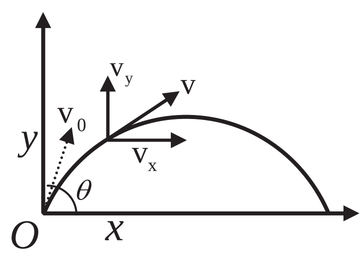
Summary of Lecture 13 – ANGULAR MOMENTUM

1. Recall the definition of angular momentum: $\vec{L} = \vec{r} \times \vec{p}$. The magnitude can be written in several different but equivalent ways,
   (a) $L = rp\sin\theta$
   (b) $L = (r\sin\theta)\,p = r_\perp p$
   (c) $L = r(p\sin\theta) = r\,p_\perp$

2. Let us use this definition to calculate the angular momentum of a projectile thrown from the ground at an angle θ. Obviously, initial angular momentum is zero (why?).



We know what the projectile's coordinates will be at time t after launch,

$$x = (\mathrm{v}_0\cos\theta)t, \quad y = (\mathrm{v}_0\sin\theta)t - \frac{1}{2}gt^2$$

as well as the velocity components,

$$\mathrm{v}_x = \mathrm{v}_0\cos\theta,\ \mathrm{v}_y = \mathrm{v}_0\sin\theta - gt.$$

Hence, $\vec{L} = \vec{r}\times\vec{p} = (x\hat{i} + y\hat{j})\times(\mathrm{v}_x\hat{i} + \mathrm{v}_y\hat{j})m = m(x\mathrm{v}_x - y\mathrm{v}_y)\mathrm{v}_x\hat{k}$

$$= m\left(\frac{1}{2}gt^2\mathrm{v}_0\cos\theta - gt^2\mathrm{v}_0\cos\theta\right)\hat{k} = -\frac{m}{2}gt^2\mathrm{v}_0\cos\theta\,\hat{k}.$$

In the above, $\hat{k} = \hat{i}\times\hat{j}$ is a unit vector perpendicular to the paper. You can see here that the angular momentum increases as t^2.

2. Momentum changes because a force makes it change. What makes angular momentum change? Answer: torque. Here is the definition again: $\vec{\tau} = \vec{r}\times\vec{F}$. Now let us establish a very important relation between torque and rate of change of L.
   Begin:
   $\vec{L} = \vec{r}\times\vec{p}$. At a slightly later time, $\vec{L} + \Delta\vec{L} = (\vec{r} + \Delta\vec{r})\times(\vec{p} + \Delta\vec{p})$

$$= \vec{r}\times\vec{p} + \vec{r}\times\Delta\vec{p} + \Delta\vec{r}\times\vec{p} + \Delta\vec{r}\times\Delta\vec{p}$$

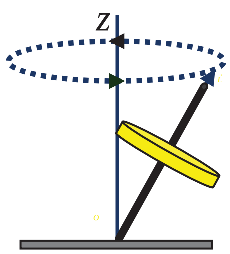
   By subtracting, $\Delta\vec{L} = \vec{r}\times\Delta\vec{p} + \Delta\vec{r}\times\vec{p}\ \dfrac{\Delta\vec{L}}{\Delta t} = \dfrac{\vec{r}\times\Delta\vec{p} + \Delta\vec{r}\times\vec{p}}{\Delta t} = \vec{r}\times\dfrac{\Delta\vec{p}}{\Delta t} + \dfrac{\Delta\vec{r}}{\Delta t}\times\vec{p}.$

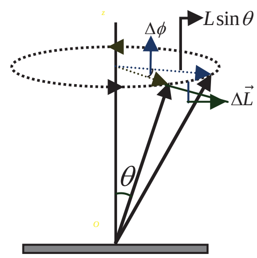
Now divide by the time difference and then take limit as $\Delta t \to 0$:

$$\because \quad \lim_{\Delta t \to 0} \frac{\Delta \vec{L}}{\Delta t} = \frac{d\vec{L}}{dt} \quad \therefore \quad \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

But $\frac{d\vec{r}}{dt}$ is $\vec{v}$ and $\vec{p} = m\vec{v}$! Also, $\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = m(\vec{v} \times \vec{v}) = 0$. So we arrive at $\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$. Now use Newton's Law, $\vec{F} = \frac{d\vec{p}}{dt}$. Hence we get the fundamental equation $\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$, or $\frac{d\vec{L}}{dt} = \vec{\tau}$. So, just as a particle's momentum changes with time because of a force, a particle's *angular* momentum changes with time because of a torque.

