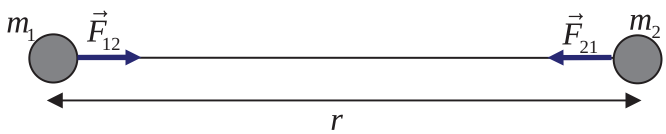
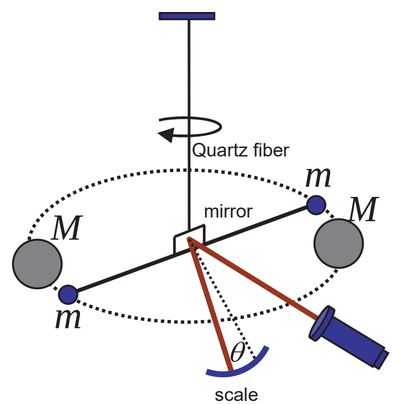
## Summary of Lecture 21 – GRAVITY

1. Newton's law of universal gravitation states that the force of attraction between two masses $m_1$ and $m_2$ is $F \propto \frac{m_1 m_2}{r^2}$ and is directed along the line joining the two bodies. Putting in a constant of proportionality, $F = G\frac{m_1 m_2}{r^2}$. Now let's be a bit careful of the direction of the force. Looking at the diagram below, $\vec{F}_{21}$ = Force on $m_2$ by $m_1$, $\vec{F}_{12}$ = Force on $m_1$ by $m_2$ , $\left|\vec{F}_{12}\right| = \left|\vec{F}_{21}\right| = F$. By Newton's Third Law, $\vec{F}_{12} = -\vec{F}_{21}$.




2. The gravitational constant G is a very small quantity and needs a very sensitive experiment. An early experiment to find $G$ involved suspending two masses and measuring the attractive force. From the figure you can see that the is gravitational torque is $2\left(\frac{GmM}{r^2}\right)\frac{L}{2}$. A thread provides the restoring torque $\kappa\theta$. The deflection $\theta$ can be measured by observing the beam of the light reflected from the small mirror. In equilibrium the torques balance, $\frac{GmML}{r^2} = \kappa\theta$. Hence $G = \frac{\kappa\theta r^2}{GmML}$. How to find $\kappa$? It can be found from observing the period of free oscillations, $T = 2\pi\sqrt{\frac{I}{\kappa}} \Rightarrow \kappa = \frac{4\pi^2 I}{T^2}$ with $I = \frac{mL^2}{2}$. The modern value is $G = 6.67259 \times 10^{-11} N.m^2/kg^2$




3. The magnitude of the force with which the Earth attracts a body of mass m towards its centre is $F = \frac{GmM_E}{R_E^2}$, where $R_E = 6400\,km$ is the radius of the Earth and $M_E$ is the mass. The material does not matter - iron, wood, leather, etc. all feel the force in proportion to their masses. If the body can fall freely, then it will accelerate. So, $F = mg = \frac{GmM_E}{R_E^2}$.

We measure g, the acceleration due to gravity, as $9.8\text{m/s}^2$. From this we can immediately deduce the Earth's mass: $M_E = \frac{gR_E^{\ 2}}{G} = 5.97\times10^{24}kg$. What a remarkable achievement!

We can do still more: the volume of the Earth $= V_{\text{E}} = \frac{4}{3}\pi R_{\text{E}}^{\ 3} = 1.08\times10^{21}m^3$. Hence the]

density of the Earth $= \rho_{\text{E}} = \frac{V_{\text{E}}}{M_{\text{E}}} = 5462\,\text{kg}\,m^{-3}$. So this is 5.462 times greater than the density of water and tells us that the earth must be quite dense inside.

4. The **gravitational potential** is an important quantity. It is the work done in moving a unit mass from infinity to a given point R, and equals $V(r) = -\frac{GM}{R}$.

Proof: Conservation of energy says, $dV = -Fdr \Rightarrow \int_{V(R)}^{0} dV = -\int_{R}^{\infty} drF(r)$

Integrate both sides: $0 - V(R) = GM\int_R^\infty \frac{dr}{r^2} = -GM\left[\frac{1}{r}\right]_R^\infty, \qquad \therefore V(R) = -\frac{GM}{R}$

5. Using the above formula, let us calculate the change in potential energy $\Delta U$ when we raise a body of mass $m$ to a height $h$ above the Earth's surface.

$$\Delta U = GMm\left(\frac{1}{R_E} - \frac{1}{R_E + h}\right) = GMm\left(1 - \frac{1}{1 + h/R_E}\right) = GMm\left(1 - \left(1 + h/R_E\right)^{-1}\right)$$

Now suppose that the distance $h$ is much smaller than the Earth's radius. So, for $h \ll R_E$, $\left(1 + h/R_E\right)^{-1} = 1 - h/R_E$. So we find $\Delta U = GMm\left(1 - \left(1 - h/R_E\right)\right) = m\left(\frac{GM}{R_E}\right)h = mgh.$

6. We can use the expression for potential energy and the law of conservation of energy to find the minimum velocity needed for a body to escape the Earth' gravity. Far away from the Earth, the potential energy is zero, and the smallest value for the kinetic energy is zero. Requiring that $\left(KE + PE\right)_{r=R} = \left(KE + PE\right)_{r=\infty}$ gives $\frac{1}{2}m\text{v}_e^2 - \frac{GMm}{R_E} = 0 + 0$. From this, $\text{v}_{\text{e}} = \sqrt{\frac{2GM}{R_E}} = \sqrt{2gR_E}$. Putting in some numbers we find that for the Earth $\text{v}_{\text{e}} = 11.2\,\text{km/s}$ and for the Sun $\text{v}_{\text{e}} = 618\,\text{km/s}$. For a Black Hole, the escape velocity is so high that nothing can escape, even if it could move with the speed of light! (Nevertheless, Black Holes can be observed because when matter falls into them, a certain kind of radiation is emitted.)

