows that when there are several particles, the rate at which the total momentum changes is equal to the total force. It makes sense!

4. A very important conclusion of the above is that if the sum of the total external forces vanishes, then the total momentum is conserved, $\sum \vec{F}_{ext} = 0 \Rightarrow \frac{d}{dt}\vec{\mathbf{P}} = 0.$ This is quite independent of what sort of forces act between the bodies - electric, gravitational, etc. - or how complicated these are. We shall see why this is so important from the following examples.

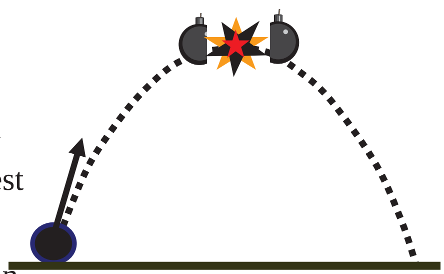
5. Two balls, which can only move along a straight line, collide with each other. The initial momentum is $P_i = m_1u_1 + m_2u_2$ and the final momentum is $P_f = m_1\text{v}_1 + m_2\text{v}_2$. Obviously one ball exerts a force on the other when they collide, so its momentum changes. But, from the fact that there is no external force acting on the balls, $P_i = P_f$, or $m_1u_1 + m_2u_2 = m_1\text{v}_1 + m_2\text{v}_2$.

6. A bomb at rest explodes into two fragments. Before the explosion the total momentum is

zero. So obviously it is zero after the explosion as well, $\mathbf{P}_f = \mathbf{0}$. During the time that the explosion happens, the forces acting upon the pieces are very complicated and changing rapidly with time. But when all is said and done, there are two pieces flying away with a total zero final momentum $\mathbf{P}_f = m_1 \mathrm{v}_1 + m_2 \mathrm{v}_2$. Hence $m_1 \mathrm{v}_1 = -m_2 \mathrm{v}_2$. In other words, the fragments fly apart with equal momentum but in opposite directions. The centre-of-mass stays at rest. So, knowing the velocity of one fragment permits knowing the velocity of the other fragment.



7. If air resistance can be ignored, then we can do some interesting calculations with what we have learned. So, suppose a shell is fired from a cannon with a speed 10 m/s at an angle 60^0 with the horizontal. At the highest point in its path it explodes into two pieces of equal masses. One of the pieces retraces its path to the cannon. Let us find the velocity of the other piece immediately after the explosion.



Solution: After the explosion: $P_{1x} = -5\frac{M}{2}$ (why?). But $P_{1x} + P_{2x} = P_x = M \times 10\cos 60$

$$\Rightarrow P_{2x} = 5M + 5\frac{M}{2}. \text{ Now use: } P_{2x} = \frac{M}{2}\mathrm{v}_{2x} \Rightarrow \mathrm{v}_{2x} = 15\, m/s.$$

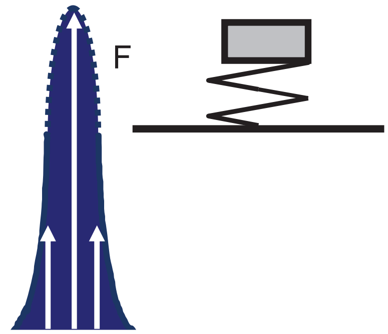
8. When you hit your thumb with a hammer it hurts, doesn't it? Why? Because a large amount of momentum has been destroyed in a short amount of time. If you wrap your thumb with foam, it will hurt less. To understand this better, remember that force is the rate of change of momentum: $F = \frac{dp}{dt} \Rightarrow dp = Fdt$. Now define the **impulse** I as:

force × time over which the force acts.

If the force changes with time between the limits , then one should define I as,

$I = \int_{t_1}^{t_2} Fdt$. Since $\int_{t_1}^{t_2} Fdt = \int_{p_i}^{p_f} dp$, therefore $I = p_f - p_i$. In words, the change of momentum equals the impulse, which is equal to the area under the curve of force versus time. Even if you wrap your thumb in foam, the impulse is the same. But the force is definitely not!