3. As you saw in the lecture, the spinning top is an excelllent application of $\frac{d\vec{L}}{dt} = \vec{\tau}$.





Start from $\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{F} = m\vec{g} \therefore \tau = Mgr\sin\theta$. But $\vec{\tau}$ is perpendicular to $\vec{L}$ and so it cannot change the magnitude of $\vec{L}$. Only the direction changes. Since $\Delta\vec{L} = \vec{\tau}\ \Delta t$, you can see from the diagram that $\Delta\phi = \frac{\Delta L}{L\sin\theta} = \frac{\tau\ \Delta t}{L\sin\theta}$. So the precession speed ω_P is: $\omega_P = \frac{\Delta\phi}{\Delta t} = \frac{\tau}{L\sin\theta} = \frac{Mgr\sin\theta}{L\sin\theta} = \frac{Mgr}{L}$. As the top slows down due to friction and L decreases, the top precesses faster and faster.

4. Now consider the case of many particles. Choose any origin with particles moving with respect to it. We want to write down the total angular momentum,

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_N = \sum_{n=1}^{N} \vec{L}_n$$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}_1}{dt} + \frac{d\vec{L}_2}{dt} + \cdots + \frac{d\vec{L}_N}{dt} = \sum_{n=1}^{N} \frac{d\vec{L}_n}{dt}$$

Since $\frac{d\vec{L}_n}{dt} = \vec{\tau}_n$, it follows that $\frac{d\vec{L}}{dt} = \sum_{n=1}^{N} \vec{\tau}_n$. Thus the time rate of change of the total angular momentum of a system of particles equals the net torque acting on the system. I showed earlier that internal forces cancel. So also do internal torques, as we shall see.

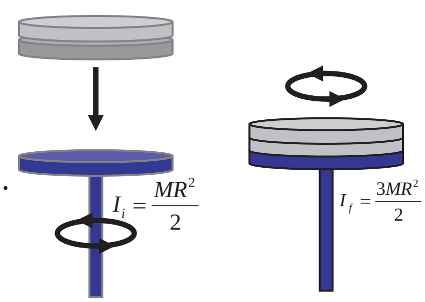
5. The torque on a system of particles can come both from external and internal forces. For example, there could be charged particles which attract/repel each other while they are all in an external gravitational field. Mathematically, $\sum \vec{\tau} = \sum \vec{\tau}_{\text{int}} + \sum \vec{\tau}_{ext}$. Now, if the forces between two particles not only are equal and opposite but are also directed along the line joining the two particles, then can easily show that the total internal torque, $\sum \vec{\tau}_{\text{int}} = 0$. Take the case of two particles,

$$\sum \vec{\tau}_{\text{int}} = \vec{\tau}_1 + \vec{\tau}_2 = \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$

But $\vec{F}_{12} = -\vec{F}_{21} = F\hat{r}_{12}, \quad \therefore \sum \vec{\tau}_{\text{int}} = \left(\vec{r}_1 - \vec{r}_2\right) \times \vec{F}_{12} = \vec{r}_{12} \times \left(F\hat{r}_{12}\right) = F\left(\vec{r}_{12} \times \hat{r}_{12}\right) = 0$

Thus net external torque acting on a system of particles is equal to the time rate of change of the of the total angular momentum of the system.

6. It follows from $\dfrac{d\vec{L}}{dt} = \vec{\tau}$ that if no net external torque acts on the system, then the angular momentum of the system does not change with the time: $\dfrac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \vec{L} =$ a constant. This is simple but extremely important. Let us apply this to the system shown here. Two stationary discs, each with $I = \dfrac{1}{2}MR^2$, fall on top of a rotating disc. The total angular momentum is unchanged so, $I_i\omega_i = I_f\omega_f \;\; \Rightarrow \;\; \omega_f = \omega_i\left(\dfrac{I_i}{I_f}\right)$.



Hence, $\omega_f = \omega_i\left(\dfrac{MR^2}{2} \times \dfrac{2}{3MR^2}\right) = \dfrac{1}{3}\omega_i$.

