Chapter Two

Velocity, acceleration and scalar angular velocity

KEY FEATURES

The key concepts in this chapter are the **velocity** and **acceleration** of a particle and the **angular velocity** of a rigid body in planar motion.

Kinematics is the study of the **motion of material bodies** without regard to the forces that cause their motion. The subject does not seek to answer the question of *why* bodies move as they do; that is the province of dynamics. It merely provides a geometrical description of the possible motions. The basic building block for bodies in mechanics is the **particle**, an idealised body that occupies only a single point of space. The important kinematical quantities in the motion of a particle are its **velocity** and **acceleration**. We begin with the simple case of straight line particle motion, where velocity and acceleration are scalars, and then progress to three-dimensional motion, where velocity and acceleration are vectors.

The other important idealisation that we consider is the **rigid body**, which we regard as a collection of particles linked by a light rigid framework. The important kinematical quantity in the motion of a rigid body is its **angular velocity**. In this chapter, we consider only those rigid body motions that are essentially two-dimensional, so that angular velocity is a scalar quantity. The general three-dimensional case is treated in Chapter 16.

2.1 STRAIGHT LINE MOTION OF A PARTICLE

Consider a particle P moving along the x-axis so that its displacement x from the origin O is a known function of the time t. Then the **mean velocity** of P over the time

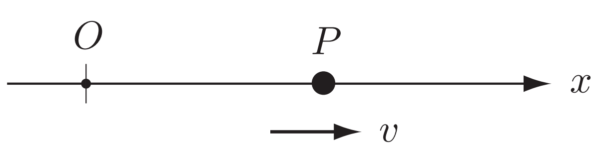

FIGURE 2.1 The particle P moves in a straight line and has displacement x and velocity v at time t.

interval $t_1 \le t \le t_2$ is defined to be the increase in the displacement of P divided by the time taken, that is,

$$\frac{x(t_2) - x(t_1)}{t_2 - t_1}. \tag{2.1}$$

Example 2.1 ***Mean velocity***

Suppose the displacement of P from O at time t is given by $x = t^2 - 6t$, where x is measured in metres and t in seconds. Find the mean velocity of P over the time interval $1 \le t \le 3$.

Solution

In this case, $x(1) = -5$ and $x(3) = -9$ so that the mean velocity of P is $((-9) - (-5))/(3 - 1) = -2\ \mathrm{m\,s^{-1}}$. ■

The mean velocity of a particle is less important to us than its *instantaneous* velocity, that is, its velocity at a given instant in time. We cannot find the instantaneous velocity of P at time t_1 merely by letting $t_2 = t_1$ in the formula (2.1), since the quotient would then be undefined. However, we can define the instantaneous velocity as the *limit* of the mean velocity as the time interval *tends* to zero, that is, as $t_2 \to t_1$. Thus $v(t_1)$, the instantaneous velocity of P at time t_1 can be defined by

$$v(t_1) = \lim_{t_2 \to t_1} \left(\frac{x(t_2) - x(t_1)}{t_2 - t_1} \right).$$

But this is precisely the definition of dx/dt, the derivative of x with respect to t, evaluated at $t = t_1$. This leads us to the official definition:

Definition 2.1 ***1-D velocity*** *The (instantaneous)* ***velocity*** *v of P, in the positive x-direction, is defined by*

$$v = \frac{dx}{dt}. \tag{2.2}$$

The ***speed*** *of P is defined to be the rate of increase of the total distance travelled and is therefore equal to $|v|$.*

Similarly, the acceleration of P, the rate of increase of v, is defined as follows:

Definition 2.2 ***1-D acceleration*** *The (instantaneous)* ***acceleration*** *a of P, in the positive x-direction, is defined by*

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \tag{2.3}$$

Example 2.2 ***Finding rectilinear velocity and acceleration***

Suppose the displacement of P from O at time t is given by $x = t^3 - 6t^2 + 4$, where x is measured in metres and t in seconds. Find the velocity and acceleration of P at time t. Deduce that P comes to rest twice and find the position and acceleration of P at the later of these two times.

Solution

Since $v = dx/dt$ and $a = dv/dt$, we obtain

$$v = 3t^2 - 12t \qquad \text{and} \qquad a = 6t - 12$$

as the velocity and acceleration of P at time t.

P comes to rest when its velocity v is zero, that is, when

$$3t^2 - 12t = 0.$$

This is a quadratic equation for t having the solutions $t = 0, 4$. Thus P is at rest when $t = 0$ s and $t = 4$ s.

When $t = 4$ s, $x = -28$ m and $a = 12\ \mathrm{m\,s^{-2}}$. Note that merely because $v = 0$ at some instant it does not follow that $a = 0$ also. ■

Example 2.3 *Reversing the process*

A particle P moves along the x-axis with its acceleration a at time t given by

$$a = 12t^2 - 6t + 6\ \mathrm{m\,s^{-2}}.$$

Initially P is at the point $x = 4$ m and is moving with speed $8\ \mathrm{m\,s^{-1}}$ in the negative x-direction. Find the velocity and displacement of P at time t.

