Chapter Seven

Orbits in a central field

including Rutherford scattering

KEY FEATURES

For motion in *general* central force fields, the key results are the **radial motion equation** and the **path equation**. For motion in the *inverse square* force field, the key formulae are the **E-formula**, the **L-formula** and the **period formula**.

The theory of orbits has a special place in classical mechanics for it was the desire to understand why the planets move as they do which provided the major stimulus in the development of mechanics as a scientific discipline. Early in the seventeenth century, Johannes Kepler * published his 'laws of planetary motion', which he deduced by analysing the accurate experimental observations made by the astronomer Tycho Brahe.†

* The German mathematician and astronomer Johannes Kepler (1571–1630) was a firm believer in the Copernican (heliocentric) model of the solar system. In 1596 he became mathematical assistant to Tycho Brahe, the foremost observational astronomer of the day, and began working on the intractable problem of the orbit of Mars. This work continued after Tycho's death in 1601 and, after much labour, Kepler showed that Tycho's observations of Mars corresponded very precisely to an elliptic orbit with the Sun at a focus. This result, together with the 'law of areas' (the second law) was published in 1609. Kepler then found similar orbits for other planets and his third law was published in 1619.

† Tycho Brahe (1546–1601) was a Danish nobleman. He had a lifelong interest in observational astronomy and developed a succession of new and more accurate instruments. The King of Denmark gave him money to create an observatory and also the island of Hven on which to build it. It was here that Tycho made his accurate observations of the planets from which Kepler was able to deduce his laws of planetary motion. Tycho's other claim to fame is that he had a metal nose. When the original was cut off in a duel, he had an artificial nose made from an alloy of silver and gold. Tycho is perhaps better remembered for his nose job than he is for a lifetime of observations.

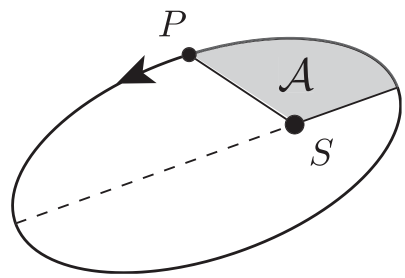

FIGURE 7.1 Each planet P moves on an elliptical path with the Sun S at one focus. The area $\mathcal{A}$ is that referred to in Kepler's second law.

Kepler's laws of planetary motion

First law Each of the planets moves on an elliptical path with the Sun at one focus of the ellipse.

Second law For each of the planets, the straight line connecting the planet to the Sun sweeps out equal areas in equal times.

Third law The squares of the periods of the planets are proportional to the cubes of the major axes of their orbits.

The problem of determining the law of force that causes the motions described by Kepler (and *proving* that it does so) was the most important scientific problem of the seventeenth century. In what must be the finest achievement in the whole history of science, Newton's publication of *Principia* in 1687 not only proved that the inverse square law of gravitation implies Kepler's laws, but also laid down the entire framework of the science of mechanics. Orbit theory is just as important today, the principal fields of application being astronomy, particle scattering and space travel.

In this chapter, we treat the problem of a particle moving in a central force field with a *fixed centre*; this is called the **one-body problem**. The assumption that the centre of force is fixed is an accurate approximation in the context of planetary orbits. The combined mass of all the planets, moons and asteroids is less than 0.2% of the mass of the Sun. We therefore expect the motion of the Sun to be comparatively small, as are inter-planetary influences.* However, we do not confine our interest to motion under the attractive inverse square field. At first, we consider motion in *any* central force field with a fixed centre. This part of the theory will then apply not only to gravitating bodies, but also (for example) to the scattering of neutrons. The important cases of inverse square attraction and repulsion are then examined in greater detail.

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* The more general **two-body problem** is treated in Chapter 10. The two-body theory must be used to analyse problems in which the *masses of the two interacting bodies are comparable*, as they are in binary stars.

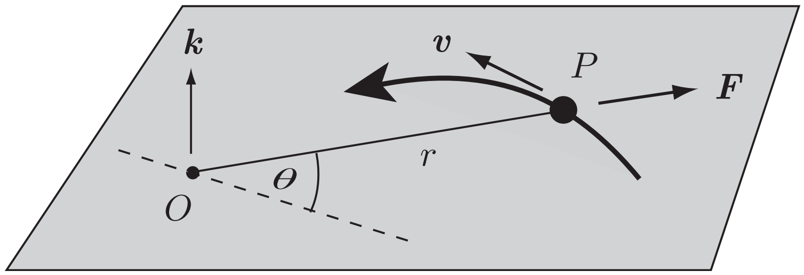

FIGURE 7.2 Each orbit of a particle P in a central force field with centre O takes place in a plane through O. The position of P in the plane of motion is specified by polar coordinates r, θ with centre at O.

7.1 THE ONE-BODY PROBLEM – NEWTON'S EQUATIONS

First we define what we mean by a central force field.

Definition 7.1 ***Central field*** *A force field $\boldsymbol{F}(\boldsymbol{r})$ is said to be a **central field** with centre O if it has the form*

$$\boldsymbol{F}(\boldsymbol{r}) = F(r)\,\widehat{\boldsymbol{r}},$$

*where $r = |\boldsymbol{r}|$ and $\widehat{\boldsymbol{r}} = \boldsymbol{r}/r$. A central field is thus **spherically symmetric** about its centre.*

A good example of a central force is the gravitational force exerted by a *fixed* point mass. Suppose P has mass m and moves under the gravitational attraction of a point mass M fixed at the origin. In this case, the force acting on P is given by the law of gravitation to be

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{mMG}{r^2}\,\widehat{\boldsymbol{r}},$$

where G is the constant of gravitation. This is a central field with

$$F(r) = -\frac{mMG}{r^2}.$$

Each orbit lies in a plane through the centre of force

The first thing to observe is that, when a particle P moves in a central field with centre O, *each orbit of P takes place in a plane through O*, as shown in Figure 7.2. This is the plane that contains O and the initial position and velocity of P. One may give a vectorial proof of this, but it is quite clear on symmetry grounds that P will never leave this plane. Each motion is therefore two-dimensional and we take polar coordinates r, θ (centred on O) to specify the position of P in the plane of motion. On using the formulae (2.14) for the components of acceleration in polar coordinates, the **Newton equations of motion** for

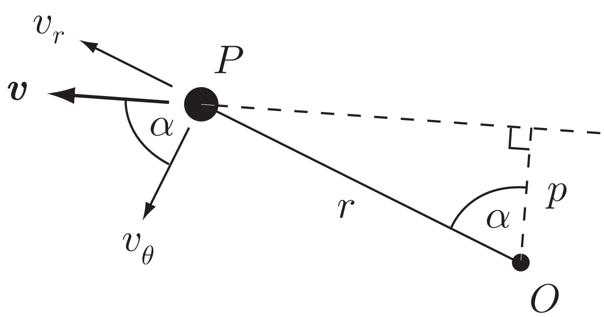

FIGURE 7.3 The angular momentum $mr^2\dot{\theta} = mp\,v$, where $v = |\boldsymbol{v}|$.

P become

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r), \tag{7.1}$$

$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0. \tag{7.2}$$

Angular momentum conservation

Equation (7.2) can be written in the form

$$\frac{1}{r}\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0,$$

which can be integrated with respect to t to give

$$mr^2\dot{\theta} = \text{constant}.$$

The quantity $mr^2\dot{\theta}$, which is a constant of the motion, is called the **angular momentum*** of P. The general theory of angular momentum (and its conservation) is described in Chapter 11, but for now it is sufficient to regard 'angular momentum' simply as a *name* that we give to the conserved quantity $mr^2\dot{\theta}$. This angular momentum has a simple kinematical interpretation. From Figure 7.3 it follows that

$$\begin{aligned} mr^2\dot{\theta} &= mr(r\dot{\theta}) = mr\,v_\theta = m(r\cos\alpha)\left(\frac{v_\theta}{\cos\alpha}\right) \\ &= mp\,v, \end{aligned}$$

where p is the perpendicular distance of O from the tangent to the path of P, and $v = |\boldsymbol{v}|$. This formula provides the usual way of calculating the constant value the angular momentum from the initial conditions.

* More precisely, it is the angular momentum of the particle about the *axis* $\{O, \boldsymbol{k}\}$, where the unit vector $\boldsymbol{k}$ is perpendicular to the plane of motion (see Figure 7.2). The angular momentum of P about the *point* O is the *vector* quantity $m\boldsymbol{r}\times\boldsymbol{v}$, but the axial angular momentum used in the present chapter is the component of this vector in the $\boldsymbol{k}$-direction.

Newton equations in specific form

It is usual and convenient to eliminate the mass m from the theory. If we write

$$F(r) = mf(r),$$

where $f(r)$ is the outward force *per unit mass*, and let L ($= r^2\dot{\theta}$) be the angular momentum *per unit mass* then the Newton equations (7.1), (7.2) reduce to the **specific form**

$$\ddot{r} - r\dot{\theta}^2 = f(r), \tag{7.3}$$

$$r^2\dot{\theta} = L, \tag{7.4}$$

where L is a constant.* Note that these equations apply to orbits in *any central field*. The second of these equations appears throughout this chapter and we will call it the *angular momentum equation*.

Angular momentum equation

$$r^2\dot{\theta} = L \tag{7.5}$$

Kepler's second law

Angular momentum conservation is equivalent to Kepler's second law. The area $\mathcal{A}$ shown in Figure 7.1 can be expressed (with an obvious choice of initial line) as

$$\mathcal{A} = \tfrac{1}{2}\int_0^\theta r^2\, d\theta.$$

Then, by the chain rule,

$$\frac{d\mathcal{A}}{dt} = \frac{d\mathcal{A}}{d\theta} \times \frac{d\theta}{dt} = \tfrac{1}{2}r^2\dot{\theta} = \tfrac{1}{2}L,$$

where L is the constant value of the angular momentum. Thus $\mathcal{A}$ increases at a constant rate, which is what Kepler's second law says. Thus Kepler's second law holds for *all* central force fields, not just the inverse square law.

7.2 GENERAL NATURE OF ORBITAL MOTION

In our first method of solution, we take as our starting point the principles of conservation of **angular momentum** and **energy**.

