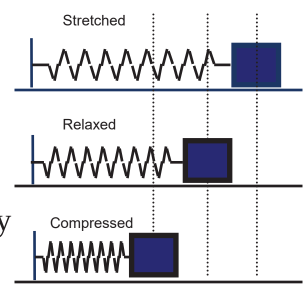
Summary of Lecture 15 – OSCILLATIONS: I

1. An oscillation is any self-repeating motion. This motion is characterized by:
   a) The period T, which is the time for completing one full cycle.
   b) The frequency $f = 1/T$, which is the number of cycles per second. (Another frequently used symbol is ν).
   c) The amplitude A, which is the maximum displacement from equilibrium (or the size of the oscillation).

2. Why does a system oscillate? It does so because a force is always directed towards a central equilibrium position. In other words, the force always acts to return the object to its equilibrium position. So the object will oscillate around the equilibrium position. The restoring force depends on the displacement $F_{\text{restore}} = -k\,\Delta x,$ where Δx is the distance away from the equilibrium point, the negative sign shows that the force acts towards the equilibrium point, and k is a constant that gives the strength of the restoring force.

3. Let us understand these matters in the context of a spring tied to a mass that can move freely over a frictionless surface. The force $F(x) = -kx$ (or $-k\Delta x$ because the extension Δx will be called x for short). In the first diagram x is positive, x is negative in the second, and zero in the middle one. The energy stored in the spring, $U(x) = \frac{1}{2}kx^2,$ is positive in the first and third diagrams and zero in the middle one. Now we will use Newton's second law to derive a differential euation that describes the motion of the mass: From $F(x) = -kx$ and $ma = F$ it follows that $m\frac{d^2x}{dt^2} = -kx,$ or $\frac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega^2 \equiv \frac{k}{m}$. This is the equation of motion of a simple harmonic oscillator (SHO) and is seen widely in many different branches of physics. Although we have derived it for the case of a mass and spring, it occurs again and again. The only difference is that $\omega,$ which is called the oscillator frequency, is defined differently depending on the situation.



4. In order to solve the SHO equation, we shall first learn how to differentiate some elementary trignometric functions. So let us first learn how to calculate $\frac{\mathrm{d}}{\mathrm{dt}}\cos\omega t$ starting from the basic definition of a derivative:

Start : $x(t) = \cos\omega t$ and $x(t+\Delta t) = \cos\omega(t+\Delta t)$. Take the difference:

$$x(t+\Delta t) - x(t) = \cos\omega(t+\Delta t) - \cos\omega t$$

$$= -\sin\omega\Delta t \sin(\omega t + \omega\Delta t/2)$$

$$\approx -\omega\Delta t \sin\omega t \quad \text{as } \Delta t \text{ becomes very small.}$$

$$\therefore \frac{d}{dt}\cos\omega t = -\omega\sin\omega t.$$

(Here you should know that $\sin\theta \approx \theta$ for small $\theta,$ easily proved by drawing triangles.)

You should also derive and remember a second important result:

$$\frac{d}{dt}\sin\omega t = \omega\cos\omega t \ .$$

(Here you should know that $\cos\theta \approx 1$ for small θ.)

5. What happens if you differentiate twice?

$$\frac{d^2}{dt^2}(\sin\omega t) = \omega\frac{d}{dt}\cos\omega t = -\ \omega^2\sin\omega t$$

$$\frac{d^2}{dt^2}(\cos\omega t) = -\omega\frac{d}{dt}\sin\omega t = -\ \omega^2\cos\omega t.$$

So twice differentiating either $\sin\omega t$ or $\cos\omega t$ gives the same function back!

6. Having done all the work above, now you can easily see that any function of the form $x(t) = a\cos\omega t + b\sin\omega t$ satisfies $\frac{d^2x}{dt^2} = -\omega^2 x$. But what do ω, a, b represent ?

a) The significance of ω becomes clear if your replace t by $t + \frac{2\pi}{\omega}$ in either $\sin\omega t$ or $\cos\omega t$. You can see that $\cos\omega\left(t + \frac{2\pi}{\omega}\right) = \cos(\omega t + 2\pi) = \cos\omega t$. That is, the function merely repeats itself after a time $2\pi/\omega$. So $2\pi/\omega$ is really the period of the motion T, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$. The frequency ν of the oscillator is the number of complete vibrations per unit time: $\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ so $\omega = 2\pi\nu = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$.