Solution

Since $a = dv/dt$ we have

$$\frac{dv}{dt} = 12t^2 - 6t + 6,$$

and integrating with respect to t gives

$$v = 4t^3 - 3t^2 + 6t + C,$$

where C is a constant of integration. This constant can be determined by using the given initial condition on v, namely, $v = -8$ when $t = 0$. This gives $C = -8$ so that the velocity of P at time t is

$$v = 4t^3 - 3t^2 + 6t - 8\ \mathrm{m\,s^{-1}}.$$

By writing $v = dx/dt$ and integrating again, we obtain

$$x = t^4 - t^3 + 3t^2 - 8t + D,$$

where D is a second constant of integration. D can now be determined by using the given initial condition on x, namely, $x = 4$ when $t = 0$. This gives $D = 4$ so that the displacement of P at time t is

$$x = t^4 - t^3 + 3t^2 - 8t + 4\ \mathrm{m}.\ ■$$

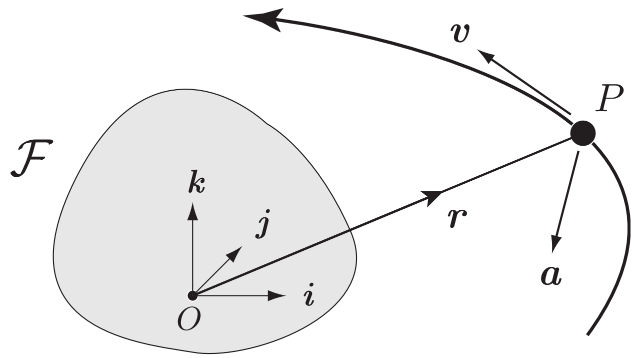

FIGURE 2.2 The particle P moves in three-dimensional space and, relative to the reference frame $\mathcal{F}$ and origin O, has position vector $\boldsymbol{r}$ at time t.

2.2 GENERAL MOTION OF A PARTICLE

When a particle P moves in two or three-dimensional space, its position can be described by its vector displacement $\boldsymbol{r}$ from an origin O that is fixed in a rigid **reference frame** $\mathcal{F}$. Whether $\mathcal{F}$ is moving or not is irrelevant here; the position vector $\boldsymbol{r}$ is simply measured *relative* to $\mathcal{F}$. Figure 2.2 shows a particle P moving in three-dimensional space with position vector $\boldsymbol{r}$ (relative to the reference frame $\mathcal{F}$) at time t.

Question *Reference frames*

What is a reference frame and why do we need one?

Answer

A rigid reference frame $\mathcal{F}$ is essentially a **rigid body** whose particles can be labelled to create reference points. The most familiar such body is the Earth. Relative to a single particle, the only thing that can be specified is distance from that particle. However, relative to a rigid body, one can specify both distance and direction. Thus the value of any vector quantity can be specified relative to $\mathcal{F}$. In particular, if we label some particle O of the body as origin, we can specify the position of any point of space by its position vector relative to the frame $\mathcal{F}$ and the origin O.

The specification of vectors relative to a reference frame is much simplified if we introduce a Cartesian coordinate system. This can be done in infinitely many different ways. Imagine that $\mathcal{F}$ is extended by a set of three mutually orthogonal planes that are *rigidly embedded* in it. The coordinates x, y, z of a point P are then the distances of P from these three planes. Let O be the origin of this coordinate system, and $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ its unit vectors. We can then conveniently refer to the frame $\mathcal{F}$, together with the embedded coordinate system $Oxyz$, by the notation $\mathcal{F}\{O\,;\,\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. ■

In general motion, the velocity and acceleration of a particle are *vector quantities* and are defined by:

Definition 2.3 ***3-D velocity and acceleration*** *The **velocity** $\boldsymbol{v}$ and **acceleration** $\boldsymbol{a}$ of P are defined by*

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} \quad \textit{and} \quad \boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}. \tag{2.4}$$

Connection with the rectilinear case

The scalar velocity and acceleration defined in section 2.1 for the case of straight line motion are simply related to the corresponding vector quantities defined above. It would be possible to use the vector formalism in all cases but, for the case of straight line motion along the x-axis, $\boldsymbol{r}$, $\boldsymbol{v}$, and $\boldsymbol{a}$ would have the form

$$\boldsymbol{r} = x\,\boldsymbol{i}, \qquad \boldsymbol{v} = v\,\boldsymbol{i}, \qquad \boldsymbol{a} = a\,\boldsymbol{i},$$

where $v = dx/dt$ and $a = dv/dt$. It is therefore sufficient to work with the scalar quantities x, v and a; use of the vector formalism would be clumsy and unnecessary.

Example 2.4 *Finding 3-D velocity and acceleration*

Relative to the reference frame $\mathcal{F}\{O\,;\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$, the position vector of a particle P at time t is given by

$$\boldsymbol{r} = (2t^2 - 3)\,\boldsymbol{i} + (4t + 4)\,\boldsymbol{j} + (t^3 + 2t^2)\,\boldsymbol{k}.$$

Find (i) the distance OP when $t = 0$, (ii) the velocity of P when $t = 1$, (iii) the acceleration of P when $t = 2$.

Solution

In this solution we will make use of the rules for differentiation of sums and products involving vector functions of the time. These rules are listed in section 1.6.

(i) When $t = 0$, $\boldsymbol{r} = -3\,\boldsymbol{i} + 4\,\boldsymbol{j}$ so that $OP = |\boldsymbol{r}| = 5$.

(ii) Relative to the reference frame $\mathcal{F}$, the unit vectors $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ are *constant* and so their time derivatives are zero. The velocity $\boldsymbol{v}$ of P is therefore

$$\boldsymbol{v} = d\boldsymbol{r}/dt = 4t\,\boldsymbol{i} + 4\,\boldsymbol{j} + (3t^2 + 4t)\,\boldsymbol{k}.$$

When $t = 1$, $\boldsymbol{v} = 4\,\boldsymbol{i} + 4\,\boldsymbol{j} + 7\,\boldsymbol{k}$.