* Without losing generality, we will take L to be positive, that is, we suppose θ is *increasing* with time. (The special case in which $L = 0$ corresponds to rectilinear motion through O.)

Energy conservation

Every central field $\boldsymbol{F} = mf(r)\widehat{\boldsymbol{r}}$ is **conservative** with potential energy $mV(r)$, where

$$f(r) = -\frac{dV}{dr}. \tag{7.6}$$

Energy conservation then implies that

$$T + V = E,$$

where T is the specific kinetic energy, V is the specific potential energy, and the constant E is the specific total energy. On replacing T by its expression in polar coordinates, we obtain

Energy equation

$$\tfrac{1}{2}\left(\dot{r}^2 + (r\dot{\theta})^2\right) + V(r) = E \tag{7.7}$$

as the **energy conservation** equation. The conservation equations (7.5), (7.7) are equivalent to the Newton equations (7.1), (7.2) and are a convenient starting point for investigating the *general* nature of orbital motion.

The radial motion equation

From the angular momentum conservation equation (7.5), we have

$$\dot{\theta} = L/r^2$$

and, on eliminating $\dot{\theta}$ from the energy conservation equation (7.7), we obtain

$$\tfrac{1}{2}\dot{r}^2 + V(r) + \frac{L^2}{2r^2} = E, \tag{7.8}$$

an ODE for the radial distance $r(t)$. We call this the **radial motion equation** for the particle P. Equation (7.8) (together with the initial conditions) is sufficient to determine the variation of r with t, and the angular momentum equation (7.5) then determines the variation of θ with t. Unfortunately, for most laws of force, this procedure cannot be carried through analytically. However, it is still possible to make important deductions about the general nature of the motion.

Equation (7.8) can be written in the form

$$\tfrac{1}{2}\dot{r}^2 + V^*(r) = E, \tag{7.9}$$

where

$$V^*(r) = V(r) + \frac{L^2}{2r^2}. \tag{7.10}$$

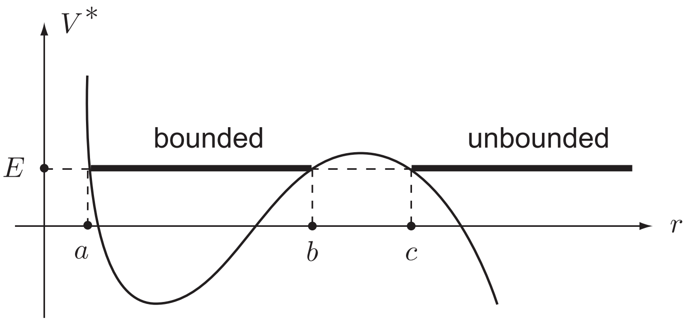

FIGURE 7.4 The effective potential V^* shown admits bounded and unbounded orbits, depending on the initial conditions.

The function $V^*(r)$ is called the **effective potential** of the radial motion and its use reduces the radial motion of P to a rectilinear problem. It must be emphasised though that the whole motion is *two*-dimensional since θ is increasing in accordance with (7.5).

Because r satisfies the radial motion equation (7.9), the variation of r with t can be analysed by using the same methods as were used in Chapter 6 for rectilinear particle motion. In particular, the general nature of the motion depends on the shape of the graph of V^* (which depends on L) and the value of E. The values of the constants L and E depend on the initial conditions.

Suppose for example that the law of force and the initial conditions are such that V^* has the form shown in Figure 7.4 and that E has the value shown. Then, since $T \geq 0$, it follows that the motion is restricted to those values r that satisfy the inequality

$$V^*(r) \leq E,$$

with equality holding when $\dot{r} = 0$. There are two possible motions, in each of which the variation of r with t is governed by the radial motion equation (7.8).

(i) a **bounded motion** in which r oscillates in the range $[a, b]$. In this motion, $r(t)$ is a periodic function.*

(ii) an **unbounded motion** in which r lies in the interval $[c, \infty)$. In this motion r is not periodic but decreases until the minimum value $r = c$ is achieved and then increases without limit.

The bounded orbit. A typical bounded orbit is shown in Figure 7.5 (left). The orbit alternately touches the inner and outer circles $r = a$ and $r = b$, which corresponds to the radial coordinate r oscillating in the interval $[a, b]$. Without losing generality, suppose that P is at the point B_1 when $t = 0$ and that OB_1 is the line $\theta = 0$. Consider the part of

* The fact that $r(t)$ is periodic does *not* mean that the whole motion must be periodic.

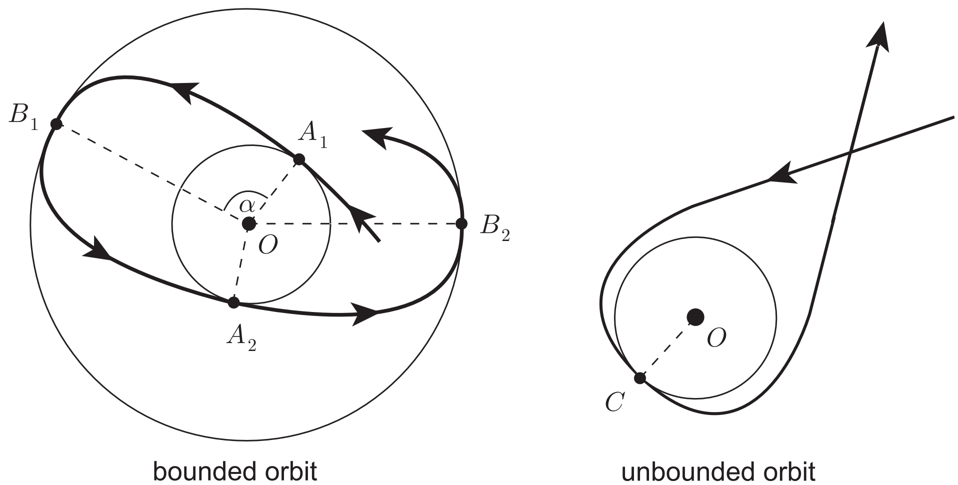

FIGURE 7.5 Typical bounded and unbounded orbits.

the orbit between A_1 and A_2. It follows from the governing equations (7.8), (7.5) that r is an *even* function of t while θ is an *odd* function of t. This means that the segment B_1A_2 of the orbit is just the reflection of the segment A_1B_1 in the line OB_1. This argument can be repeated to show that the segment A_2B_2 is the reflection of the segment B_1A_2 in the line OA_2, and so on. Thus the whole orbit can be constructed from a knowledge of a single segment such as A_1B_1.

It follows from what has been said that the angles $A_1\widehat{O}B_1$, $B_1\widehat{O}A_2$, $A_2\widehat{O}B_2$ (and so on) are all equal. Let α be the common value of these angles. Then the orbit will eventually close itself if some integer multiple of α is equal to some whole number of complete revolutions, that is, if α/π is a *rational number*. There is no reason to expect this condition to hold and, in general, it does not. It follows that these bounded orbits are *not generally closed*. The closed orbits associated with the attractive inverse square field are therefore exceptional, rather than typical!

The unbounded orbit. In the unbounded case there are just two segments both of which are semi-infinite (see Figure 7.5 (right)). The segment in which P recedes from O is the reflection of the segment in which P approaches O in the line OC.

Apses and apsidal distances

The points at which an orbit touches its bounding circles are important and are given a special name:

Definition 7.2 ***Apse, apsidal distance, apsidal angle*** *A point of an orbit at which the distance OP achieves its maximum or minimum value is called an* ***apse*** *of the orbit. These maximum and minimum distances are called the* ***apsidal distances*** *and the angular displacement between successive apses (the angle α in Figure 7.5 (left)) is called the* ***apsidal angle****.*

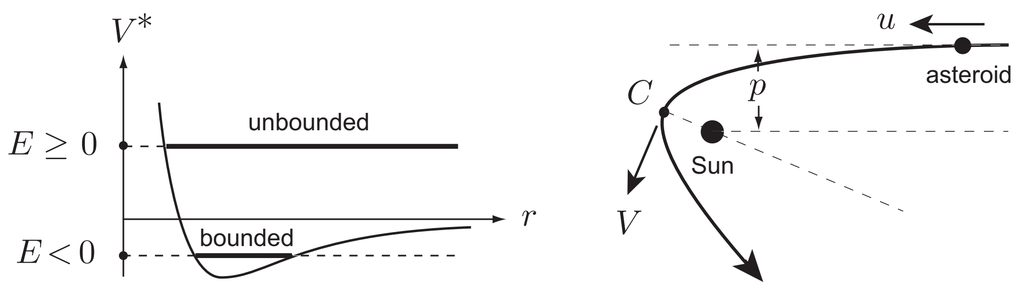

FIGURE 7.6 **Left:** The effective potential V^* for the attractive inverse square force. **Right:** The path of the asteroid around the Sun. C is the point of closest approach.

In the special case of orbits around the Sun, the point of closest approach is called the **perihelion** and the point of maximum distance the **aphelion**. The corresponding terms for orbits around the Earth are **perigee** and **apogee**.

The **apsidal distances**, the maximum and minimum distances of P from O, are easily found from the radial motion equation (7.8). At an apse, $\dot{r} = 0$ and so r must satisfy

$$V(r) + \frac{L^2}{2r^2} = E. \qquad (7.11)$$

The positive roots of this equation are the apsidal distances.

Example 7.1 *Asteroid deflected by the Sun*

A particle P of mass m moves in the central force field $-(m\gamma/r^2)\widehat{\boldsymbol{r}}$, where γ is a positive constant. Show that bounded and unbounded orbits are possible depending on the value of E.

An asteroid is approaching the Sun from a great distance. At this time it has constant speed u and is moving in a straight line whose perpendicular distance from the Sun is p. Find the equation satisfied by the apsidal distances of the subsequent orbit. For the special case in which $u^2 = 4M_\odot G/3p$ (where $M_\odot$ is the mass of the Sun), find (i) the distance of closest approach of the asteroid to the Sun, and (ii) the speed of the asteroid at the time of closest approach.

Solution

For this law of force, $V = -\gamma/r$ and the effective potential V^* is

$$V^* = -\frac{\gamma}{r} + \frac{L^2}{2r^2}.$$

This V^* has the form shown in Figure 7.6 (left), from which it is clear that the orbit will be

(i) **bounded** if $E < 0$,

(ii) **unbounded** if $E \geq 0$,

whatever the value of L.