7. You shoud be aware of the similarities and differences between the equations for linear and rotational motion: $\vec{F} = \dfrac{d\vec{p}}{dt} \Leftrightarrow \vec{\tau} = \dfrac{d\vec{L}}{dt}$. One big difference is that for momentum, $\vec{p} = m\vec{\text{v}}$, it does not matter where you pick the origin. But $\vec{L}$ definitely depends on the choice of the origin. So changing $\vec{r}$ to $\vec{c} + \vec{r}$ changes $\vec{L}$ to $\vec{L}'$:

$$\vec{L}' = \vec{r}' \times \vec{p} = \left(\vec{c} + \vec{r}\right) \times \vec{p} = \vec{c} \times \vec{p} + \vec{L}.$$

8. Linear and angular acceleration:

$$\vec{\text{a}} = \frac{d\vec{\text{v}}}{dt} = \frac{d}{dt}\left(\vec{\omega} \times \vec{r}\right) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$
$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\text{v}}$$

So the acceleration has a tangential and radial part, $\vec{\text{a}} = \vec{\text{a}}_{\text{T}} + \vec{\text{a}}_{\text{R}}$.

QUESTIONS AND EXERCISES – 13

Q1. A car goes along a section of road shaped like the letter S. The speed of the car stays constant. Discuss how the acceleration changes, and make a rough plot.

Q2. All bodies dropped together fall to the ground in the same amount of time. But if two spheres roll down an inclined plane, one may take more time than the other. Explain why.

Q3. A solid cylinder of mass M, radius R, and moment of inertia I rolls without slipping down an inclined plane of length L and height h. Find the speed of its centre of mass when the cylinder reaches the bottom.

Q4. If the radius of the earth, assumed to be a perfect sphere, suddenly shrinks to half its present value, the mass of the Earth remaining unchanged, what will be the duration of one day?

Q5. A uniform solid cylinder of radius R = 12cm and mass M = 3.2kg is given an initial clockwise angular velocity ω_0 of 15rev/s and then lowered on to a flat horizontal surface. The coefficient of kinetic friction between the surface and the cylinder is $\mu = 0.21$. Initially, the cylinder slips as it moves along the surface, but after a time t pure rolling without slipping begins.

(a) What is the velocity v_{cm}?

(b) What is the value of t?

[Hint: find the acceleration, and hence the force. When slipping stops, the frictional force produces acceleration]

Summary of Lecture 14 – EQUILIBRIUM OF RIGID BODIES

1. A rigid body is one where all parts of the body are fixed relative to each other (for example, a pencil). Fluids and gases are non-rigid.

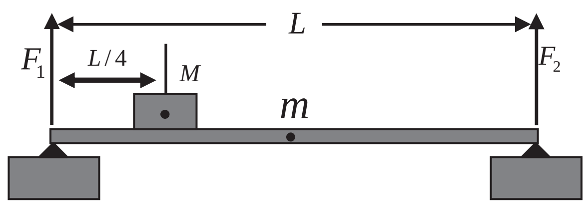
2. The translational motion of the centre of mass of a rigid body is governed by:

$$\frac{d\vec{P}}{dt} = \vec{F} \text{ where } \vec{F} = \sum \vec{F}_{ext} \text{ is the } net \text{ external force.}$$

Similarly, for rotational motion, $\frac{d\vec{L}}{dt} = \vec{\tau}$ where $\vec{\tau} = \sum \vec{\tau}_{ext}$ is the net external torque.

3. A rigid body is in mechanical equilibrium if **both** the linear momentum $\vec{P}$ and angular momentum $\vec{L}$ have a constant value. i.e., $\frac{d\vec{P}}{dt} = 0$ **and** $\frac{d\vec{L}}{dt} = 0$. **Static equilibrium** refers to $\vec{P} = 0$ and $\vec{L} = 0$.

4. As an example of static equilibrium, consider a beam resting on supports:



We want to find the forces F_1 and F_2 with which the supports push on the rod in the upwards direction. First, balance forces in the vertical y direction:

$$\sum F_y = F_1 + F_2 - Mg - mg = 0$$

Now demand that the total torque vanishes:

$$\sum \tau_y = (F_1)(0) + (F_2)(L) - (Mg)(L/4) - (mg)(L/2) = 0$$

From these two conditions you can solve for F_1 and F_2,

$$F_1 = \frac{(3M + 2m)g}{4}, \quad \text{and} \quad F_2 = \frac{(M + 2m)g}{4}.$$

5. Angular momentum and torque depend on where you choose the origin of your coordinates. However, I shall now prove that for a body in equilibrium, the choice of origin does not matter. Let's start with the origin O and calculate the torque about O,