(iii) Relative to the reference frame $\mathcal{F}$, the acceleration $\boldsymbol{a}$ of P is

$$\boldsymbol{a} = d\boldsymbol{v}/dt = 4\,\boldsymbol{i} + (6t + 4)\,\boldsymbol{k}.$$

When $t = 2$, $\boldsymbol{a} = 4\,\boldsymbol{i} + 16\,\boldsymbol{k}$. ■

Interpretation of the vectors $\boldsymbol{v}$ and $\boldsymbol{a}$

The velocity vector $\boldsymbol{v}$ has a simple interpretation. Suppose that s is the arc-length travelled by P, measured from some fixed point of its path, and that s is increasing with time.*

* The arguments that follow assume a familiarity with the unit tangent and normal vectors to a general curve, as described in section 1.7

Then, by the chain rule,

$$\begin{aligned} \boldsymbol{v} &= \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds} \times \frac{ds}{dt} \\ &= v\,\boldsymbol{t} \end{aligned}$$

where $\boldsymbol{t}$ is the **unit tangent vector** to the path and v $(= ds/dt)$ is the **speed*** of P. Thus, at each instant, *the direction of the velocity vector* $\boldsymbol{v}$ *is along the tangent to its path, and* $|\boldsymbol{v}|$ *is the speed of* P.

The acceleration vector $\boldsymbol{a}$ is harder to picture. This is partly because we are too accustomed to the special case of straight line motion. However, in general,

$$\begin{aligned} \boldsymbol{a} &= \frac{d\boldsymbol{v}}{dt} = \frac{d(v\,\boldsymbol{t})}{dt} = \frac{dv}{dt}\boldsymbol{t} + v\frac{d\boldsymbol{t}}{dt} = \left(\frac{dv}{dt}\right)\boldsymbol{t} + v\left(\frac{d\boldsymbol{t}}{ds} \times \frac{ds}{dt}\right) \\ &= \left(\frac{dv}{dt}\right)\boldsymbol{t} + \left(\frac{v^2}{\rho}\right)\boldsymbol{n}, \end{aligned} \tag{2.5}$$

where $\boldsymbol{n}$ is the unit normal vector to the path of P and ρ $(= \kappa^{-1})$ is its radius of curvature. Hence, *the acceleration vector* $\boldsymbol{a}$ *has a component* dv/dt *tangential to the path and a component* v^2/ρ *normal to the path*.

This formula is surprising. Since each small segment of the path is 'approximately straight' one might be tempted to conclude that only the first term $(dv/dt)\boldsymbol{t}$ should be present. However, what we have shown is that the acceleration vector of P does not generally point along the path but has a component *perpendicular* to the local path direction. The full meaning of formula (2.5) will become clear when we have treated particle motion in polar coordinates.

Uniform circular motion

The simplest example of non-rectilinear motion is motion in a circle. Circular motion is important in practical applications such as rotating machinery. Here we consider the special case of *uniform* circular motion, that is, circular motion with *constant speed*.

Consider a particle P moving with *constant speed* u in the anti-clockwise direction around a circle centre O and radius b, as shown in Figure 2.3. At time $t = 0$, P is at the point $B(b, 0)$. What are its velocity and acceleration vectors at time t?

The first step is to find the position vector of P at time t. Since P moves with constant speed u, the *arc* length BP travelled in time t must be ut. It follows that the angle θ shown in Figure 2.3 is given by $\theta = ut/b$. The position vector of P at time t is therefore

$$\begin{aligned} \boldsymbol{r} &= b\cos\theta\,\boldsymbol{i} + b\sin\theta\,\boldsymbol{j}, \\ &= b\cos(ut/b)\,\boldsymbol{i} + b\sin(ut/b)\,\boldsymbol{j}. \end{aligned}$$

* As in the rectilinear case, *speed* means the rate of increase of the total distance travelled, which, in the present context, is ds/dt, the rate of increase of arc length along the path of P.

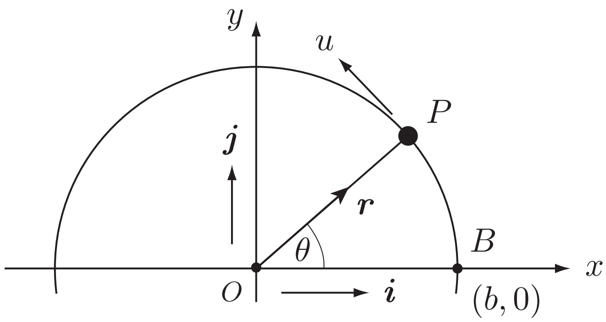

FIGURE 2.3 Particle P moves with constant speed u around a circle of radius b.

It follows that the **velocity** and **acceleration** of P at time t are given by

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = -u\sin(ut/b)\,\boldsymbol{i} + u\cos(ut/b)\,\boldsymbol{j},$$
$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -\frac{u^2}{b}\cos(ut/b)\,\boldsymbol{i} - \frac{u^2}{b}\sin(ut/b)\,\boldsymbol{j}.$$

We note that the speed of P, calculated from $\boldsymbol{v}$, is

$$|\boldsymbol{v}| = \left(u^2\cos^2(ut/b) + u^2\sin^2(ut/b)\right)^{1/2} = u,$$

which is what it was specified to be.

The *magnitude* of the acceleration $\boldsymbol{a}$ is given by

$$|\boldsymbol{a}| = \left(\left(\frac{u^2}{b}\right)^2\cos^2(ut/b) + \left(\frac{u^2}{b}\right)^2\sin^2(ut/b)\right)^{1/2} = \frac{u^2}{b}$$

and, since $\boldsymbol{a} = -(u^2/b^2)\boldsymbol{r}$, the *direction* of $\boldsymbol{a}$ is opposite to that of $\boldsymbol{r}$. This proves the following important result:

Uniform circular motion

When a particle P moves with constant speed u around a fixed circle with centre O and radius b, its acceleration vector is in the direction $\overrightarrow{PO}$ and has constant magnitude u^2/b.

This result is consistent with the general formula (2.5). In this special case, we have $v = u$ and $\rho = b$ so that $dv/dt = 0$ and $\boldsymbol{a} = (u^2/b)\boldsymbol{n}$.