In the asteroid example, the constant $\gamma = M_{\odot}G$, where $M_{\odot}$ is the mass of the Sun and G is the constant of gravitation. With the given initial conditions, $L = p\,u$ and $E = u^2/2$, so that $E > 0$ and the orbit is **unbounded**.

The equation (7.11) for the **apsidal distances** becomes

$$-\frac{\gamma}{r} + \frac{p^2u^2}{2r^2} = \tfrac{1}{2}u^2,$$

that is,

$$u^2r^2 + 2\gamma r - p^2u^2 = 0,$$

where $\gamma = M_{\odot}G$.

For the special case in which $u^2 = 4M_{\odot}G/3p$, this equation simplifies to

$$2r^2 + 3pr - 2p^2 = 0.$$

The **distance** of closest approach of the asteroid is the *positive* root of this quadratic equation, namely $r = p/2$.

The **speed** V of the asteroid at closest approach is easily deduced from angular momentum conservation. Initially, $L = pu$ and, at closest approach, $L = (p/2)V$. It follows that $V = 2u$. ■

7.3 THE PATH EQUATION

In principle, the method of the last section allows us to determine the complete motion of the orbiting body as a function of the time. However, the procedure is usually too difficult to be carried through analytically. We can make the problem easier (and make more progress) by seeking just the **equation of the path** taken by the body, and not enquiring where the body is on this path at any particular time.

We start from the Newton equation (7.3) and try to eliminate the time by using the angular momentum equation (7.5). In doing this it is helpful to introduce the new dependent variable u, given by

$$u = 1/r. \tag{7.12}$$

This transformation has a magically simplifying effect. We begin by transforming $\dot{r}$ and $\ddot{r}$. By the chain rule,

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2} \times \frac{du}{d\theta} \times \frac{d\theta}{dt} = -\left(r^2\dot{\theta}\right)\frac{du}{d\theta}$$

which, on using the angular momentum equation (7.5), gives

$$\dot{r} = -L\frac{du}{d\theta}. \tag{7.13}$$

A second differentiation with respect to t then gives

$$\ddot{r} = -L\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -L\frac{d^2u}{d\theta^2} \times \frac{d\theta}{dt} = -L^2u^2\frac{d^2u}{d\theta^2}, \tag{7.14}$$

on using the angular momentum equation again.

The term $r\dot{\theta}^2 = L^2u^3$ so that the Newton equation (7.3) is transformed into

$$-L^2u^2\frac{d^2u}{d\theta^2} - L^2u^3 = f(1/u),$$

that is,

The path equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{L^2u^2} \tag{7.15}$$

This is the **path equation**. Its solutions are the polar equations of the paths that the body can take when it moves under the force field $\boldsymbol{F} = mf(r)\widehat{\boldsymbol{r}}$.

Despite the appearance of the left side of equation (7.15), the path equation is **not linear** in general. This is because the right side is a function of u, the *dependent* variable. Only for the **inverse square** and **inverse cube** laws does the path equation become linear. It is a remarkable piece of good luck that the inverse square law (the most important case by far) is one of only two cases that can be solved easily.

Initial conditions for the path equation

Suitable initial conditions for the path equation are provided by specifying the values of u and $du/d\theta$ when $\theta = \alpha$, say. Since $u = 1/r$, the initial value of u is given directly by the initial data. The value of $du/d\theta$ is not given directly but can be deduced from equation (7.13) in the form

$$\frac{du}{d\theta} = -\frac{\dot{r}}{L}, \tag{7.16}$$

where $\dot{r}$ and L are obtained from the initial data.

Example 7.2 *Path equation for the inverse cube law*

The engines of the starship Enterprise have failed and the ship is moving in a straight line with speed V. The crew calculate that their present course will miss the planet B – Zar by a distance p. However, B – Zar is known to exert the force

$$\boldsymbol{F} = -\frac{m\gamma}{r^3}\widehat{\boldsymbol{r}}$$

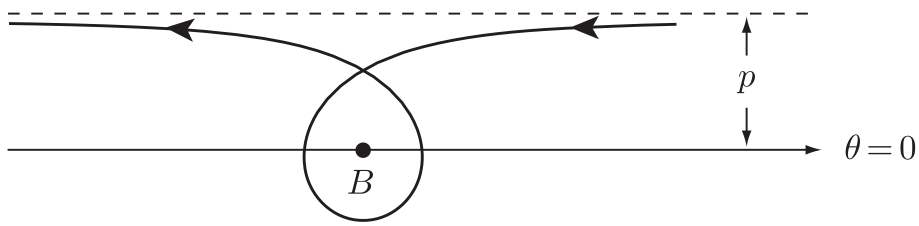

FIGURE 7.7 The path of the Enterprise around the planet B – Zar (B).

on any mass m in its vicinity. A measurement of the constant γ reveals that

$$\gamma = \frac{8p^2V^2}{9}.$$

Show that the crew of the Enterprise will get a free tour around B –Zar before continuing along their *original* path. What is the distance of closest approach and what is the speed of the Enterprise at that instant?

Solution

For the given law of force, $f(r) = -\gamma/r^3$ so that $f(1/u) = -\gamma u^3$. Also, from the initial conditions, $L = pV$. The path equation is therefore

$$\frac{d^2u}{d\theta^2} + u = \frac{\gamma u^3}{p^2V^2u^2},$$

which simplifies to

$$\frac{d^2u}{d\theta^2} + \frac{u}{9} = 0,$$

on using the stated value of γ. The general solution of this equation is

$$u = A\cos(\theta/3) + B\sin(\theta/3).$$

The constants A and B can now be determined from the initial conditions. Take the initial line $\theta = 0$ as shown in Figure 7.7. Then:

(i) The initial condition $r = \infty$ when $\theta = 0$ implies that $u = 0$ when $\theta = 0$. It follows that $A = 0$.

(ii) The initial condition on $du/d\theta$ is given by (7.16) to be

$$\frac{du}{d\theta} = -\frac{\dot{r}}{L} = -\left(\frac{-V}{pV}\right) = \frac{1}{p}$$

when $\theta = 0$. It follows that $B = 3/p$.

The required solution is therefore

$$u = \frac{3}{p}\sin(\theta/3),$$

that is

$$r = \frac{p}{3\sin(\theta/3)}.$$

This is the polar equation of the **path** of the Enterprise, as shown in Figure 7.7. The Enterprise recedes to infinity when $\sin(\theta/3) = 0$ again, that is when $\theta = 3\pi$. Thus the Enterprise makes one circuit of B–Zar before continuing on as before.

The distance of **closest approach** is $p/3$ and is achieved when $\theta = 3\pi/2$. By angular momentum conservation, the **speed** of the Enterprise at that instant is $3V$. ■

7.4 NEARLY CIRCULAR ORBITS

Although the path equation cannot be solved exactly for most laws of force, it is possible to obtain approximate solutions when the body is slightly perturbed from a *known* orbit. In particular, this can always be done when the unperturbed orbit is a circle with centre O.

Suppose that a particle P moves in a circular orbit of radius a under the *attractive* force $f(r)$ per unit mass. This is only possible if its speed v satisfies $v^2/a = f(a)$, in which case its angular momentum L is given by $L^2 = a^3 f(a)$. Suppose that P is now slightly disturbed by a small *radial* impulse. The angular momentum is unchanged but P now moves along some new path

$$u = \frac{1}{a}(1 + \xi(\theta)),$$

where ξ is a **small perturbation**. In terms of ξ, the path equation becomes

$$\frac{d^2\xi}{d\theta^2} + 1 + \xi = + \frac{(1+\xi)^{-2}}{f(a)} f\left(\frac{a}{1+\xi}\right).$$

This exact equation for ξ is non-linear, but we will now approximate it by expanding the right side in powers of ξ. On expanding the function $f(r)$ in a Taylor series about $r = a$ we obtain

$$\begin{aligned} f\left(\frac{a}{1+\xi}\right) &= f\left(a - \frac{a\xi}{1+\xi}\right) \\ &= f(a) - \left(\frac{a\xi}{1+\xi}\right) f'(a) + O\left(\frac{\xi}{1+\xi}\right)^2 \\ &= f(a) - af'(a)\,\xi + O\left(\xi^2\right), \end{aligned}$$

and a simple binomial expansion gives

$$(1+\xi)^{-2} = 1 - 2\xi + O\left(\xi^2\right).$$

On combining these results together, the constant terms cancel and we obtain

$$\frac{d^2\xi}{d\theta^2} + \left(3 + \frac{af'(a)}{f(a)}\right)\xi = 0, \tag{7.17}$$

on neglecting terms of order $O(\xi^2)$. This is the approximate **linearised equation** satisfied by the perturbation $\xi(\theta)$.

The general behaviour of the solutions of equation (7.17) depends on the *sign* of the coefficient of ξ.

(i) If

$$3 + \frac{af'(a)}{f(a)} < 0, \tag{7.18}$$

then the solutions are linear combinations of *real* exponentials, one of which has a positive exponent. In this case, the solution for ξ will not remain small, contrary to assumption. The conclusion is that the original circular orbit is **unstable**.

(ii) Alternatively, if

$$\Omega^2 \equiv 3 + \frac{af'(a)}{f(a)} > 0, \tag{7.19}$$

then the solutions are linear combinations of real cosines and sines, which remain bounded. The conclusion is that the original circular orbit is **stable** (at least to small radial impulses).