9. Sometimes we only know the force numerically (i.e. there is no expression like F=something). But we still know what the integral means: it is the area under the curve of force versus time. The curve here is that of a hammer striking a table. Before the hammer strikes, the force is zero, reaches a peak, and goes back to zero.



QUESTIONS AND EXERCISES – 9

Q.1 A stream of bullets, each of mass m, is fired horizontally with a speed v into a large wooden block of mass M that is at rest on a horizontal table. If the coefficient of friction is μ, how many bullets must be fired per second so that the block just begins to move? [Hint, calculate the momentum destroyed in one second.]

Q. 2 In point 5, we have one equation but two unknowns v_1, v_2. Obviously more information needs to be supplied. So consider two extreme cases. In both cases, find the final velocities:

a) The collision is perfectly elastic, meaning that the sum of initial kinetic energies is exactly equal to the sum of final kinetic energies.

b) The collision results in the two bodies sticking together and moving off as one body.

Q.3 a) Would you rather land with your legs bending or stiff ?

b) Why do cricket fielders move their hands backwards when catching a fast ball?

c) Why do railway carriages have dampers at the front and back?

Q.4 A rocket of mass M_0 is at rest in space. Then at $t = 0$ it starts to eject hot gas at speed v from the nozzle and the mass of the rocket is $m(t) = M_0 - \alpha t$. Find the acceleration of the rocet at time $t = 0$.

Q.5 Sand drops onto an open railway carriage at ρ kg/sec. If the engine pulling it is working at power P_0 watts at time $t = 0,$ find the additional power of the engine pulling it such that the speed of the train remans constant at v_0.

Summary of Lecture 10 – COLLISIONS

1. Collisions are extremely important to understand because they happen all the time - electrons collide with atoms, a bat with a ball, cars with trucks, star galaxies with other galaxies,...In every case, the sum of the initial momenta equals the sum of the final momenta. This follows directly from Newton's Second Law, as we have already seen.

2. Take the simplest collision: two bodies of mass m_1 and m_2 moving with velocities u_1 and u_2. After the collision they are moving with velocities v_1 and v_2. From momentum conservation,

$$m_1u_1 + m_2u_2 = m_1\mathrm{v}_1 + m_2\mathrm{v}_2 \Rightarrow m_1(u_1 - \mathrm{v}_1) = m_2(\mathrm{v}_2 - u_2)$$

This is as far as we can go. There are two unknowns but only one equation. However, if the collision is elastic then,

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 = \tfrac{1}{2}m_1\mathrm{v}_1^2 + \tfrac{1}{2}m_2\mathrm{v}_2^2 \Rightarrow \tfrac{1}{2}m_1(u_1^2 - \mathrm{v}_1^2) = \tfrac{1}{2}m_2(\mathrm{v}_2^2 - u_2^2).$$

Combine the two equations above,

$$u_1 + \mathrm{v}_1 = \mathrm{v}_2 + u_2 \Rightarrow u_1 - u_2 = \mathrm{v}_2 - \mathrm{v}_1.$$

In words, this says that in an elastic collision the relative speed of the incoming particles equals the relative speed of the outgoing particles.

3. One can solve for v_1 and v_2 (please do it!) easily and find that:

$$\mathrm{v}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$\mathrm{v}_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

Notice that if $m_1 = m_2$, then $\mathrm{v}_1 = u_2$ and $\mathrm{v}_2 = u_1$. So this says that after the collision, the bodies will just reverse their velocities and move on as before.

4. What if one of the bodies is much heavier than the other body, and the heavier body is at rest? In this case, $m_2 >> m_1$ and $u_2 = 0$. e can immediately see that $\mathrm{v}_1 = -u_1$ and $\mathrm{v}_2 = 0$. This makes a lot of sense: the heavy body continues to stay at rest and the light body just bounces back with the same speed. In the lecture, you saw a demonstration of this!