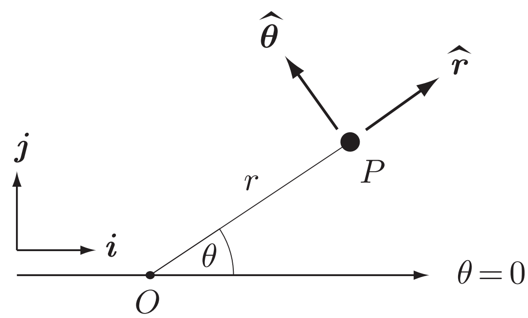

FIGURE 2.4 The plane polar co-ordinates r, θ of the point P and the polar unit vectors $\widehat{\boldsymbol{r}}$ and $\widehat{\boldsymbol{\theta}}$ at P.

Example 2.5 ***Uniform circular motion***

A body is being whirled round at $10\ \mathrm{m\,s^{-1}}$ on the end of a rope. If the body moves on a circular path of 2 m radius, find the magnitude and direction of its acceleration.

Solution

The acceleration is directed towards the centre of the circle and its magnitude is $10^2/2 = 50\ \mathrm{m\,s^{-1}}$, five times the acceleration due to Earth's gravity! ■

2.3 PARTICLE MOTION IN POLAR CO-ORDINATES

When a particle is moving in a plane, it is sometimes very convenient to use polar co-ordinates r, θ in the analysis of its motion; the case of circular motion is an obvious example. Less obviously, polar co-ordinates are used in the analysis of the orbits of the planets. This famous problem stimulated Newton to devise his laws of mechanics.

Figure 2.4 shows the polar co-ordinates r, θ of a point P and the **polar unit vectors** $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ at P. The directions of the vectors $\widehat{\boldsymbol{r}}$ and $\widehat{\boldsymbol{\theta}}$ are called the **radial** and **transverse** directions respectively at the point P. As P moves around, the polar unit vectors do not remain constant. They have constant magnitude (unity) but their directions depend on the θ co-ordinate of P; they are however independent of the r co-ordinate.* In other words, $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ are *vector functions of the scalar variable* θ.

We will now evaluate the two derivatives $d\widehat{\boldsymbol{r}}/d\theta$, $d\widehat{\boldsymbol{\theta}}/d\theta$. These will be needed when we derive the formulae for the velocity and acceleration of P in polar co-ordinates. First we expand† $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ in terms of the Cartesian basis vectors $\{\boldsymbol{i},\ \boldsymbol{j}\}$. This gives

$$\widehat{\boldsymbol{r}} = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}, \tag{2.6}$$

$$\widehat{\boldsymbol{\theta}} = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}. \tag{2.7}$$

Since $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ are now expressed in terms of the *constant* vectors $\boldsymbol{i}$, $\boldsymbol{j}$, the differentiations with respect to θ are simple and give

* If this is not clear, sketch the directions of the polar unit vectors for P in a few different positions.

† Recall that *any* vector $\boldsymbol{V}$ lying in the plane of $\boldsymbol{i}$, $\boldsymbol{j}$ can be expanded in the form $\boldsymbol{V} = \alpha\,\boldsymbol{i} + \beta\,\boldsymbol{j}$, where the coefficients α, β are the components of $\boldsymbol{V}$ in the $\boldsymbol{i}$- and $\boldsymbol{j}$-directions respectively.

$$\boxed{\frac{d\widehat{\boldsymbol{r}}}{d\theta} = \widehat{\boldsymbol{\theta}} \qquad \frac{d\widehat{\boldsymbol{\theta}}}{d\theta} = -\widehat{\boldsymbol{r}}} \tag{2.8}$$

Suppose now that P is a moving particle with polar co-ordinates r, θ that are functions of the time t. The position vector of P relative to O has magnitude $OP = r$ and direction $\widehat{\boldsymbol{r}}$ and can therefore be written

$$\boldsymbol{r} = r\,\widehat{\boldsymbol{r}}. \tag{2.9}$$

In what follows, one must distinguish carefully between the position vector $\boldsymbol{r}$, which is the *vector* $\overrightarrow{OP}$, the co-ordinate r, which is the *distance* OP, and the polar unit vector $\widehat{\boldsymbol{r}}$.

To obtain the polar formula for the velocity of P, we differentiate formula (2.9) with respect to t. This gives

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}\left(r\,\widehat{\boldsymbol{r}}\right) = \left(\frac{dr}{dt}\right)\widehat{\boldsymbol{r}} + r\left(\frac{d\widehat{\boldsymbol{r}}}{dt}\right) \tag{2.10}$$

$$= \dot{r}\,\widehat{\boldsymbol{r}} + r\left(\frac{d\widehat{\boldsymbol{r}}}{dt}\right) \tag{2.11}$$

We will use the **dot notation** for time derivatives throughout this section; $\dot{r}$ means dr/dt, $\dot{\theta}$ means $d\theta/dt$, $\ddot{r}$ means d^2r/dt^2 and $\ddot{\theta}$ means $d^2\theta/dt^2$.

Now $\widehat{\boldsymbol{r}}$ is a function of θ which is, in its turn, a function of t. Hence, by the chain rule and formula (2.8),

$$\frac{d\widehat{\boldsymbol{r}}}{dt} = \frac{d\widehat{\boldsymbol{r}}}{d\theta} \times \frac{d\theta}{dt} = \widehat{\boldsymbol{\theta}} \times \dot{\theta} = \dot{\theta}\,\widehat{\boldsymbol{\theta}}.$$

If we now substitute this formula into equation (2.11) we obtain

$$\boldsymbol{v} = \dot{r}\,\widehat{\boldsymbol{r}} + \left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}}, \tag{2.12}$$

which is the polar formula for the **velocity** of P.