Closure of the perturbed orbits

From now on we will assume that the stability condition (7.19) is satisfied. The general solution of equation (7.17) then has the form

$$\xi = A \cos \Omega\theta + B \sin \Omega\theta.$$

We see that the perturbed orbit will **close** itself after one revolution if Ω is a **positive integer**. When the law of force is the **power law**

$$f(r) = kr^{\nu},$$

the perturbed orbit is stable for $\nu > -3$ and will close itself after one revolution if

$$\nu = m^2 - 3,$$

where m is a positive integer. The case $m = 1$ corresponds to inverse square attraction and $m = 2$ corresponds to simple harmonic attraction. The exponents $\nu = 6, 13, \ldots$ are also predicted to give closed orbits. It should be remembered though that these are only the predictions of the approximate linearised theory.* It is possible (but not pretty) to improve on the linear approximation by including quadratic terms in ξ as well as linear ones. The result of this refined theory is that the powers $\nu = -2$ and $\nu = 1$ still give

* It makes no sense to say that an orbit *approximately* closes itself!

closed orbits, but the powers $\nu = 6,\ 13,\ \ldots$ do not. This shows that the power laws with $\nu = 6,\ 13,\ \ldots$ do *not* give perturbed orbits that close after one revolution, but the cases $\nu = -2$ and $\nu = 1$ are still not finally decided. Mercifully, there is no need to carry the approximation procedure any further because all the paths corresponding to both inverse square and simple harmonic attraction can be calculated exactly. It is found that, for these two laws of force, *all bounded orbits close after one revolution*.* There remains the possibility that the perturbed orbits might close themselves after more than one revolution, but a similar analysis shows that this does not happen. We have therefore shown that *the only power laws for which all bounded orbits are closed are the simple harmonic and inverse square laws*. This result is actually true for all central fields (not just power laws) and is known as **Bertrand's theorem**.

Precession of the perihelion of Mercury

The fact that the inverse square law leads to closed orbits, whilst very similar laws do not, provides an extremely sensitive test of the law of gravitation. Suppose for instance that the attractive force experienced by a planet were

$$f(r) = \frac{\gamma}{r^{2+\epsilon}}$$

(per unit mass), where $\gamma > 0$ and $|\epsilon|$ is small. Then the value of Ω for a nearly circular orbit is

$$\Omega = (1-\epsilon)^{1/2} = 1 - \tfrac{1}{2}\epsilon + O(\epsilon^2).$$

This perturbed orbit does not close but has **apsidal angle** α, where

$$\alpha = \frac{\pi}{\Omega} = \frac{\pi}{1 - \frac{1}{2}\epsilon + O(\epsilon^2)} = \pi(1 + \tfrac{1}{2}\epsilon) + O(\epsilon^2).$$

Hence successive perihelions of the planet will not occur at the same point, but the **perihelion will advance** 'annually' by the small angle $\pi\epsilon$. The position of the perihelion of a planet can be measured with great accuracy. For the planet Mercury it is found (after all known perturbations have been subtracted out) that the perihelion advances by 43 (± 0.5) seconds of arc per century, or 5×10^{-7} radians per revolution. This correponds to $\epsilon = 1.6 \times 10^{-7}$ and a power of -2.00000016 instead of -2. Miniscule though this discrepancy from the inverse square law seems, it is considerably greater than the error in the observations and for a considerable time was something of a puzzle.

This puzzle was resolved in a striking fashion by the theory of **general relativity**, published by Einstein in 1915. Einstein showed that one consequence of his theory was that planetary orbits *should* precess slightly and that, in the case of Mercury, the rate of precession should be 43 seconds of arc per century!

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* In the inverse square case, the bounded orbits are ellipses with a *focus* at O, and, in the simple harmonic case, they are ellipses with the *centre* at O.

7.5 THE ATTRACTIVE INVERSE SQUARE FIELD

Because of its many applications to **astronomy**, the attractive inverse square field is the most important force field in the theory of orbits. The same field occurs in particle scattering when the two particles carry unlike electric charges. Because of these important applications, we will treat the inverse square field in more detail than other fields. In particular, we will obtain formulae that enable inverse square problems to be solved quickly and easily without referring to the equations of motion at all.

The paths

Suppose that $f(r) = -\gamma/r^2$ where $\gamma > 0$. Then $f(1/u) = -\gamma u^2$ and the path equation becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\gamma}{L^2},$$

where L is the angular momentum of the orbit. This has the form of the SHM equation with a constant on the right. The general solution is

$$u = A\cos\theta + B\sin\theta + \frac{\gamma}{L^2},$$

which can be written in the form

$$\frac{1}{r} = \frac{\gamma}{L^2}\Big(1 + e\cos(\theta - \alpha)\Big), \qquad (7.20)$$

where e, α are constants with $e \geq 0$. This is the **polar equation of a conic** of eccentricity e and with one focus at O; α is the angle between the major axis of the conic and the initial line $\theta = 0$. If $e < 1$, then the conic is an **ellipse**; if $e = 1$ then the conic is a **parabola**; and when $e > 1$ the conic is the *near* branch of a **hyperbola**. The neccessary geometry of the ellipse and hyperbola is summarised in Appendix A at the end of the chapter; the special case of the parabolic orbit is of marginal interest and we will make little mention of it.

Kepler's first law

It follows from the above that the only bounded orbits in the attractive inverse square field are **ellipses** with one **focus at the centre of force**. This is Kepler's first law, which is therefore a consequence of inverse square law attraction by the Sun. It would not be true for other laws of force.

The L-formula and the E-formula

By comparing the path formula (7.20) with the standard polar forms given in Appendix A, we see that the angular momentum L of the orbit is related to the conic parameters a,

b by the formula

$$\frac{\gamma}{L^2} = \frac{a}{b^2},$$

that is,

The L-formula

$$L^2 = \gamma b^2/a \tag{7.21}$$

We will call this result the **L-formula**. It applies to both elliptic and hyperbolic orbits. It is the first of two important formulae that relate L, E, the dynamical constants of the motion, to the conic parameters of the resulting orbit.

The second such formula involves the energy E. At the point of closest approach $r = c$,

$$E = \tfrac{1}{2}V^2 - \frac{\gamma}{c},$$

where V is the speed of P when $r = c$. Since P is moving transversely at the point of closest approach, it follows that $cV = L$, so that E may be written

$$E = \frac{L^2}{2c^2} - \frac{\gamma}{c} = \frac{\gamma b^2}{ac^2} - \frac{\gamma}{c}$$

on using the L-formula.

From this point on, the different types of conic must be treated separately. When the orbit is an ellipse, $c = a(1-e)$, where e is the eccentricity, and a, b and e are related by the formula

$$e^2 = 1 - \frac{b^2}{a^2}.$$

Then E can be written

$$\begin{aligned} E &= \frac{\gamma a^2(1-e^2)}{2a^3(1-e)^2} - \frac{\gamma}{a(1-e)} \\ &= -\frac{\gamma}{2a}. \end{aligned}$$

Thus the total energy E in the orbit is directly connected to a, the semi-major axis of the elliptical orbit. The parabolic and hyperbolic orbits are treated similarly and the full result, which we will call the **E-formula**, is

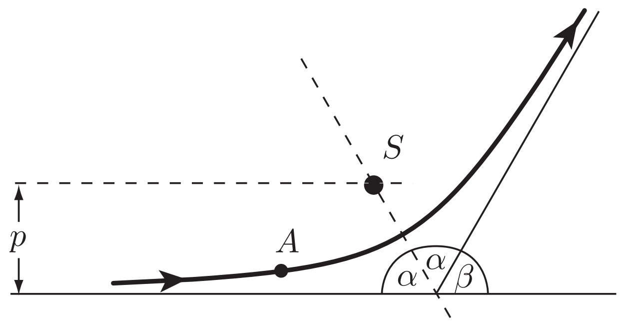

FIGURE 7.8 The asteroid A moves on a hyperbolic orbit around the Sun S as a focus and is deflected through the angle β.

The E-formula

Ellipse:	$E < 0$	$E = -\dfrac{\gamma}{2a}$
Parabola:	$E = 0$	
Hyperbola:	$E > 0$	$E = +\dfrac{\gamma}{2a}$

(7.22)

Note that the type of orbit is determined solely by the *sign* of the total energy E. It follows that that the **escape condition** (the condition that the body should eventually go off to infinity) is simply that $E \geq 0$. This useful result is true only for the inverse square law.

Example 7.3 *Asteroid deflected by the Sun*

An asteroid approaches the Sun with speed V along a line whose perpendicular distance from the Sun is p. Find the angle through which the asteroid is deflected by the Sun.

Solution

In this case we have the attractive inverse square field with $\gamma = M_\odot G$, where $M_\odot$ is the mass of the Sun. This problem can be solved from first principles by using the path equation, but here we make short work of it by using the L- and E-formulae.

From the initial conditions, $L = pV$ and $E = \frac{1}{2}V^2$. Since $E > 0$, the orbit is the near branch of a **hyperbola** and the L- and E-formulae give

$$p^2V^2 = \frac{M_\odot G b^2}{a} \qquad \text{and} \qquad \tfrac{1}{2}V^2 = +\frac{M_\odot G}{2a}.$$

It follows that

$$a = \frac{M_\odot G}{V^2}, \qquad b = p.$$

The semi-angle α between the asymptotes of the hyperbola is then given (see Appendix A) by

$$\tan\alpha = \frac{b}{a} = \frac{pV^2}{M_\odot G}.$$

Let β be the angle through which the asteroid is deflected. Then (see Figure 7.8) $\beta = \pi - 2\alpha$ and

$$\tan(\beta/2) = \tan(\pi/2 - \alpha) = \cot\alpha = \frac{M_\odot G}{pV^2}. \blacksquare$$

Period of the elliptic orbit

Whatever the law of force, once the path of P has been found, the progress of P along that path can be deduced from the angular momentum equation

$$r^2\dot{\theta} = L.$$

If we take $\theta = 0$ when $t = 0$, then the time t taken for P to progress to the point of the orbit with polar coordintes r, θ is given by

$$t = \frac{1}{L}\int_0^\theta r^2\, d\theta, \tag{7.23}$$

where $r = r(\theta)$ is the equation of the path. In particular then, the period τ of the elliptic orbit is given by

$$\tau = \frac{1}{L}\int_0^{2\pi} r^2\, d\theta,$$

where the path $r = r(\theta)$ is given by

$$\frac{1}{r} = \frac{a}{b^2}\left(1 + e\cos\theta\right). \tag{7.24}$$

Fortunately there is no need to evaluate the above integral since, for any path that closes itself after one circuit,

$$\tfrac{1}{2}\int_0^{2\pi} r^2\, d\theta = A,$$

where A is the area enclosed by the path. For the elliptical path, $A = \pi ab$ so that

$$\tau = \frac{2\pi ab}{L},$$

and on using the L-formula, the period of the elliptic orbit is given by:

The period formula

$$\tau = 2\pi\left(\frac{a^3}{\gamma}\right)^{1/2} \tag{7.25}$$

Kepler's third law

In the case of the planetary orbits, $\gamma = M_{\odot}G$, where $M_{\odot}$ is the mass of the Sun. Equation (7.25) can then be written

$$\tau^2 = \left(\frac{4\pi^2}{M_{\odot}G}\right)a^3. \tag{7.26}$$

This is Kepler's third law, which is therefore a consequence of inverse square law attraction by the Sun and would not be true for other laws of force.