5. And what if the lighter body (rickshaw) is at rest and is hit by the heavier body (truck)? In this case, $m_2 << m_1$ and $u_2 = 0$. From the above equation we see that $\mathrm{v}_1 = u_1$ and $\mathrm{v}_2 = 2u_1$. So the truck's speed is unaffected, but the poor rickshaw is thrust in the direction of the truck at twice the truck's speed!

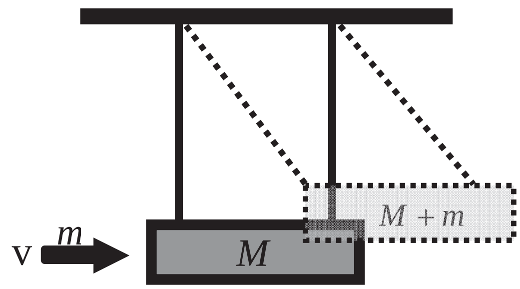
6. Sometimes we wish to slow down particles by making them collide with other particles.

In a nuclear reactor, neutrons can be slowed down in this way. let's calculate the fraction by which the kinetic energy of a neutron of mass m_1 decreases in a head-on collision with an atomic nucleus of mass m_2 that is initially at rest:

Solution: $\frac{K_i - K_f}{K_i} = 1 - \frac{K_f}{K_i} = 1 - \frac{v_f^2}{v_i^2}$

For a target at rest: $v_f = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_i$ $\quad \therefore \frac{K_i - K_f}{K_i} = \frac{4m_1 m_2}{(m_1 + m_2)^2}.$

7. A bullet with mass m, is fired into a block of wood with mass M, suspended like a pendulum and makes a completely inelastic collision with it. After the impact, the block swings up to a maximum height y. What is the initial speed of the bullet?



Solution:

By conservation of momentum in the direction of the bullet,

$m\text{v} = (m + M)V, \;\Rightarrow\; \text{v} = \frac{(m+M)}{m} V$

The block goes up by distance y, and so gains potential energy. Now we can use the conservation of energy to give, $\frac{1}{2}(m+M)V^2 = (m+M)gy,$ where V is the velocity acquired by the block+bullet in the upward direction just after the bullet strikes. Now use $V = \sqrt{2gy}$. So finally, the speed of the bullet is: $\text{v} = \frac{(m+M)}{m}\sqrt{2gy}.$

8. In 2 or 3 dimensions, you must apply conservation of momentum in each direction separately. The equation $\vec{P}_i = \vec{P}_f$ looks as if it is one equation, but it is actually 3 separate equations: $p_{ix} = p_{fx}, p_{iy} = p_{fy}, p_{iz} = p_{fz}$. On the other hand, suppose you had an elastic collision. In that case you would have only one extra equation coming from energy conservation, not three.

9. What happens to energy in an inelastic collision? Let's say that one body smashes into another body and breaks it into 20 pieces. To create 20 pieces requires doing work against the intermolecular forces, and the initial kinetic energy is used up for this.

QUESTIONS AND EXERCISES – 10

Q.1 Give two examples of nearly elastic collisions, and two examples of nearly inelastic collisions that are not from the lecture.

Q.2 In the lecture demonstration you saw that there are 3 balls of equal mass placed in a straight line. The first strikes the second and comes to rest. The second moves and hits the third and comes to rest. The third one then moves off alone.

a) What would happen if the second one was twice the mass of the first or the third, and if the collision was elastic?

b) What would happen if the second was infinitely heavy, and the collision was elastic?

c) What would happen if the collision of the first and second resulted in these two getting stuck to each other? With what speed would they strike the third one?