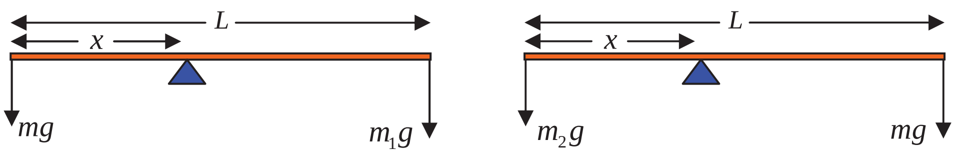
$$\vec{\tau}_O = \vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_N = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N$$

Now, if we take a second point P, then all distances will be measured from P and each vector will be shifted by an amount $\vec{r}_P$. Hence the torque about P is,

$$\begin{aligned}\vec{\tau}_P &= (\vec{r}_1 - \vec{r}_P)\times\vec{F}_1 + (\vec{r}_2 - \vec{r}_P)\times\vec{F}_2 + \cdots + (\vec{r}_N - \vec{r}_P)\times\vec{F}_N \\ &= \left[\vec{r}_1\times\vec{F}_1 + \vec{r}_2\times\vec{F}_2 + \cdots + \vec{r}_N\times\vec{F}_N\right] - \left[\vec{r}_P\times\vec{F}_1 + \vec{r}_P\times\vec{F}_2 + \cdots + \vec{r}_P\times\vec{F}_N\right] \\ &= \vec{\tau}_O - \left[\vec{r}_P\times\left(\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N\right)\right] \\ &= \vec{\tau}_O - \left[\vec{r}_P\times\left(\sum\vec{F}_{ext}\right)\right]\end{aligned}$$

but $\sum\vec{F}_{ext} = 0,$ for a body in translational equilibrium $\therefore\quad \vec{\tau}_P = \vec{\tau}_O.$

6. Let us use the equilibrium conditions to do something of definite practical importance. Consider the balance below which is in equilibrium when two known weights are hung as shown. We want to know m in terms of m_1 and m_2.



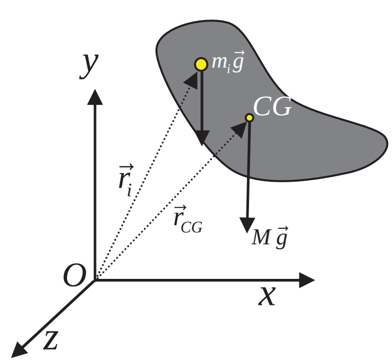
Taking the torques about the knife edge in the two cases, we have:

$$mgx = m_1g(L-x) \text{ and } m_2gx = mg(L-x)$$

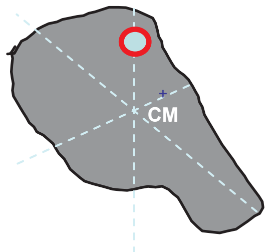
$$\Rightarrow \frac{m}{m_2} = \frac{m_1}{m} \text{ or } m = \sqrt{m_1m_2}.$$

Remarkably, we do not need the values of x or L.

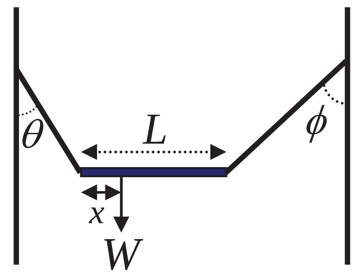
7. **Centre of Gravity.** The centre of gravity is the average location of the weight of an object. This is not quite the same as the centre of mass of a body (see lecture 12) but suppose the gravitational acceleration $\vec{g}$ has the same value at all points of a body. Then: 1) The weight is equal to $M\vec{g}$, and 2)the centre of gravity coincides with the centre of mass. Remember that weight is force, so the CG is really the centre of gravitational force acting on the body. The net force on the whole body = sum of forces over all individual particles, $\sum\vec{F} = \sum m_i\vec{g}$. If $\vec{g}$ has the same value at all points of the body, then $\sum\vec{F} = \vec{g}\sum m_i = M\vec{g}$. So the net torque about the origin O is $\sum\vec{\tau} = \sum(\vec{r}_i\times m_i\vec{g}) = \sum(m_i\vec{r}_i\times\vec{g})$ Hence, $\sum\vec{\tau} = M\vec{r}_{cm}\times\vec{g} = \vec{r}_{cm}\times M\vec{g}$. So the torque due to gravity about the centre of mass of a body (i.e. at $\vec{r}_{cm} = 0$) is zero !!