Masses of celestial bodies

Once the constant of gravitation G is known, the formula (7.26) provides an accurate way to find the mass of the Sun. The same method applies to *any celestial body that has a satellite*. All that is needed is to measure the major axis $2a$ and the period τ of the satellite's orbit.*

Question ***Finding the mass of Jupiter***

The Moon moves in a nearly circular orbit of radius 384,000 km and period 27.32 days. Callisto, the fourth moon of the planet Jupiter, moves in a nearly circular orbit of radius 1,883,000 km and period 16.69 days. Estimate the mass of Jupiter as a multiple of the mass of the Earth.

Answer

$M_J = 316M_E$.

Astronomical units

For astronomical problems, it is useful to write the period equation (7.26) in **astronomical units**. In these units, the unit of mass is the mass of the Sun ($M_{\odot}$), the unit of length (the AU) is the semi-major axis of the Earth's orbit, and the unit of time is the (Earth) year. On

* It should be noted that here we are neglecting the motion of the centre of force. We will see later that, when this is taken into account, formula (7.26) actually gives the *sum* of the masses of the body and its satellite. Usually, the satellite has a much smaller mass than the body and its contribution can be disregarded.

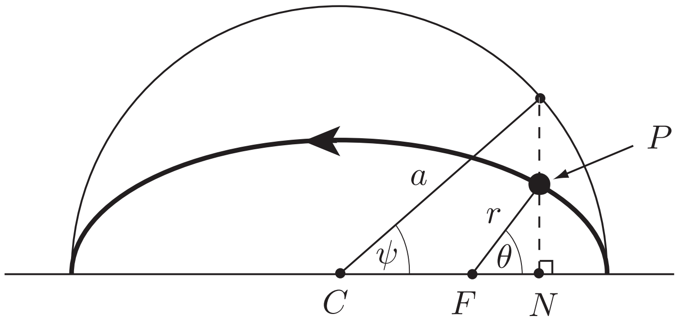

FIGURE 7.9 The eccentric angle ψ corresponding to the polar angle θ.

substituting the data for the Earth and Sun into equation (7.26), we find that $G = 4\pi^2$ in astronomical units. Hence, in **astronomical units** the period formula becomes

$$\tau^2 = \frac{a^3}{M}.$$

Question ***The major axis of the orbit of Pluto***

The period of Pluto is 248 years. What is the semi-major axis of its orbit?

Answer

39.5 AU. ■

Time dependence of the motion – Kepler's equation

The formula (7.23) can be used to find how long it takes for P to progress to a general point of the orbit. However, although the integration with respect to θ can be done in closed form, it is a *very* complicated expression. In order to obtain a manageable formula, we make a cunning change of variable, replacing the polar angle θ by the **eccentric angle** ψ. The relationship between these two angles is shown in Figure 7.9. Since $CN = CF + FN$, it follows that

$$a \cos\psi = ae + r\cos\theta,$$

and, on using the polar equation for the ellipse (7.24) together with the formula $b^2 = a^2(1 - e^2)$, the relation between ψ and θ can be written in the symmetrical form

$$(1 - e\cos\psi)(1 + e\cos\theta) = \frac{b^2}{a^2}. \tag{7.27}$$

Implicit differentiation of equation (7.27) with respect to ψ then gives

$$\frac{d\theta}{d\psi} = \frac{b}{a\,(1 - e\cos\psi)}, \tag{7.28}$$

after more manipulation.

We can now make the change of variable from θ to ψ. From (7.23) and (7.24)

$$\begin{aligned} t &= \frac{b^4}{a^2 L}\int_0^{\theta} \frac{d\theta}{(1+e\cos\theta)^2} \\ &= \frac{b^4}{a^2 L}\int_0^{\psi} \frac{1}{(1+e\cos\theta)^2}\left(\frac{d\theta}{d\psi}\right) d\psi \\ &= \frac{ab}{L}\int_0^{\psi} (1-e\cos\psi)\, d\psi, \\ &= \frac{ab}{L}\left(\psi - e\sin\psi\right), \end{aligned}$$

on using (7.27), (7.28). Finally, on making use of the L-formula $L^2 = \gamma b^2/a$, we obtain

Kepler's equation

$$t = \frac{\tau}{2\pi}\left(\psi - e\sin\psi\right) \tag{7.29}$$

where τ (given by (7.25)) is the period of the orbit. This is **Kepler's equation** which gives the time as a function of position on the elliptical orbit.

If one needs to calculate the position of the orbiting body after a *given time*, then equation (7.29) must be solved numerically for the eccentric angle ψ. The corresponding value of θ is then given by equation (7.27) and the r value by equation (7.24) which, in view of (7.27), can be written in the form

$$r = a\left(1 - e\cos\psi\right). \tag{7.30}$$

The need to solve Kepler's equation for the unknown ψ was a major stimulus in the development of approximate numerical methods for finding roots of equations.

Example 7.4 *Kepler's equation*

A body moving in an inverse square attractive field traverses an elliptical orbit with eccentricity e and period τ. Find the time taken for the body to traverse the half of the orbit that is nearer the centre of force.

Solution

The half of the orbit nearer the centre of force corresponds to the range $-\pi/2 \le \psi \le \pi/2$. The time taken is therefore

$$\frac{\tau}{\pi}\left(\frac{\pi}{2} - e\right) = \tau\left(\frac{1}{2} - \frac{e}{\pi}\right).$$

For example, Halley's comet moves on an elliptic orbit whose eccentricity is almost unity. It therefore spends only about 18% of its time on the half of its orbit that is nearer the Sun.

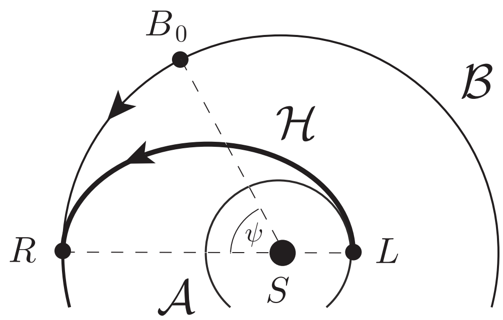

FIGURE 7.10 Two planets move on the circular orbits $\mathcal{A}$ and $\mathcal{B}$. A spacecraft is required to depart from one planet and rendezvous with the other planet at some point of its orbit. The Hohmann orbit $\mathcal{H}$ achieves this with the least expenditure of fuel.

7.6 SPACE TRAVEL – HOHMANN TRANSFER ORBITS

An important problem in space travel, and one that nicely illustrates the preceding theory, is that of transferring a spacecraft from one planet to another (from Earth to Jupiter say). In order to simplify the analysis, we will assume that both of the planetary orbits are circular. We will also suppose that the spacecraft has already effectively been removed from Earth's gravity, but is still in the vicinity of the Earth and is orbiting the Sun on the same orbit as the Earth. The object is to use the rocket motors to transfer the spacecraft to the vicinity of Jupiter, orbiting the Sun on the same orbit as Jupiter. Like everything else on board a spacecraft, fuel has to be transported from Earth at huge cost, so the transfer from Earth to Jupiter must be achieved using the *least mass of fuel*. In our analysis we will neglect the time during which the rocket engines are firing so that the engines are regarded as delivering an impulse to the spacecraft, resulting in a sudden change of velocity. After the initial firing impulse, the spacecraft is assumed to move freely under the Sun's gravitation until it reaches the orbit of Jupiter, when a second firing impulse is required to circularise the orbit. This is called a **two-impulse transfer**.

If the two firings produce velocity changes of $\Delta \boldsymbol{v}^A$ and $\Delta \boldsymbol{v}^B$ respectively, then the quantity Q that must be minimised if the least fuel is to be used is

$$Q = |\Delta \boldsymbol{v}^A| + |\Delta \boldsymbol{v}^B|.$$

The orbit that connects the two planetary orbits and minimises Q is called the **Hohmann transfer orbit*** and is shown in Figure 7.10. It has its perihelion at the lift-off point L and its aphelion at the rendezvous point R. It is not at all obvious that this is the optimal orbit; a proof is given in Appendix B at the end of the chapter. However, it is quite easy to find its properties.

Since the perihelial and aphelial distances in the Hohmann orbit are A and B (the radii of the orbits of Earth and Jupiter), it follows that

$$A = a(1-e), \qquad B = a(1+e),$$

* After Walter Hohmann, the German space research pioneer.

so that the geometrical parameters of the orbit are given by

$$a = \tfrac{1}{2}(B + A), \qquad e = \frac{B - A}{B + A}.$$

The angular momentum L of the orbit is then given by the L-formula to be

$$L^2 = \frac{\gamma b^2}{a} = \gamma\left(1 - e^2\right)a = \frac{\gamma B A}{B + A},$$

where $\gamma = M_\odot G$.

From L we can find the **speed** V^L of the spacecraft just after the lift-off firing, and the **speed** V^R at the rendezvous point just before the second firing. These are

$$V^L = \left(\frac{2\gamma B}{A(B + A)}\right)^{1/2}, \qquad V^R = \left(\frac{2\gamma A}{B(B + A)}\right)^{1/2}.$$

The **travel time** T, which is half the period of the Hohmann orbit, is given by

$$T^2 = \frac{\pi^2 a^3}{\gamma} = \frac{\pi^2 (B + A)^3}{8\gamma}.$$

Finally, in order to rendezvous with Jupiter, the lift-off must take place when Earth and Jupiter have the correct relative positions, so that Jupiter arrives at the meeting point at the right time. Since the speed of Jupiter is $(\gamma/B)^{1/2}$ and the travel time is now known, the angle ψ in Figure 7.10 must be

$$\psi = \pi\left(\frac{B + A}{2B}\right)^{3/2}.$$

Numerical results for the Earth–Jupiter transfer

In astronomical units, $G = 4\pi^2$, $A = 1$ AU and, for Jupiter, $B = 5.2$ AU. A speed of 1 AU per year is 4.74 km per second. Simple calculations then give:

(i) The travel time is 2.73 years, or 997 days.
(ii) V^L is 8.14 AU per year, which is 38.6 km per second. This is the speed the spacecraft must have after the lift-off firing.
(iii) V^R is 1.56 AU per year, which is 7.4 km per second. This is the speed with which the spacecraft arrives at Jupiter before the second firing.
(iv) The angle ψ at lift-off must be 83°.