Q.3 Suppose a ball loses 10% of its energy when it is bounced off the ground. If it dropped from a height h initially, find the maximum height it reaches after

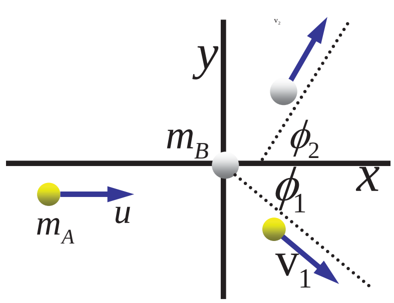
a) The first bounce

b) The second bounce

c) The $n'th$ bounce

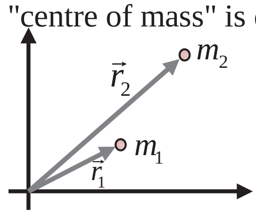
Q.4 In the example in point 7 above, suppose that the bullet bounces back with speed v/2 instead of lodging itself into the wooden block. In that case, calculate how high the block will rise.

Q.5 The figure shows an elastic collision of two bodies A,B on a frictionless table. A has mass $m_A = 0.5$ kg, and B has mass $m_B = 0.3$ kg. A has an initial velocity of 4 m/s in the the positive x-direction and a final velocity of 2 m/s in an unknown direction. B is initially at rest. Find the final velocity of B and the angles in the figure.



Summary of Lecture 12 – PHYSICS OF MANY PARTICLES

1. A body is made of a collection of particles. We would like to think of this body having a "centre". For two masses the "centre of mass" is defined as: $\vec{r}_{cm} \equiv \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$.



In 2 dimensions (i.e. a plane) this is actually two equations:

$x_{cm} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ and $y_{cm} = \dfrac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$. These give the coordinates of the centre of mass of the two-particle system.

2. Example: one mass is placed at $x = 2cm$ and a second mass, equal to the first, is placed at $x = 6cm$. The cm position lies halfway between the two as you can see from:

$$x_{cm} = \frac{mx_1 + mx_2}{m + m} = \frac{2m + 6m}{2m} = 4cm.$$

Note that there is no physical body that is actually located at $x_{cm} = 4cm$! So the centre of mass can actually be a point where there is no matter. Now suppose that the first mass is three times bigger than the first:

$$x_{cm} = \frac{(3m)x_1 + mx_2}{3m + m} = \frac{2(3m) + 6m}{4m} = 3cm$$

This shows that the cm lies closer to the heavier body. This is always true.

3. For N masses the obvious generalization of the centre of mass position is the following:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N} = \frac{1}{M}\left(\sum_{n=1}^{n=N} m_n \vec{r}_n\right).$$

In words, this says that the following: choose any origin and draw vectors $\vec{r}_1, \vec{r}_2, \cdots \vec{r}_N$ that connect to the masses $m_1, m_2, \cdots m_N$. Heavier masses get more importance in the sum. So suppose that m_2 is much larger than any of the others. If so, $\vec{r}_{cm} \approx \dfrac{m_2\vec{r}_2}{m_2} = \vec{r}_2$. Hence, the cm is very close to the position vector of m_2.

4. For symmetrical objects, it is easy to see where the cm position lies: for a sphere or circle it lies at the centre; for a cylinder it is on the axis halfway between the two faces, etc.

5. Our definition of the cm allows Newton's Second Law to be written for entire collection

of particles:

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{v}_n\right)$$

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{a}_n\right)$$

$$\therefore \quad M\,\vec{a}_{cm} = \sum \vec{F}_n = \sum\left(\vec{F}_{ext} + \vec{F}_{int}\right) \qquad \text{use } \sum \vec{F}_{int} = 0$$

$$\Rightarrow \quad \sum \vec{F}_{ext} = M\,\vec{a}_{cm} \text{ (the sum of external forces is what causes acceleration)}$$

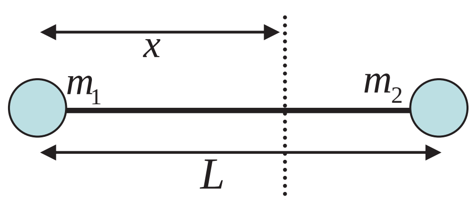
In the above we have used Newton's Third Law as well: $\vec{F}_{12} + \vec{F}_{21} = 0$ etc.