To obtain the polar formula for acceleration, we differentiate the velocity formula (2.12) with respect to t. This gives*

$$\begin{aligned}
\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} &= \frac{d}{dt}\left(\dot{r}\,\widehat{\boldsymbol{r}}\right) + \frac{d}{dt}\left(\left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}}\right)\\
&= \ddot{r}\,\widehat{\boldsymbol{r}} + \dot{r}\,\frac{d\widehat{\boldsymbol{r}}}{dt} + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\widehat{\boldsymbol{\theta}} + \left(r\dot{\theta}\right)\frac{d\widehat{\boldsymbol{\theta}}}{dt}\\
&= \ddot{r}\,\widehat{\boldsymbol{r}} + \dot{r}\left(\frac{d\widehat{\boldsymbol{r}}}{d\theta} \times \frac{d\theta}{dt}\right) + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\widehat{\boldsymbol{\theta}} + \left(r\dot{\theta}\right)\left(\frac{d\widehat{\boldsymbol{\theta}}}{d\theta} \times \frac{d\theta}{dt}\right)\\
&= \ddot{r}\,\widehat{\boldsymbol{r}} + \left(\dot{r}\dot{\theta}\right)\widehat{\boldsymbol{\theta}} + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\widehat{\boldsymbol{\theta}} - \left(r\dot{\theta}^2\right)\widehat{\boldsymbol{r}}\\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\widehat{\boldsymbol{\theta}},
\end{aligned}$$

* Be a hero. Obtain this formula yourself without looking at the text.

which is the polar formula for the **acceleration** of P. These results are summarised below:

Polar formulae for velocity and acceleration

If a particle is moving in a plane and has polar coordinates r, θ at time t, then its velocity and acceleration vectors are given by

$$\boldsymbol{v} = \dot{r}\,\widehat{\boldsymbol{r}} + \left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}}, \tag{2.13}$$

$$\boldsymbol{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\widehat{\boldsymbol{\theta}}. \tag{2.14}$$

The formula (2.13) shows that the velocity of P is the vector sum of an outward radial velocity $\dot{r}$ and a transverse velocity $r\dot{\theta}$; in other words $\boldsymbol{v}$ is just the sum of the velocities that P *would* have if r and θ varied separately. This is *not* true for the acceleration as it will be observed that adding together the separate accelerations would not yield the term $2\dot{r}\dot{\theta}\widehat{\boldsymbol{\theta}}$. This 'Coriolis term' is certainly present however, but is difficult to interpret intuitively.

Example 2.6 *Velocity and acceleration in polar coordinates*

A particle sliding along a radial groove in a rotating turntable has polar coordinates at time t given by

$$r = ct \qquad \theta = \Omega t,$$

where c and Ω are positive constants. Find the velocity and acceleration vectors of the particle at time t and find the speed of the particle at time t.

Deduce that, for $t > 0$, the angle between the velocity and acceleration vectors is always acute.

Solution

From the polar formulae (2.13), (2.14) for velocity and acceleration, we obtain

$$\boldsymbol{v} = c\,\widehat{\boldsymbol{r}} + (ct)\Omega\widehat{\boldsymbol{\theta}} = c\left(\widehat{\boldsymbol{r}} + \Omega t\,\widehat{\boldsymbol{\theta}}\right)$$

and

$$\boldsymbol{a} = \left(0 - (ct)\Omega^2\right)\widehat{\boldsymbol{r}} + (0 + 2c\Omega)\,\widehat{\boldsymbol{\theta}} = c\Omega\left(-\Omega t\,\widehat{\boldsymbol{r}} + 2\widehat{\boldsymbol{\theta}}\right).$$

The *speed* of the particle at time t is thus given by $|\boldsymbol{v}| = c\left(1 + \Omega^2 t^2\right)^{1/2}$.

To find the angle between $\boldsymbol{v}$ and $\boldsymbol{a}$, consider

$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{a} &= c^2\Omega(-\Omega t + 2\Omega t) = c^2\Omega^2 t \\ &> 0\end{aligned}$$

for $t > 0$. Hence, for $t > 0$, the angle between $\boldsymbol{v}$ and $\boldsymbol{a}$ is acute. ■

General circular motion

An important application of polar coordinates is to circular motion. We have already considered the special case of *uniform* circular motion, but now we suppose that P moves in any manner (not necessarily with constant speed) around a circle with centre O and radius b. If we take O to be the origin of polar coordinates, the condition $r = b$ implies that $\dot{r} = \ddot{r} = 0$ and the formula (2.13) for the **velocity** of P reduces to

$$\boldsymbol{v} = \big(b\dot{\theta}\big)\,\widehat{\boldsymbol{\theta}}. \tag{2.15}$$

This result is depicted in Figure 2.5. The transverse velocity component $b\dot{\theta}$ (which is not necessarily the speed of P since $\dot{\theta}$ may be negative) is called the **circumferential velocity** of P. Circumferential velocity will be important when we study the motion of a rigid body rotating about a fixed axis; in this case, each particle of the rigid body moves on a circular path.

The corresponding formula for the acceleration of P is

$$\begin{aligned}
\boldsymbol{a} &= \Big(0 - b\dot{\theta}^2\Big)\,\widehat{\boldsymbol{r}} + + \big((b\ddot{\theta} + 0\big)\,\widehat{\boldsymbol{\theta}} \\
&= -\Big(b\dot{\theta}^2\Big)\,\widehat{\boldsymbol{r}} + + \big((b\ddot{\theta}\big)\,\widehat{\boldsymbol{\theta}} \\
&= -\left(\frac{v^2}{b}\right)\widehat{\boldsymbol{r}} + + \dot{v}\,\widehat{\boldsymbol{\theta}}
\end{aligned}$$

where v is the circumferential velocity $b\dot{\theta}$. These results are summarised below:

General circular motion

Suppose a particle P moves in any manner around the circle $r = b$, where r, θ are plane polar coordinates. Then the velocity and acceleration vectors of P are given by

$$\boldsymbol{v} = v\,\widehat{\boldsymbol{\theta}}, \tag{2.16}$$

$$\boldsymbol{a} = -\left(\frac{v^2}{b}\right)\widehat{\boldsymbol{r}} + \dot{v}\,\widehat{\boldsymbol{\theta}}, \tag{2.17}$$

where v $(= b\dot{\theta})$ is the circumferential velocity of P.