Sometimes ω is also called the angular frequency. Note that $\dim[\omega] = T^{-1}$, from it is clear that the unit of ω is radian/second.

b) To understand what a and b mean let us note that from $x(t) = a\cos\omega t + b\sin\omega t$ it follows that $x(0) = a$ and that $\frac{\mathrm{d}}{\mathrm{dt}}x(t) = -\omega a\sin\omega t + \omega b\cos\omega t = \omega b$ (at $t = 0$). Thus, a is the initial position, and b is the initial velocity divided by ω.

c) To understand what a and b mean let us note that from $x(t) = a\cos\omega t + b\sin\omega t$ it follows that $x(0) = a$ and that $\frac{\mathrm{d}}{\mathrm{dt}}x(t) = -\omega a\sin\omega t + \omega b\cos\omega t = \omega b$ (at $t = 0$). Thus, a is the initial position, and b is the initial velocity divided by ω.

d) The solution can also be written as: $x(t) = x_m \cos(\omega t + \phi)$. Since cos and sin never beome bigger than 1, or less than -1, it follows that $-x_m \le x \le +x_m$. For obvious reason x_m is called the amplitude of the motion. The frequency of the simple harmonic motion is independent of the amplitude of the motion.

e) The quantity $\theta = \omega t + \phi$ is called the phase of the motion. The constant ϕ is called the *phase constant*. A different value of ϕ just means that the origin of time has been chosen differently.

7. **Energy of simple harmonic motion**. Put $\phi = 0$ for convenience, and so imagine a mass whose position oscillates like $x = x_m \cos\omega t$. Let us first calculate the potential energy:

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2\cos^2\omega t.$$

Now calculate the kinetic energy:

$$K = \frac{1}{2}m\mathrm{v}^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\omega^2 x_m^2\sin^2\omega t = \frac{1}{2}kx_m^2\sin^2\omega t$$

The sum of potential + kinetic is:

$$E = K + U = \frac{1}{2}kx_m^2\cos^2\omega t + \frac{1}{2}kx_m^2\sin^2\omega t$$
$$= \frac{1}{2}kx_m^2\left(\cos^2\omega t + \sin^2\omega t\right) = \frac{1}{2}kx_m^2.$$

Note that this is independent of time and energy goes from kinetic to potential, then back to kinetic etc.

8. From the above, you can see that $\mathrm{v} = \frac{dx}{dt} = \pm\sqrt{\frac{k}{m}\left(x_m^2 - x^2\right)}$. From this it is clear that the speed is maximum at $x = 0$ and that the speed is zero at $x = \pm x_m$.

9. Putting two springs in parallel makes it harder to stretch them, and $k_{eff} = k_1 + k_2$. In series they are easier to stretch, and $k_{eff} = \left(\frac{k_1 k_2}{k_1 + k_2}\right)$. So a mass will oscillate faster in the first case as compared to the second.

QUESTIONS AND EXERCISES – 15

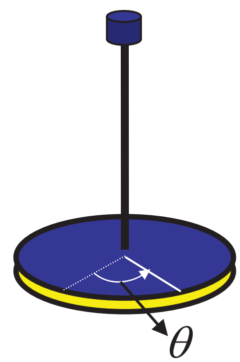
Q.1 A bottle is half filled with water (so that it floats upright) and then pushed a little into the water. As you can see, it oscillates up and down.

a) Where does the restoring force come from?

b) Suppose that you filled the bottle 3/4 full. What would happen to the oscillation frequency?

c) Why does the bottle eventually stop oscillating?

Q.2 A disc is suspended from a string. The equilibrium position is that for which there is no twist in the string, i.e. at $\theta = 0$. When it is slightly moved off from equilibrium, there is a restoring force $F = -\kappa\theta$.



a) Show that $\frac{d^2\theta}{dt^2} = -\left(\frac{\kappa}{I}\right)\theta$ where I is the moment of inertia.

b) If the disc is at $\theta = 0$ at $t = 0$ and suddenly given a twist so that $\frac{d\theta}{dt} = \omega_0$, find how long it takes to return to its initial position. Where will be at time t?

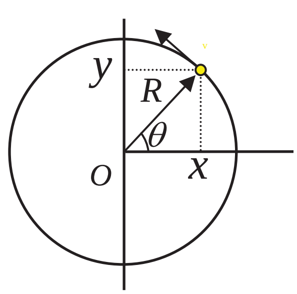
c) Why does the motion eventually cease? List all the ways in which this disc loses energy.

Q.3 Referring to the figure below, you can see that the coordinates of a particle going around a circle are given by $(x, y) = (R\cos\theta, R\sin\theta)$ where $\theta = \omega t$.

a) On the same axes, plot x and y as a function of time. Obviously, here is a case of two harmonic oscillations. What is the phase difference between the two?

b) Find $\dot{x}^2 + \dot{y}^2$, where $\dot{x}$ is $\frac{dx}{dt}$ (this is a very popular way of denoting time derivatives because it is short, so you should be familiar with it.