6. Consider rotational motion now for a rigid system of N particles. Rigid means that all particles have a fixed distance from the origin. The kinetic energy is,

$$\begin{aligned} K &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 + \cdots \\ &= \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + \frac{1}{2}m_3 r_3^2\omega^2 + \cdots \\ &= \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2 \end{aligned}$$

Now suppose that we define the "moment of inertia" $I \equiv \sum m_i r_i^2$. Then clearly the kinetic energy is $K = 1/2\, I\omega^2$. How similar this is to $K = \frac{1}{2}Mv^2$!

7. To familiarize ourselves with I, let us consider the following: Two particles m_1 and m_2 are connected by a light rigid rod of length L. Neglecting the mass of the rod, find the rotational inertia I of this system about an axis perpendicular to the rod and at a distance x from m_1.



Answer: $I = m_1 x^2 + m_2 (L - x)^2$. Of course, this was quite trivial. Now we can ask a more interesting question: For what x is I the largest? Now, near a maximum, the slope of a function is zero. So calculate $\frac{dI}{dx}$ and then put it equal to zero:

$$\therefore \quad \frac{dI}{dx} = 2m_1 x - 2m_2(L - x) = 0 \quad \Rightarrow \quad x_{\max} = \frac{m_2 L}{m_1 + m_2}.$$

8. Although matter is made up of discrete atoms, even if one takes small pieces of any body, there are billions of atoms within it. So it is useful to think of matter as being continuously distributed. Since a sum $\sum$ becomes an integral $\int$, it is obvious that the new definitions of I and $\vec{\mathrm{R}}_{\mathrm{cm}}$ becom:

$$I \equiv \int r^2 dm \quad \text{and} \quad \vec{\mathrm{R}}_{\mathrm{cm}} \equiv \frac{1}{M}\int \vec{r} dm.$$

9. A simple application: suppose there is a hoop with mass distributed uniformly over it. The moment of inertia is: $I = \int r^2 dm = R^2 \int dm = MR^2$.

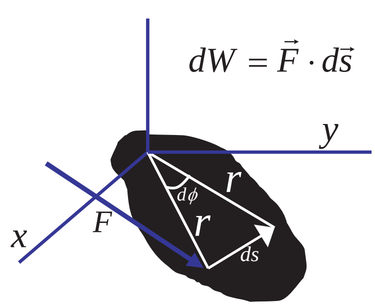
10. A less trivial application: instead of a hoop as above, now consider a solid plate:

$$I = \int r^2 dm \quad (dm = 2\pi r dr \rho_0)$$

$$= \int_0^R 2\pi r^3 dr \rho_0 = \frac{1}{2}(\pi R^2 \rho_0)R^2 = \frac{1}{2} M R^2$$

11. You have seen that it is easier to turn things (e.g. a nut, when changing a car's tyre after a puncture) when the applied force acts at a greater distance. This is because the *torque* τ is greater. We define $\vec{\tau} = \vec{r} \times \vec{F}$ from the magnitude is $\tau = rF\sin\theta$. Here θ is the angle between the radius vector and the force.

12. Remember that when a force $\vec{F}$ acts through a distance $d\vec{r}$ it does an amount of work equal to $\vec{F} \cdot d\vec{r}$. Now let us ask how much work is done when a torque acts through a certain angle as in the diagram below:



The small amount of work done is:

$$dW = \vec{F} \cdot d\vec{s} = F\cos\theta\, ds = \left(F\cos\theta\right)\left(r d\phi\right) = \tau d\phi$$

Add the contributions coming from from all particles,

$$dW_{net} = \left(F_1 \cos\theta_1\right) r_1 d\phi + \left(F_2 \cos\theta_2\right) r_2 d\phi + \cdots + \left(F_n \cos\theta_n\right) r_n d\phi$$

$$= \left(\tau_1 + \tau_2 + \cdots + \tau_n\right) d\phi$$

$$\therefore dW_{net} = \left(\sum \tau_{\text{ext}}\right) d\varphi = \left(\sum \tau_{\text{ext}}\right) \omega dt \qquad .....(1)$$

Now consider the change in the kinetic energy K,

$$dK = d\left(\frac{1}{2} I\omega^2\right) = I\omega d\omega = (I\alpha)\omega dt \qquad ......(2)$$

By conservation of energy, the change in K must equal the work done, and so:

$$dW_{net} = dK \quad \Rightarrow \quad \sum \tau_{ext} = I\alpha$$

In words this says that the total torque equals the moment of inertia times the angular acceleration. This is just like Newton's second law, but for rotational motion !