8. In the demonstration I showed, you saw how to find the CG of an irregular object by simply suspending it on a pivot,



9. Let's solve a problem of static equilibrium: A non-uniform bar of weight W is suspended at rest in a horizontal position by two light cords. Find the distance x from the left-hand end to the center of gravity.



Solution : Call the tensions T_1 and T_2 . Put the forces in both directions equal to zero,

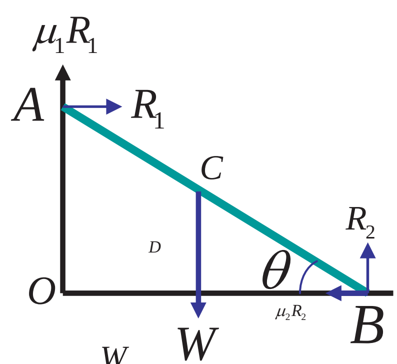
a) $T_2 \sin\phi - T_1 \sin\theta = 0$ (horizontal)

b) $T_2 \cos\phi + T_1 \cos\theta - W = 0$ (vertical) $\Rightarrow T_2 = \dfrac{W}{\sin(\theta+\phi)}$

The torque about any point must vanish. Let us choose that point to be one end of the bar,

$$-Wx + (T_2 \cos\phi)L = 0 \Rightarrow x = \frac{(T_2\cos\phi)L}{W} = \frac{L\cos\phi}{\sin(\theta+\phi)}$$

10.Here is another problem of the same kind: find the least angle θ at which the rod can lean to the horizontal without slipping.



Solution : Considering the translational equilibrium of the rod, $R_1 = \mu_2 R_2$ and $R_2 + \mu_1 R_1 = W$. This gives, $R_2 = \dfrac{W}{(1+\mu_1\mu_2)}$. Now consider rotational equilibrium about the point A: $R_2 \times OB = W \times OD + \mu_2 R_2 \times OA$ or, $R_2 \times AB\cos\theta = W \times \dfrac{AB\cos\theta}{2} + \mu_2 R_2 \times AB\sin\theta.$

This gives $\cos\theta\left(R_2 - \dfrac{W}{2}\right) = \mu_2 R_2 \sin\theta$ from which $\tan\theta = \dfrac{R_2 - \frac{W}{2}}{\mu_2 R_2}$ with $R_2 = \dfrac{W}{(1+\mu_1\mu_2)}$.

Using this value of R_2, we get $\tan\theta = \dfrac{1\text{-}\mu_1\mu_2}{2\mu_2}$.

10. **Types of Equilibrium.** In the lecture you heard about:

a)Stable equilibrium: object returns to its original position if displaced slightly.

b)Unstable equilibrium: object moves farther away from its original position if displaced slightly.

c)Neutral equilibrium: object stays in its new position if displaced slightly.

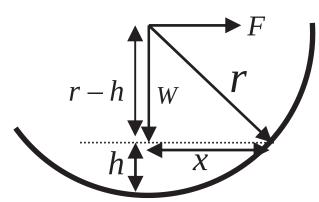
QUESTIONS AND EXERCISES – 14

Q.1 Give three examples of each of the following that are *not* given in either the lecture or these notes:

a) Static equilibrium
b) Dynamic equilibrium
c) Stable equilibrium
d) Neutral equilibrium
e) Unstable equilibrium

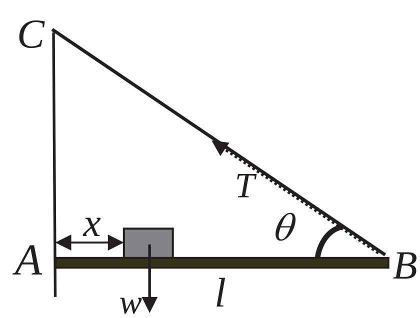
Q.2 Ship designers want to keep the CG of their ships as low as possible. Why? Discuss why there is a contradiction between this requirement, and other requirements.

Q.3 Work through the example given in the lecture where you are asked to find the minimum force F applied horizontally at the axle of the wheel in order to raise it over an obstacle of height *h*.



Q.4 A thin bar AB of negligible weight is pinned to a vertical wall at A and supported by a thin wire BC. A weight *w* can be moved along the bar.

a) Find T as a function of *x*.
b) Find the horizontal and vertical components of the force exerted on the bar by the pin at *A*.



[Hint: Since the system is in rotational equilibrium, the net torque about A is zero.]