The formula (2.17) shows that, in *general* circular motion, the acceleration of P is the (vector) sum of an inward radial acceleration v^2/b and a transverse acceleration $\dot{v}$. This is consistent with the general formula (2.5). Indeed, what the formula (2.5) says is that, when P moves along a completely general path, its acceleration vector is the same *as if* it were moving on the circle of curvature at each point of its path.

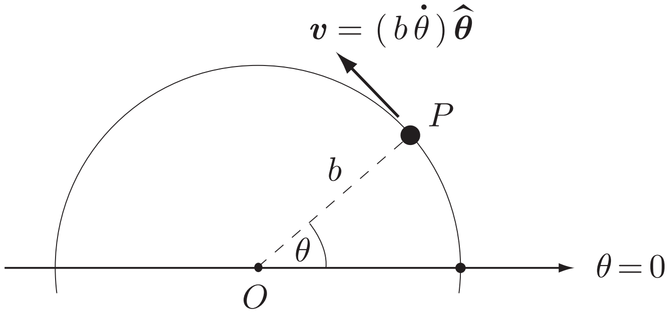

FIGURE 2.5 The particle P moves on the circle with centre O and radius b. At time t its angular displacement is θ and its circumferential velocity is $b\dot{\theta}$.

Example 2.7 *Pendulum motion*

The bob of a certain pendulum moves on a vertical circle of radius b and, when the string makes an angle θ with the downward vertical, the circumferential velocity v of the bob is given by

$$v^2 = 2gb\cos\theta,$$

where g is a positive constant. Find the acceleration of the bob when the string makes angle θ with the downward vertical.

Solution

From the acceleration formula (2.17), we have

$$\boldsymbol{a} = -\left(\frac{v^2}{b}\right)\widehat{\boldsymbol{r}} + \dot{v}\,\widehat{\boldsymbol{\theta}} = -\,(2g\cos\theta)\,\widehat{\boldsymbol{r}} + \dot{v}\,\widehat{\boldsymbol{\theta}}.$$

It remains to express $\dot{v}$ in terms of θ. On differentiating the formula $v^2 = 2gb\cos\theta$ with respect to t, we obtain

$$2v\dot{v} = -\,(2gb\sin\theta)\,\dot{\theta},$$

and, since $b\dot{\theta} = v$, we find that

$$\dot{v} = -g\sin\theta.$$

Hence the **acceleration** of the bob when the string makes angle θ with the downward vertical is

$$\boldsymbol{a} = -\,(2g\cos\theta)\,\widehat{\boldsymbol{r}} - (g\sin\theta)\,\widehat{\boldsymbol{\theta}}. \blacksquare$$

2.4 RIGID BODY ROTATING ABOUT A FIXED AXIS

Some objects that we find in everyday life, such as a brick or a thick steel rod, are so difficult to deform that their shape is virtually unchangeable. We model such an

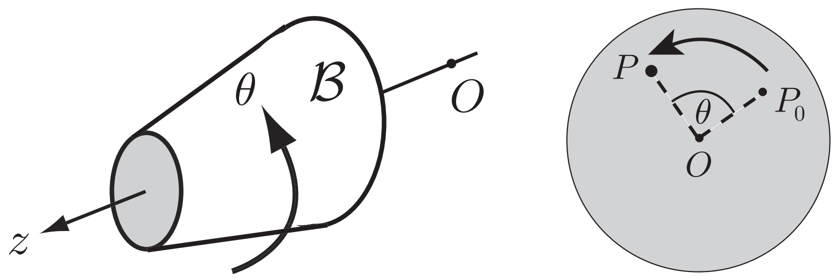

FIGURE 2.6 The rigid body $\mathcal{B}$ rotates about the fixed axis Oz and has angular displacement θ at time t. Each particle P of $\mathcal{S}$ moves on a circular path; the point P_0 is the reference position of P.

object by a **rigid body**, a collection of particles forming a perfectly rigid framework. Any motion of the rigid body must maintain this framework.

An important type of rigid body motion is **rotation about a fixed axis**; a spinning fan, a door opening on its hinges and a playground roundabout are among the many examples of this type of motion. Suppose $\mathcal{B}$ is a rigid body which is constrained to rotate about the fixed axis Oz as shown in Figure 2.6. (This means that the particles of $\mathcal{B}$ that lie on Oz are held fixed. Rotation about Oz is then the only motion of $\mathcal{B}$ consistent with rigidity.) At time t, $\mathcal{B}$ has **angular displacement** θ measured from some reference position. The angular displacement θ is the rotational counterpart of the Cartesian displacement x of a particle in straight line motion. By analogy with the rectilinear case, we make the following definitions:

Definition 2.4 ***Angular velocity*** *The* ***angular velocity*** ω *of* $\mathcal{B}$ *is defined to be* $\omega = d\theta/dt$ *and the absolute value of* ω *is called the* ***angular speed*** *of* $\mathcal{B}$.

Units. Angular velocity (and angular speed) are measured in radians per second (rad s^{-1}).

Example 2.8 *Spinning crankshaft 1*

The crankshaft of a motorcycle engine is spinning at 6000 revolutions per minute. What is its angular speed in S.I. units?

Solution

6000 revolutions per minute is 100 revolutions per second which is 200π radians per second. This is the angular speed in S.I. units. ■

Particle velocities in a rotating rigid body

In rotational motion about a fixed axis, each particle P of $\mathcal{B}$ moves on a circle of some radius ρ, where ρ is the (fixed) perpendicular distance of P from the rotation axis. It then follows from (2.16) that the **circumferential velocity** v of P is given by $\rho\dot{\theta}$, that is

$$\boxed{v = \omega\,\rho} \qquad (2.18)$$

Example 2.9 ***Spinning crankshaft 2***

In the crankshaft example above, find the speed of a particle of the crankshaft that has perpendicular distance 5 cm from the rotation axis. Find also the magnitude of its acceleration.