13.A comparison between linear and rotational motion quantities and formulae:

LINEAR	ROTATIONAL
x, M	ϕ, I
$\text{v} = \frac{dx}{dt}$	$\omega = \frac{d\phi}{dt}$
$\text{a} = \frac{d\text{v}}{dt}$	$\alpha = \frac{d\omega}{dt}$
$F = M\text{a}$	$\tau = I\alpha$
$K = \frac{1}{2} M\text{v}^2$	$K = \frac{1}{2} I\omega^2$
$W = \int F dx$	$W = \int \tau d\phi$

14. Rotational and translational motion can occur simultaneously. For example a car's wheel rotates and translates. In this case the total kinetic energy is clearly the sum of the energies of the two motions: $K = \frac{1}{2} M\, \text{v}_{\text{cm}}^2 + \frac{1}{2} I_{cm} \omega^2$.

15. It will take a little work to prove the following fact that I simply stated above: for a system of N particles, the total kinetic energy divides up neatly into the kinetic energy of rotation and translation. Start with the expression for kinetic energy and write $\vec{\text{v}}_{\text{cm}} + \vec{\text{v}}_i'$ where $\vec{\text{v}}_i'$ is the velocity of a particle with respect to the cm frame,

$$K = \sum \frac{1}{2} m_i \text{v}_i{}^2 = \sum \frac{1}{2} m_i \vec{\text{v}}_i \cdot \vec{\text{v}}_i$$

$$= \sum \frac{1}{2} m_i \left(\vec{\text{v}}_{\text{cm}} + \vec{\text{v}}_i'\right) \cdot \left(\vec{\text{v}}_{\text{cm}} + \vec{\text{v}}_i'\right)$$

$$= \sum \frac{1}{2} m_i \left(\text{v}_{\text{cm}}^2 + 2\vec{\text{v}}_{\text{cm}} . \vec{\text{v}}_{\text{i}}' + \text{v}_{\text{i}}'^2\right)$$

Now, $\sum m_i \vec{v}_{cm} \cdot \vec{v}'_i = \vec{v}_{cm} \cdot \sum m_i \vec{v}'_i = \vec{v}_{cm} \cdot \sum \vec{p}'_i = 0$. Why? because the total momentum is xero in the cm frame! So this brings us to our result that,

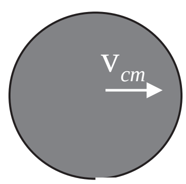
$$K = \sum \frac{1}{2} m_i v_{cm}^2 + \sum \frac{1}{2} m_i v_i'^2$$
$$= \frac{1}{2} M v_{cm}^2 + \sum \frac{1}{2} m_i r_i'^2 \omega^2.$$

QUESTIONS AND EXERCISES – 12

Q.1 A triangle has 3 equal masses placed at each of its vertices. Locate the centre of mass if: a)All 3 angles are equal, b)Two sides are equal and there is one right angle.

Q.2 Suppose that two concentric rings, each with equal mass per unit length are joined so that they rotate together. Find the moment of inertia.

Q.3 A wheel of mass M, radius R, and moment of inertia I is on a surface and moves towards the right as shown.



a) What is the kinetic energy if the wheel slips and does not rotate?
b) What is the kinetic energy if it does not slip? Write your answer only in terms of the quantities specified above.

Q.4 Find the moment of inertia of a solid cylinder that rotates about the (long) axis of symmetry.

Q.5 A sphere rolls down the inclined plane shown here. The total kinetic energy is $K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$. Use this to show that the speed at which it reaches the lowest point is $v_{cm} = \sqrt{\frac{4}{3} gh}$. Assume that there is only rolling and no slipping.