The speeds V^L and V^R should be compared with the speeds of Earth and Jupiter in their orbits. These are 29.8 km/sec and 13.1 km/sec respectively. Thus the first firing must boost the speed of the spacecraft from 29.8 to 38.6 km/sec, and the second firing must boost the speed from 7.4 to 13.1 km/sec. The sum of these speed increments, 14.5 km/sec, is greater than the speed increment needed (12.4 km/sec) to escape from the Earth's orbit to infinity. Thus it takes more fuel to transfer a spacecraft from Earth's orbit to Jupiter's orbit than it does to escape from the solar system altogether!

7.7 THE REPULSIVE INVERSE SQUARE FIELD

The force field with $f(r) = +\gamma/r^2$, ($\gamma > 0$), is the **repulsive inverse square field**. It occurs in the interaction of charged particles carrying *like* charges and is required for the analysis of Rutherford scattering. Below we summarise the important properties of orbits in a repulsive inverse square field. These results are obtained in exactly the same way as for the attractive case.

The paths

The path equation is

$$\frac{d^2u}{d\theta^2} + u = -\frac{\gamma}{L^2},$$

where L is the angular momentum of the orbit. Its general solution can be written in the form

$$\frac{1}{r} = \frac{\gamma}{L^2}\left[-1 + e\cos(\theta - \alpha)\right],$$

where e, α are constants with $e \geq 0$. By comparing this path with the standard polar forms of conics given in Appendix A, we see that the path can only be the *far* branch of a **hyperbola** with focus at the centre O.

The L- and E-formulae

The formulae relating L, E, the dynamical constants of the orbit, to the hyperbola parameters are

$$L^2 = \gamma b^2/a, \tag{7.31}$$

$$E = +\gamma/2a. \tag{7.32}$$

7.8 RUTHERFORD SCATTERING

The most celebrated application of orbits in a repulsive inverse square field is Rutherford's* famous experiment in which a beam of alpha particles was scattered by gold nuclei in a sheet of gold leaf. We will analyse Rutherford's experiment in detail, beginning with the basic problem of a single alpha particle being deflected by a single fixed gold nucleus.

Alpha particle deflected by a heavy nucleus

An alpha particle A of mass m and charge q approaches a gold nucleus B of charge Q (see Figure 7.11). B is initially at rest and A is moving with speed V along a line whose

* Ernest Rutherford (1871–1937), a New Zealander, was one of the greatest physicists of the twentieth century. His landmark work on the structure of the nucleus in 1911 (and with Geiger and Marsden in 1913) was conducted at the University of Manchester, England.

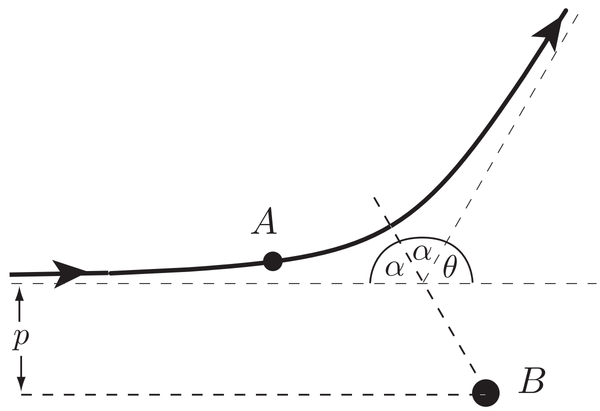

FIGURE 7.11 The alpha particle A of mass m and charge q is repelled by the fixed nucleus B of charge Q and moves on a hyperbolic orbit with the nucleus at the far focus. The alpha particle is deflected through the angle θ.

perpendicular distance from B is p. In the present treatment, we neglect the motion of the gold nucleus. This is justified since the mass of the gold nucleus is about fifty times larger than that of the alpha particle. A then moves in the electrostatic field due to B, which we now suppose to be fixed at the origin O. The force exerted on A is then

$$\boldsymbol{F} = +\frac{qQ}{r^2}\widehat{\boldsymbol{r}}$$

in cgs units. This is the **repulsive inverse square** field with $\gamma = qQ/m$.

We wish to find θ, the angle through which the alpha particle is deflected. This is obtained in exactly the same way as that of the asteroid in Example 7.1. From the initial conditions, $L = pV$ and $E = \frac{1}{2}V^2$. The L-formula (7.31) and the E-formula (7.32) then give

$$p^2V^2 = \frac{\gamma b^2}{a}, \qquad \frac{1}{2}V^2 = +\frac{\gamma}{2a}.$$

It follows that

$$a = \frac{\gamma}{V^2}, \qquad b = p.$$

The semi-angle α between the asymptotes of the hyperbola is then given (see Appendix A) by

$$\tan\alpha = \frac{b}{a} = \frac{pV^2}{\gamma}.$$

Hence, θ, the angle through which the asteroid is deflected, is given by

$$\tan(\theta/2) = \tan(\pi/2 - \alpha) = \cot\alpha = \frac{\gamma}{pV^2}.$$

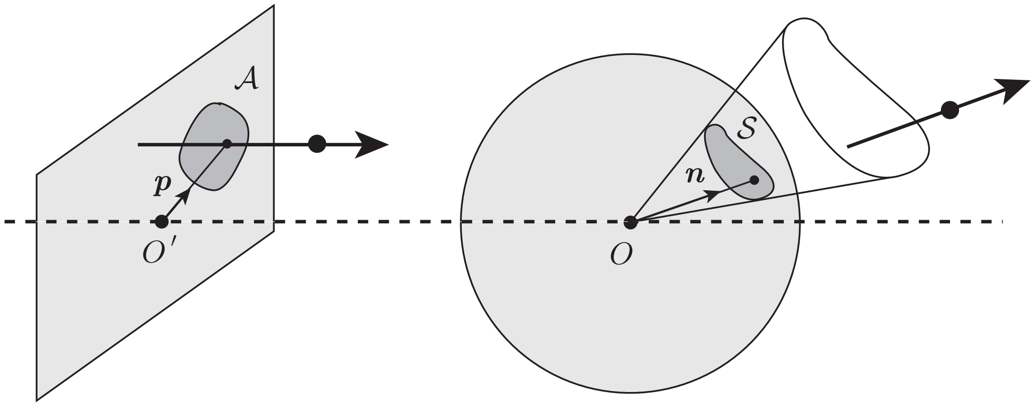

FIGURE 7.12 General scattering. A typical particle crosses the reference plane at the point $\boldsymbol{p}$ and finally emerges in the direction of the unit vector $\boldsymbol{n}$. Particles that cross the reference plane within the region $\mathcal{A}$ emerge within the (generalised) cone shown.

On writing $\gamma = qQ/m$, we obtain

$$\tan(\theta/2) = \frac{qQ}{mpV^2}. \qquad (7.33)$$

as the formula for the **deflection angle** of the alpha particle. The quantity p, the distance by which the incident particle would miss the scatterer if there were no interaction, is called the **impact parameter** of the particle.

The deflection formula (7.33) cannot be confirmed directly by experiment since this would require the observation of a single alpha particle, a single nucleus, and a knowledge of the impact parameter p. What is actually done is to irradiate a gold target by a uniform beam of alpha particles of the same energy. Thus the target consists of many gold nuclei together with their associated electrons. However, the electrons have masses that are very small compared to that of an alpha particle and so their influence can be disregarded. Also, the gold target is taken in the form of thin foil to minimise the chance of multiple collisions. If multiple collisions are eliminated, then the gold nuclei act as *independent* scatterers and the problem reduces to that of a *single* fixed gold nucleus irradiated by a *uniform beam* of alpha particles. In this problem the alpha particles come in with different values of the impact parameter p and are scattered through different angles in accordance with formula (7.33). What *can* be measured is the **angular distribution** of the scattered alpha particles.

Differential scattering cross-section

The angular distribution of scattered particles is expressed by a function $\sigma(\boldsymbol{n})$, called the **differential scattering cross section**, where the unit vector $\boldsymbol{n}$ specifies the final direction of emergence of a particle from the scatterer O. One may imagine the values of $\boldsymbol{n}$ corresponding to points on the surface of a sphere with centre O and unit radius, as shown in Figure 7.12. Then values of $\boldsymbol{n}$ that lie in the shaded patch $\mathcal{S}$ correspond to particles whose final direction of emergence lies inside the (generalised) cone shown.

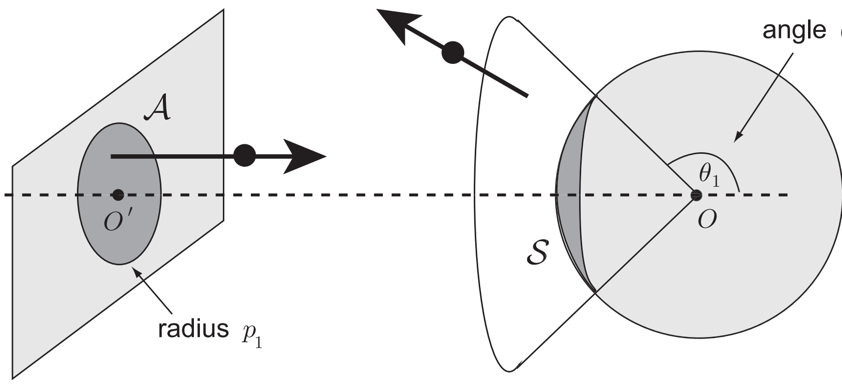

FIGURE 7.13 Axisymmetric scattering. Particles crossing the reference plane within the shaded circular disk are scattered and emerge in directions within the circular cone.