Solution

In this case, $|\omega| = 200\pi$ and $\rho = 1/20$ so that the particle speed (the magnitude of the circumferential velocity v) is $10\pi \approx 31.4\ \mathrm{m\,s^{-1}}$.

Since the circumferential velocity is constant, $|\boldsymbol{a}| = v^2/\rho = (10\pi)^2/0.05 \approx 2000\ \mathrm{m\,s^{-2}}$, which is two hundred times the value of the Earth's gravitational acceleration! ■

2.5 RIGID BODY IN PLANAR MOTION

We now consider a more general form of rigid body motion called **planar motion**.

Definition 2.5 ***Planar motion*** *A rigid body $\mathcal{B}$ is said to be in* ***planar motion*** *if each particle of $\mathcal{B}$ moves in a fixed plane and all these planes are parallel to each other.*

Planar motion is quite common. For instance, any flat-bottomed rigid body sliding on a flat table is in planar motion. Another example is a circular cylinder rolling on a rough flat table.

The **particle velocities** in planar motion can be calculated by the following method; the proof is given in Chapter 16. First select some particle C of the body as the reference particle. The velocity of a general particle P of the body is then the vector sum of

(i) a **translational contribution** equal to the velocity of C (as if the body did not rotate) and
(ii) a **rotational contribution** (as if C were fixed and the body were rotating with angular velocity ω about a fixed axis through C).

This result is illustrated in Figure 2.7, where the body is a rectangular plate and the reference particle C is at a corner of the plate. The velocity $\boldsymbol{v}$ of P is given by $\boldsymbol{v} = \boldsymbol{v}^C + \boldsymbol{v}^R$, where the translational contribution $\boldsymbol{v}^C$ is the velocity of C and the rotational contribution $\boldsymbol{v}^R$ is caused by the angular velocity ω about C. Although the reference particle can be any particle of the body, it is usually taken to be the centre of mass or centre of symmetry of the body.

Example 2.10 ***The rolling wheel***

A circular wheel of radius b rolls in a straight line with speed u on a fixed horizontal table. Find the velocities of its particles.

Solution

This is an instance of planar motion and so the particle velocities can be found by the method above. Let the position of the wheel at some instant be that shown in

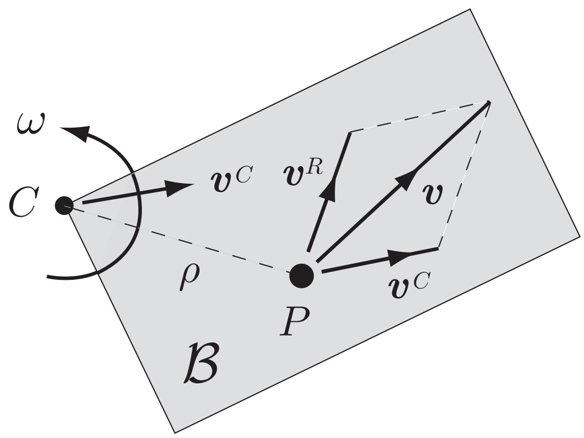

FIGURE 2.7 The velocity of the particle P belonging to the rigid body $\mathcal{B}$ is the sum of the translational contribution $\boldsymbol{v}^C$ and the rotational contribution $\boldsymbol{v}^R$. The reference particle C can be *any* particle of the body.

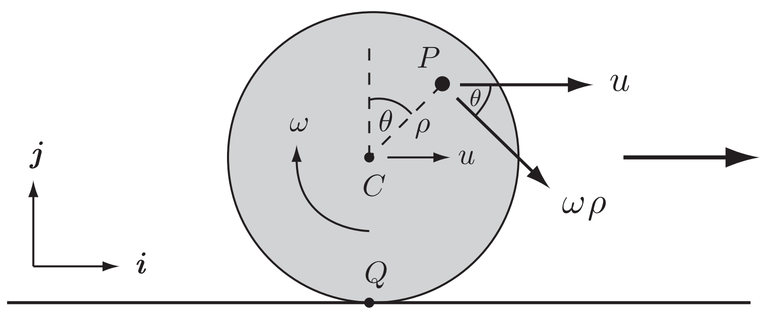

FIGURE 2.8 The circular wheel rolls from left to right on a fixed horizontal table. The reference particle C is taken to be the centre of the wheel and the velocity of a typical particle P is the sum of the two velocities shown.

Figure 2.8. The reference particle C is taken to be the centre of the wheel, and the wheel is supposed to have some angular velocity ω about C. The velocity $\boldsymbol{v}^P$ of a typical particle P is then the sum of the two velocities shown. In terms of the vectors $\{\boldsymbol{i}, \boldsymbol{j}\}$

$$\begin{aligned}\boldsymbol{v}^P &= u\,\boldsymbol{i} + \omega\rho\,(\cos\theta\,\boldsymbol{i} - \sin\theta\,\boldsymbol{j})\\ &= (u + \omega\rho\cos\theta)\,\boldsymbol{i} - (\omega\rho\sin\theta)\,\boldsymbol{j}.\end{aligned} \tag{2.19}$$

In particular, on taking $\rho = b$ and $\theta = \pi$, the velocity $\boldsymbol{v}^Q$ of the contact particle Q is given by

$$\boldsymbol{v}^Q = (u - \omega b)\,\boldsymbol{i}. \tag{2.20}$$

If the wheel is allowed to slip as it moves across the table, there is no restriction on $\boldsymbol{v}^Q$ so that u and ω are unrelated. But rolling, by definition, requires that

$$\boldsymbol{v}^Q = \boldsymbol{0}. \tag{2.21}$$

On applying this **rolling condition** to our formula (2.20) for $\boldsymbol{v}^Q$, we find that ω must be related to u by

$$\omega = \frac{u}{b}, \tag{2.22}$$

and on using this value of ω in (2.19) we find that the **velocity** of the typical particle P is given by

$$\boldsymbol{v}^P = u\left(1 + \frac{\rho}{b}\cos\theta\right)\boldsymbol{i} - u\left(\frac{\rho}{b}\sin\theta\right)\boldsymbol{j}. \tag{2.23}$$