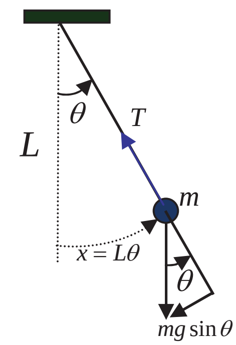
c) Repeat the above for $\ddot{x}^2 + \ddot{y}^2$, where we now have second derivatives instead.



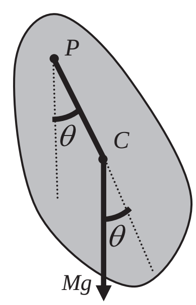
Summary of Lecture 16 – OSCILLATIONS: II

1. In this chapter we shall continue with the concepts developed in the previous chapter that relate to simple harmonic motion and the simple harmonic oscillator (SHO). It is really very amazing that the SHO occurs again and again in physics, and in so many different branches.

2. As an example illustrating the above, consider a mass suspended a string. From the diagram, you can see that $F = -mg\sin\theta$. For small values of θ we know that $\sin\theta \approx \theta$. Using $x = L\theta$ (length of arc), we have $F = -mg\theta = -mg\frac{x}{L} = -\left(\frac{mg}{L}\right)x$. So now we have a restoring force that is proportional to the distance away from the equilibrium point. Hence we have a SHO with $\omega = \sqrt{g/L}$. What if we had not made the small θ approximation? We would still have an oscillator (i.e. the motion would be self repeating) but the solutions of the differential equation would be too complicated to discuss here.



3. If you take a common object (like a piece of cardboard) and pivot it at some point, it will oscillate when disturbed. But this is not the simple pendulum discussed above because all the mass is not concentrated at one point. So now let us use the ideas of torque and angular momentum discussed earlier for many particle systems. You can see that $\tau = -Mgd\sin\theta$. For small θ, $\sin\theta \approx \theta$ and so $\tau = -Mgd\theta$. But we also know that $\tau = I\alpha$ where I is the moment of inertia and α is the angular acceleration, $\alpha = \frac{d^2\theta}{dt^2}$. Hence, we have $I\frac{d^2\theta}{dt^2} = -Mgd\theta$, or, $\frac{d^2\theta}{dt^2} = -\left(\frac{Mgd}{I}\right)\theta$. From this we immediately see that the oscillation frequency is $\omega = \sqrt{\frac{Mgd}{I}}$. Of course, we have used the small angle approximation over here again. Since all variables except I are known, we can use this formula to tell us what I is about any point. Note that we can choose to put the pivot at any point on the body. However, if you put the pivot exactly at the centre of mass then it will not oscillate. Why? Because there is no restoring force and the torque vanishes at the cm position, as we saw earlier.



4. Suppose you were to put the pivot at point P which is at a distance L from the centre of mass of the irregular object above. What should L be so that you get the same formula as for a simple pendulum?

Answer: $T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{I}{Mgd}} \Rightarrow L = \frac{I}{Md}$

P is then called the centre of gyration - when suspended from this point it appears as if all the mass is concentrated at the cm position.

5. Sum of two simple harmonic motions of the same period along the same line:

$$x_1 = A_1 \sin\omega t \text{ and } x_2 = A_2 \sin(\omega t + \phi)$$

Let us look at the sum of x_1 and x_2,

$$x = x_1 + x_2 = A_1 \sin\omega t + A_2 \sin(\omega t + \phi)$$
$$= A_1 \sin\omega t + A_2 \sin\omega t \cos\phi + A_2 \sin\phi \cos\omega t$$
$$= \sin\omega t (A_1 + A_2 \cos\phi) + \cos\omega t (A_2 \sin\phi)$$

Let $A_1 + A_2 \cos\phi = R\cos\theta$ and $A_2 \sin\phi = R\sin\theta$. Using some simple trigonometry, you can put x in the form, $x = R\sin(\omega t + \theta)$. It is easy to find R and θ:

$R = \sqrt{A_1^2 + A_2^2 + A_1 A_2 \cos\phi}$ and $\tan\theta = \frac{A_2 \sin\phi}{A_1 + A_2 \cos\phi}$.

Note that if $\phi = 0$ then $R = \sqrt{A_1^2 + A_2^2 + A_1 A_2} = \sqrt{(A_1 + A_2)^2} = A_1 + A_2$ and $\tan\theta = 0$ $\Rightarrow \theta = 0$. So we get $x = (A_1 + A_2)\sin\omega t$. This is an example of *constructive interference*. If $\phi = \pi$ then $R = \sqrt{A_1^2 + A_2^2 - A_1 A_2} = \sqrt{(A_1 - A_2)^2} = A_1 - A_2$ and $\tan\theta = 0$ $\Rightarrow \theta = 0$. Now we get $x = (A_1 - A_2)\sin\omega t$. This is *destructive interference.*

6. Composition of two simple harmonic motions of the same period but now at right angles to each other:

Suppose $x = A\sin\omega t$ and $y = B\sin(\omega t + \phi)$. These are two independent motions. We can write $\sin\omega t = \frac{x}{A}$ and $\cos\omega t = \sqrt{1 - x^2/A^2}$.