Take a reference plane far to the left of the scatterer and perpendicular to the incident beam, as shown in Figure 7.12. Suppose that there is a uniform flux of incomimg particles crossing the reference plane such that N particles cross any unit area of the reference plane in unit time. When these particles have been scattered, they will emerge in different directions and some of the particles will emerge with directions lying within the (generalised) cone shown in Figure 7.12. The **differential scattering cross section** is defined to be that function $\sigma(\boldsymbol{n})$ such that the flux of particles that emerge with directions lying within the cone is given by the surface integral

$$N \int_{\mathcal{S}} \sigma(\boldsymbol{n})\, dS. \tag{7.34}$$

It is helpful to regard $\sigma(\boldsymbol{n})$ as a **scattering density**, analogous to a probability density, that must be integrated to give the flux of particles scattered within any given solid angle.

The particles that finally emerge within the cone must have crossed the reference plane within some region $\mathcal{A}$ as shown in Figure 7.12. A typical particle crosses the reference plane at the point $\boldsymbol{p}$ (relative to O') and eventually emerges in the direction $\boldsymbol{n}$ lying within the cone. However because the incoming beam is uniform, the flux of these particles across $\mathcal{A}$ is just $N|\mathcal{A}|$, where $|\mathcal{A}|$ is the **area** of the region $\mathcal{A}$. On equating the incoming and outgoing fluxes, we obtain the relation

$$\int_{\mathcal{S}} \sigma(\boldsymbol{n})\, dS = |\mathcal{A}|. \tag{7.35}$$

This is the general relation that *any* differential scattering cross section must satisfy; it simply expresses the equality of incoming and outgoing fluxes of particles. However, Rutherford scattering is axisymmetric and this provides a major simplification.

Axisymmetric scattering and Rutherford's formula

Rutherford scattering is simpler than the general case outlined above in that the problem is **axisymmetric** about the axis $O'O$. Thus σ depends on θ (the angle between $\boldsymbol{n}$ and the axis $O'O$), but is independent of ϕ (the azimuthal angle measured around the axis). In this case $\sigma(\theta)$ can be determined by using the axisymmetric regions shown in Figure 7.13. Particles that cross the reference plane within the *circle* centre O' and radius p_1 emerge within the *circular* cone $\theta_1 \le \theta \le \pi$, where p_1 and θ_1 are related by the deflection formula for a single particle, in our case formula (7.33). On applying equation (7.35) to the present case, we obtain

$$\int_{\mathcal{S}} \sigma(\theta)\, dS = \pi p_1^2.$$

We evaluate the surface integral using θ, ϕ coordinates. The element of surface area on the unit sphere is given by $dS = \sin\theta\, d\theta d\phi$ so that

$$\begin{aligned}\int_{\mathcal{S}} \sigma(\theta)\, dS &= \int_{\theta_1}^{\pi} \left\{ \int_0^{2\pi} \sigma(\theta) \sin\theta\, d\phi \right\} d\theta \\ &= 2\pi \int_{\theta_1}^{\pi} \sigma(\theta) \sin\theta\, d\theta.\end{aligned}$$

Hence

$$\begin{aligned}2\pi \int_{\theta_1}^{\pi} \sigma(\theta) \sin\theta\, d\theta &= \pi p_1^2 \\ &= 2\pi \int_0^{p_1} p\, dp \\ &= -2\pi \int_{\theta_1}^{\pi} p \frac{dp}{d\theta}\, d\theta,\end{aligned}$$

on changing the integration variable from p to θ. Here the impact parameter p is regarded as a function of the scattering angle θ. Now the above equality holds for all choices of the integration limit θ_1 and this can only be true if the two *integrands* are equal. Hence:

Axisymmetric scattering cross section

$$\sigma(\theta) = -\left(\frac{p}{\sin\theta}\right) \frac{dp}{d\theta} \qquad (7.36)$$

This is the formula for the differential scattering cross section σ in any problem of **axisymmetric scattering**. All that is needed to evaluate it in any particular case is the expression for the impact parameter p in terms of the scattering angle θ.

In the case of **Rutherford scattering**, the expression for p in terms of θ is provided by solving equation (7.33) for p, which gives

$$p = \frac{qQ}{mV^2} \tan(\theta/2).$$

On substituting this function into the formula (7.36), we obtain

Rutherford's scattering cross-section

$$\sigma(\theta) = \frac{q^2 Q^2}{16E^2}\left(\frac{1}{\sin^4(\theta/2)}\right) \tag{7.37}$$

where $E(=\frac{1}{2}mV^2)$ is the energy of the incident alpha particles. This is **Rutherford's formula** for the angular distribution of the scattered alpha particles.

Significance of Rutherford's experiment

In the above description we have used the term 'nucleus' for convenience. What we really mean is '*the positively charged part of the atom that carries most of the mass*'. If this positive charge is distributed in a spherically symmetric manner, then the above results still hold, irrespective of the radius of the charge, provided that the alpha particles *do not penetrate* into the charge itself. What Rutherford found was that, when using alpha particles from a radium source, the formula (7.37) held even for particles that were scattered through angles close to π. These are the particles that get closest to the nucleus, the distance of closest approach being qQ/E. This meant that the nuclear radius of gold must be smaller than this distance, which was about 10^{-12} cm in Rutherford's experiment. The radius of an atom of gold is about 10^{-8} cm. This result completely contradicted the Thompson model, in which the positive charge was distributed over the whole volume of the atom, by showing that the nucleus (as it became known) must be a very small and very dense core at the centre of the atom.

Note on two-body scattering problems

Throughout this section we have neglected the motion of the target nucleus. This will introduce only small errors when the target nucleus is much heavier than the incident particles, as it was in Rutherford's experiment. However, if lighter nuclei are used as the target, then the motion of the nucleus cannot be neglected and we have a **two-body scattering problem**. Such problems are treated in Chapter 10.

Appendix A The geometry of conics

Ellipse

(i) In **Cartesian coordinates**, the standard ellipse with **semi-major axis** a and **semi-minor axis** b ($b \leq a$) has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

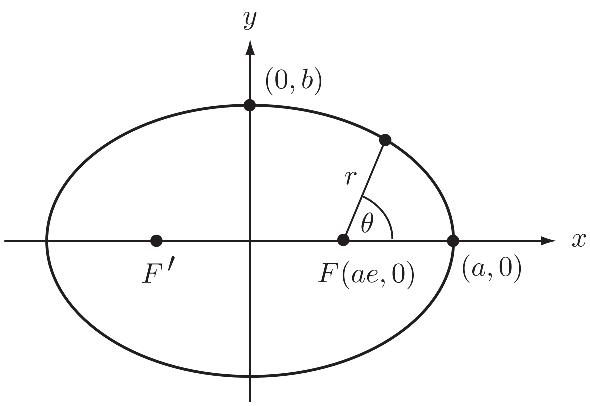

FIGURE 7.14 The standard ellipse $x^2/a^2 + y^2/b^2 = 1$.

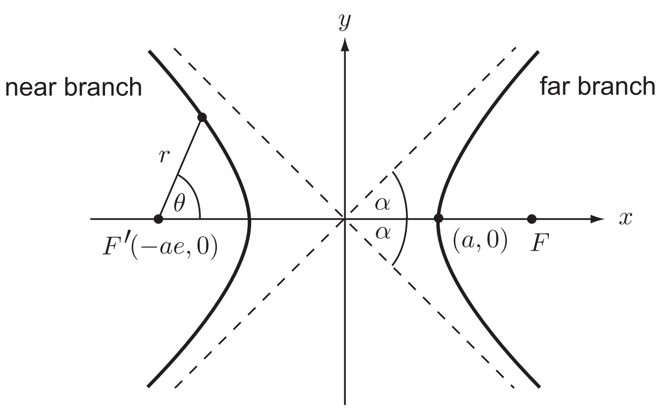

FIGURE 7.15 The standard hyperbola $x^2/a^2 - y^2/b^2 = 1$. The near and far branches are relative to the focus F', which is the origin of polar coordinates.

(ii) The **eccentricity** e of the ellipse is defined by

$$e^2 = 1 - \frac{b^2}{a^2}$$

and lies in the range $0 \le e < 1$. When $e = 0$, $b = a$ and the ellipse is a circle.

(iii) The **focal points** F, F' of the ellipse lie on the major axis at $(\pm ae, 0)$.

(iv) In **polar coordinates** with origin at the focus F and with initial line in the positive x-direction, the equation of the ellipse is

$$\frac{1}{r} = \frac{a}{b^2}(1 + e\cos\theta).$$

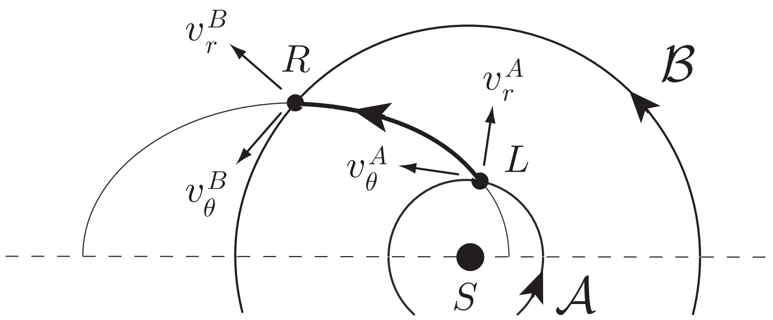

FIGURE 7.16 The circular orbits $\mathcal{A}$ and $\mathcal{B}$ are the orbits of the two planets. The elliptical orbit shown is a possible path for the spacecraft, which travels along the arc LR. The velocities shown are those *after* the first firing at L and *before* the second firing at R.

Hyperbola

(i) In **Cartesian coordinates**, the standard hyperbola has the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (a, b > 0)$$

so that the angle 2α between the asymptotes is given by

$$\tan\alpha = \frac{b}{a}.$$

(ii) The **eccentricity** e of the hyperbola is defined by

$$e^2 = 1 + \frac{b^2}{a^2}$$

and lies in the range $e > 1$.

(iii) The **focal points** F, F' of the hyperbola lie on the x-axis at $(\pm ae, 0)$.

(iv) In **polar coordinates** with origin at the focus F' and with initial line in the positive x-direction, the equations of the near and far branches of the hyperbola are

$$\frac{1}{r} = \frac{a}{b^2}(1 + e\cos\theta), \qquad \frac{1}{r} = \frac{a}{b^2}(-1 + e\cos\theta),$$

respectively.