When P lies on the circumference of the wheel, this formula simplifies to

$$\boldsymbol{v}^P = u\,(1 + \cos\theta)\,\boldsymbol{i} - u\sin\theta\,\boldsymbol{j}, \tag{2.24}$$

in which case the *speed* of P is given by

$$|\,\boldsymbol{v}^P\,| = 2u\cos(\theta/2), \qquad\qquad (-\pi \le \theta \le \pi).$$

Thus the highest particle of the wheel has the largest speed, $2u$, while the contact particle has speed zero, as we already know. ■

2.6 REFERENCE FRAMES IN RELATIVE MOTION

A reference frame is simply a rigid coordinate system that can be used to specify the positions of points in space. In practice it is convenient to regard a reference frame as being embedded in, or attached to, some rigid body. The most familiar case is that in which the rigid body is the Earth but it could instead be a moving car, or an orbiting space station. In principle, any event, the motion of an aircraft for example, can be observed from any of these reference frames and the motion will appear different to each observer. It is this difference that we now investigate.

Let the motion of a particle P be observed from the reference frames $\mathcal{F}\,\{\mathcal{O}\,;\,\boldsymbol{i},\,\boldsymbol{j},\,\boldsymbol{k}\}$ and $\mathcal{F}'\,\{O'\,;\,\boldsymbol{i},\,\boldsymbol{j},\,\boldsymbol{k}\}$ as shown in Figure 2.9. Here we are supposing that the frame $\mathcal{F}'$ *does not rotate* relative to $\mathcal{F}$. This is why, without losing generality, we can suppose that $\mathcal{F}$ and $\mathcal{F}'$ have the same set of unit vectors $\{\boldsymbol{i},\,\boldsymbol{j},\,\boldsymbol{k}\}$. For example, P could be an aircraft, $\mathcal{F}$ could be attached to the Earth, and $\mathcal{F}'$ could be attached to a car driving along a *straight* road.

Then, $\boldsymbol{r}$, $\boldsymbol{r}'$, the position vectors of P relative to $\mathcal{F}$, $\mathcal{F}'$ are connected by

$$\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{D}, \tag{2.25}$$

where $\boldsymbol{D}$ is the position vector of O' relative to $\mathcal{F}$.

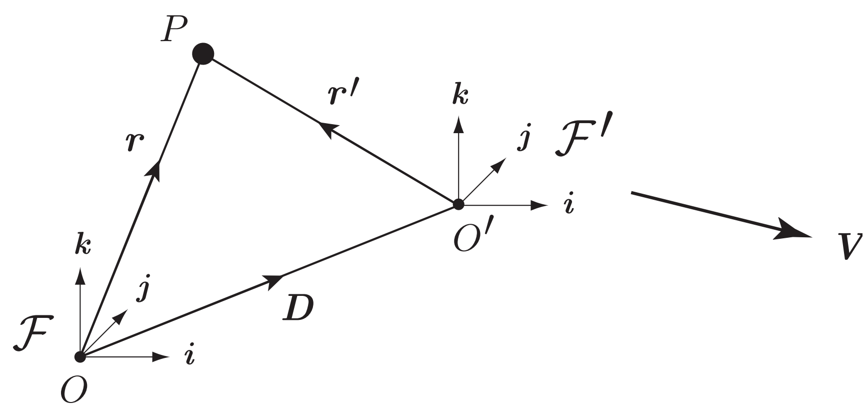

FIGURE 2.9 The particle P is observed from the two reference frames $\mathcal{F}$ and $\mathcal{F}'$.

We now differentiate this equation with respect to t, a step that requires some care. Let us consider the rates of change of the vectors in equation (2.25), as observed from the frame $\mathcal{F}$. Then

$$\boldsymbol{v} = \left(\frac{d\boldsymbol{r}'}{dt}\right)_{\mathcal{F}} + \boldsymbol{V}, \tag{2.26}$$

where $\boldsymbol{v}$ is the velocity of P observed in $\mathcal{F}$ and $\boldsymbol{V}$ is the velocity of $\mathcal{F}'$ relative to $\mathcal{F}$.

Now when two different reference frames are used to observe the same vector, the observed *rates of change* of that vector will generally be different. In particular, it is *not* generally true that

$$\left(\frac{d\boldsymbol{r}'}{dt}\right)_{\mathcal{F}} = \left(\frac{d\boldsymbol{r}'}{dt}\right)_{\mathcal{F}'}.$$

However, as we will show in Chapter 17, these two rates of change *are* equal if the frame $\mathcal{F}$ *does not rotate* relative to $\mathcal{F}$. Hence, in our case, we do have

$$\left(\frac{d\boldsymbol{r}'}{dt}\right)_{\mathcal{F}} = \left(\frac{d\boldsymbol{r}'}{dt}\right)_{\mathcal{F}'} = \boldsymbol{v}',$$

where $\boldsymbol{v}'$ is the velocity of P observed in $\mathcal{F}'$.

Equation (2.26) can then be written

$$\boxed{\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{V}} \tag{2.27}$$

Thus *the velocity of P observed in $\mathcal{F}$ is the sum of the velocity of P observed in $\mathcal{F}'$ and the velocity of the frame $\mathcal{F}'$ relative to $\mathcal{F}$.* This result applies only when $\mathcal{F}'$ *does not rotate* relative to $\mathcal{F}$.