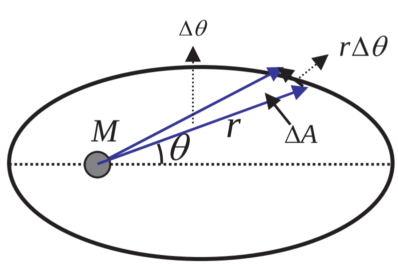
7. **Satellite problems**: A satellite is in circular orbit over the Earth's surface. The condition for equilibrium, $\frac{m\mathrm{v}_o^2}{r}=\frac{GMm}{r^2}\Rightarrow \mathrm{v}_\mathrm{o}=\sqrt{\frac{GM}{r}}$. If $R_E$ is the Earth's radius, and $h$ is the height of the satellite above the ground, then $r = R_E + h$. Hence, $\mathrm{v}_\mathrm{o}=\sqrt{\frac{GM}{R_E+h}}$. For $h << R_E$, we can approximate $\mathrm{v}_\mathrm{o}=\sqrt{\frac{GM}{R_E}}=\sqrt{gR_E}$. We can easily calculate the time for one complete revolution, $T=\frac{2\pi}{\omega}=\frac{2\pi r}{\mathrm{v}_\mathrm{o}}=2\pi r\sqrt{\frac{r}{GM}}=\frac{2\pi}{\sqrt{GM}}r^{3/2}$. This gives the important result, observed by Kepler nearly 3 centuries ago that $T^2=\frac{4\pi^2}{GM}r^3$, or $T^2 \propto r^3$.

8. What is the total energy of a satellite moving in a circular orbit around the earth? Clearly, it has two parts, kinetic and potential. Remeber that the potential energy is negative. So, $E = KE + PE = \frac{1}{2}m\mathrm{v}_o^2 - \frac{GM_E m}{r}$. But, $\mathrm{v}_o^2=\frac{GM}{r}$ as we saw earlier and therefore, $E=\frac{1}{2}\frac{GMm}{r}-\frac{GMm}{r}=-\frac{1}{2}\frac{GMm}{r}$. Note that the magnitude of the potential energy is larger than the kinetic energy. If it wasn't, the satellite would not be bound to the Earth!

9. A famous discovery of the astronomer Johann Kepler some 300 years ago says that the line joining a planet to the Sun sweeps out equal areas in equal intervals of time. We can easily see this from the conservation of angular momentum. Call $\Delta A$ the area swept out in time $\Delta t$. Then from the diagram below you can see that $\Delta A=\frac{1}{2}r\left(r\Delta\theta\right)$. Divide this by $\Delta t$ and then take the limit where it becomes very small,

$$\frac{dA}{dt}=\lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t}=\frac{1}{2}r^2\left(\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}\right)=\frac{1}{2}r^2\omega=\frac{L}{2m}.$$

Since $L$ is a constant, we have proved one of Kepler's laws (with so little effort)!

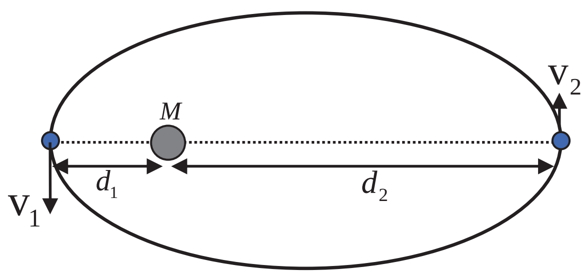


## QUESTIONS AND EXERCISES – 21

1. The acceleration due to gravity $g$ is, as you know, about 9.8 metres/second$^2$ on the surface of the earth. However, this is not a constant!
   a) Why does $g$ vary a little bit from place to place on the surface of the Earth even if the height at which it is being measured remains constant?
   b) What happens to $g$ as you go higher and higher?
   c) What happens to $g$ as you approach the centre of the Earth?
   d) Where do you expect $g$ will have its maximum value, and why?

2. An Earth satellite so positioned that it appears stationary to an observer on Earth is called a geostationary satellite. Its time period of revolution is one day.
   a) What is the height above the ground of the satellite?
   b) Calculate the energy required to lift it from Earth into orbit.
   c) What would be the advantage to a mobile phone company in having such a satellite? The disadvantage?

3. A planet moves in an elliptical orbit (see below) and has a speed $v_1$ at a distance $d_1$ from the Sun.




   (a) What will be its speed at distance $d_2$? [Ans: $v_2 = \frac{v_1 d_1}{d_2}$ ]

   (b) Show that the value of its (constant) angular momentum is, $m\sqrt{\frac{2GMd_1 d_2}{(d_1 + d_2)}}$.