From this, $\frac{y}{B} = \sin\omega t \cos\phi + \sin\phi \cos\omega t = \frac{x}{A}\cos\phi + \sin\phi\sqrt{1 - x^2/A^2}$. Now square and rearrange terms to find:

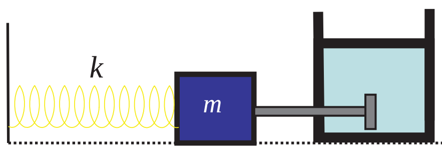
$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - 2\frac{xy}{AB}\cos\phi = \sin^2\phi$$

This is the equation for an ellipse (see questions at the end of this section).

7. If two oscillations of different frequencies at right angles are combined, the resulting motion is more complicated. It is not even periodic unless the two frequencies are in the ratio of integers. This resulting curve are called Lissajous figures. Specifically, if $x = A\sin\omega_x t$ and $y = B\sin\left(\omega_y t + \phi\right)$, then periodic motion requires $\frac{\omega_x}{\omega_y} =$ integers. You should look up a book for more details.

8. **Damped harmonic motion**: Typically the frictional force due to air resistance, or in a liquid, is proportional to the speed. So suppose that the damping force $= -b\frac{dx}{dt}$ (why negative sign?). Now apply Newton's law to a SHO that is damped: $-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$

Rearrange slightly to get the equation for a damped SHO: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$



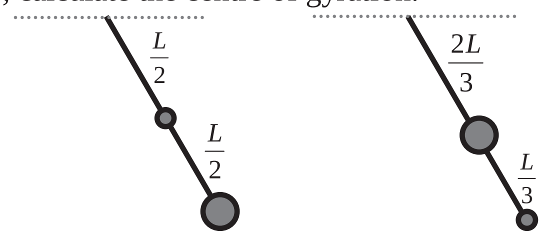
Its solution for $\frac{k}{m} \geq \left(\frac{b}{2m}\right)^2$ is $x = x_m e^{-bt/2m}\cos\left(\omega' t + \phi\right)$. The frequency is now changed: $\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$. The damping causes the amplitude to decrease with time and when $bt/2m = 1$, the amplitude is $1/e \approx 1/2.7$ of its initial value.

8. **Forced oscillation and resonance.** There is a characteristic value of the driving frequency ω at which the amplitude of oscillation is a maximum. This condition is called resonance. For negligible damping resonance occurs at $\omega = \omega_0$. Here ω_0 is the natural frequency of the system and is given by $\omega_0 = \sqrt{\frac{k}{m}}$. The equation of motion is: $m\frac{d^2x}{dt^2} + kx = F_0\cos\omega t$. You should check that this is solved by putting $x = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos\omega t$ (just subsitute into the equation and see!). Note that the amplitude "blows up" when $\omega \to \omega_0$. This is because we have no damping term here. With damping, the amplitude is large when $\omega \to \omega_0$ but remains finite.

QUESTIONS AND EXERCISES – 16

Q.1 For the equation derived in point 6, sketch the curves on an x-y plot for:
a) $\phi = 0$, and, b) $\phi = \pi/2$. Take $A = 1,\ B = 2$.

Q.2 A light rod of length L has two masses M and $3M$ attached to it as shown in the diagrams below.
a) In each case, calculate the frequency of small oscillations.
b) In each case, calculate the centre of gyration.



Q.3 In each of the two cases below, eliminate the time t. In other words, find a relation between x and y which does not involve t.
a) $x = \sin t,\ y = 2\sin t$
b) $x = \sin t,\ y = \cos t$
c) $x = \cos t,\ y = \sin 2t$

Q.4 Verify that $x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi)$ is a solution of the damped SHO equation. Plot $x(t)$ from $t = 0$ to $t = 2$ for the following case: $x_m = 1,\ b = 2,\ m = 1,\ \phi = 0$.

Q.5 A SHO is driven by a force $F(t)$ that depends upon time and obeys the equation,
$m\dfrac{d^2x}{dt^2} + kx = F(t)$. Suppose that $F(t) = F_0\cos 2\omega_0 t + 2F_0 \sin 3\omega_0 t,$ where $\omega_0 = \sqrt{\dfrac{k}{m}}$.
a) Find the general solution $x(t)$.
b) Find that particular solution which has $x(t=0) = 0$ and $\dot{x}(t=0) = 1$.