Appendix B The Hohmann orbit is optimal

The result that the Hohmann orbit is the connecting orbit that minimises Q is not at all obvious and correct proofs are rare.* Hopefully, the proof given below *is* correct!

* It is sometimes stated that the optimality requirement is to minimise the *energy* of the connecting orbit, which is not true. In any case, the Hohmann orbit is *not* the connecting orbit of minimum energy!

Proof of optimality Consider the general two-impulse transfer orbit LR shown in Figure 7.16, where the orbit is regarded as being generated by the velocity components v_θ^A, v_r^A of the spacecraft after the first impulse. Then, by angular momentum and energy conservation,

$$Av_\theta^A = Bv_\theta^B,$$

$$\left(v_r^A\right)^2 + \left(v_\theta^A\right)^2 - \frac{2\gamma}{A} = \left(v_r^B\right)^2 + \left(v_\theta^B\right)^2 - \frac{2\gamma}{B},$$

where A, B are the radii of the circular orbits of Earth and Jupiter and $\gamma = M_\odot G$. Thus

$$v_\theta^B = \frac{A}{B} v_\theta^A,$$

$$\left(v_r^B\right)^2 = \left(1 - \frac{A^2}{B^2}\right)\left(v_\theta^A\right)^2 + \left(v_r^A\right)^2 - 2\gamma\left(\frac{1}{A} - \frac{1}{B}\right).$$

Since the orbital speeds of Earth and Jupiter are $(\gamma/A)^{1/2}$ and $(\gamma/B)^{1/2}$, it follows that the velocity changes $\Delta\boldsymbol{v}^A$, $\Delta\boldsymbol{v}^B$ required at L and R have magnitudes given by

$$|\Delta\boldsymbol{v}^A|^2 = \left(v_\theta^A - \left(\frac{\gamma}{A}\right)^{1/2}\right)^2 + \left(v_r^A\right)^2,$$

$$\begin{aligned}
|\Delta\boldsymbol{v}^B|^2 &= \left(\left(\frac{\gamma}{B}\right)^{1/2} - v_\theta^B\right)^2 + \left(v_r^B\right)^2 \\
&= \left(\left(\frac{\gamma}{B}\right)^{1/2} - \frac{A}{B}v_\theta^A\right)^2 + \left(1 - \frac{A^2}{B^2}\right)\left(v_\theta^A\right)^2 + \left(v_r^A\right)^2 - 2\gamma\left(\frac{1}{A} - \frac{1}{B}\right) \\
&= \left(v_\theta^A - \frac{\gamma^{1/2}A}{B^{3/2}}\right)^2 + \left(v_r^A\right)^2 + \gamma\left(\frac{3}{B} - \frac{2}{A} - \frac{A^2}{B^3}\right).
\end{aligned}$$

It is evident that, with v_θ^A fixed, both $|\Delta\boldsymbol{v}^A|$ and $|\Delta\boldsymbol{v}^B|$ are *increasing* functions of v_r^A. Thus Q may be reduced by reducing v_r^A provided that the resulting orbit still meets the circle $r = B$. Q can be thus reduced until either

(i) v_r^A is reduced to zero, or
(ii) the orbit shrinks until it touches the circle $r = B$ and any further reduction in v_r^A would mean that the orbit would not meet $r = B$.

In the first case, L becomes the perihelion of the orbit and, in the second case, R becomes the aphelion of the orbit. We will proceed assuming the first case, the second case being treated in a similar manner and with the same result.

Suppose then that L is the perihelion of the connecting orbit. Then $v_r^A = 0$ and, from now on, we will simply write v instead of v_θ^A. The velocity v must be such that the orbit reaches the circle $r = B$, which now means that the major axis of the orbit must not be less than $A + B$. On using the E-formula, this implies that v must satisfy

$$v^2 \geq \frac{2\gamma B}{A(A+B)}.$$

The formulae for $|\Delta\boldsymbol{v}^A|$ and $|\Delta\boldsymbol{v}^B|$ now simplify to

$$|\Delta\boldsymbol{v}^A|^2 = \left(v - \left(\frac{\gamma}{A}\right)^{1/2}\right)^2,$$

$$|\Delta\boldsymbol{v}^B|^2 = \left(v - \frac{\gamma^{1/2}A}{B^{3/2}}\right)^2 + \gamma\left(\frac{3}{B} - \frac{2}{A} - \frac{A^2}{B^3}\right)$$

from which it is evident that, for v in the permitted range, both of $|\Delta \boldsymbol{v}^A|$ and $|\Delta \boldsymbol{v}^B|$ are *increasing* functions v. Hence the minimum value of Q is achieved when v takes its smallest permitted value, namely

$$v = \left(\frac{2\gamma B}{A(A+B)} \right)^{1/2}.$$

With this value of v, the orbit touches the circle $r = B$ and so has its aphelion at R. Hence *the optimum orbit has its perihelion at L and its aphelion at R*. This is precisely the **Hohmann orbit**. ■

Problems on Chapter 7

Answers and comments are at the end of the book.

Harder problems carry a star (∗).

Radial motion equation, apses

7.1 A particle P of mass m moves under the repulsive inverse cube field $\boldsymbol{F} = (m\gamma/r^3)\,\widehat{\boldsymbol{r}}$. Initially P is at a great distance from O and is moving with speed V towards O along a straight line whose perpendicular distance from O is p. Find the equation satisfied by the apsidal distances. What is the distance of closest approach of P to O?

7.2 A particle P of mass m moves under the attractive inverse square field $\boldsymbol{F} = -(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$. Initially P is at a point C, a distance c from O, when it is projected with speed $(\gamma/c)^{1/2}$ in a direction making an acute angle α with the line OC. Find the apsidal distances in the resulting orbit.

Given that the orbit is an ellipse with O at a focus, find the semi-major and semi-minor axes of this ellipse.

7.3 A particle of mass m moves under the attractive inverse square field $\boldsymbol{F} = -(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$. Show that the equation satisfied by the apsidal distances is

$$2Er^2 + 2\gamma r - L^2 = 0,$$

where E and L are the specific total energy and angular momentum of the particle. When $E < 0$, the orbit is known to be an ellipse with O as a focus. By considering the sum and product of the roots of the above equation, establish the elliptic orbit formulae

$$L^2 = \gamma b^2/a, \qquad E = -\gamma/2a.$$

7.4 A particle P of mass m moves under the simple harmonic field $\boldsymbol{F} = -(m\Omega^2 r)\,\widehat{\boldsymbol{r}}$, where Ω is a positive constant. Obtain the radial motion equation and show that all orbits of P are bounded.

Initially P is at a point C, a distance c from O, when it is projected with speed Ωc in a direction making an acute angle α with OC. Find the equation satisfied by the apsidal distances. Given that the orbit of P is an ellipse with centre O, find the semi-major and semi-minor axes of this ellipse.

Path equation

7.5 A particle P moves under the attractive inverse square field $\boldsymbol{F} = -(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$. Initially P is at the point C, a distance c from O, and is projected with speed $(3\gamma/c)^{1/2}$ perpendicular to OC. Find the polar equation of the path make a sketch of it. Deduce the angle between OC and the final direction of departure of P.

7.6 A comet moves under the gravitational attraction of the Sun. Initially the comet is at a great distance from the Sun and is moving towards it with speed V along a straight line whose perpendicular distance from the Sun is p. By using the path equation, find the angle through which the comet is deflected and the distance of closest approach.

7.7 A particle P of mass m moves under the attractive inverse cube field $\boldsymbol{F} = -(m\gamma^2/r^3)\,\widehat{\boldsymbol{r}}$, where γ is a positive constant. Initially P is at a great distance from O and is projected towards O with speed V along a line whose perpendicular distance from O is p. Obtain the path equation for P.

For the case in which

$$V = \frac{15\gamma}{\sqrt{209}\,p},$$

find the polar equation of the path of P and make a sketch of it. Deduce the distance of closest approach to O, and the final direction of departure.

7.8∗ A particle P of mass m moves under the central field $\boldsymbol{F} = -(m\gamma^2/r^5)\,\widehat{\boldsymbol{r}}$, where γ is a positive constant. Initially P is at a great distance from O and is projected towards O with speed $\sqrt{2}\gamma/p^2$ along a line whose perpendicular distance from O is p. Show that the polar equation of the path of P is given by

$$r = \frac{p}{\sqrt{2}} \coth\left(\frac{\theta}{\sqrt{2}}\right).$$

Make a sketch of the path.

7.9∗ A particle of mass m moves under the central field

$$\boldsymbol{F} = -m\gamma^2\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)\widehat{\boldsymbol{r}},$$

where γ and a are positive constants. Initially the particle is at a distance a from the centre of force and is projected at right angles to the radius vector with speed $3\gamma/\sqrt{2}a$. Find the polar equation of the resulting path and make a sketch of it.

Find the time taken for the particle to reach the centre of force.

Nearly circular orbits

7.10 A particle of mass m moves under the central field

$$\boldsymbol{F} = -m\left(\frac{\gamma e^{-\epsilon r/a}}{r^2}\right)\widehat{\boldsymbol{r}},$$

where γ, a and ϵ are positive constants. Find the apsidal angle for a nearly circular orbit of radius a. When ϵ is small, show that the perihelion of the orbit advances by approximately $\pi\epsilon$ on each revolution.

7.11 ***Solar oblateness*** A planet of mass m moves in the equatorial plane of a star that is a uniform oblate spheroid. The planet experiences a force field of the form

$$\boldsymbol{F} = -\frac{m\gamma}{r^2}\left(1 + \frac{\epsilon\, a^2}{r^2}\right)\widehat{\boldsymbol{r}},$$

approximately, where γ, a and ϵ are positive constants and ϵ is small. If the planet moves in a nearly circular orbit of radius a, find an approximation to the 'annual' advance of the perihelion. [It has been suggested that oblateness of the Sun might contribute significantly to the precession of the planets, thus undermining the success of general relativity. This point has yet to be resolved conclusively.]

7.12 Suppose the solar system is embedded in a dust cloud of uniform density ρ. Find an approximation to the 'annual' advance of the perihelion of a planet moving in a nearly circular orbit of radius a. (For convenience, let $\rho = \epsilon M/a^3$, where M is the solar mass and ϵ is small.)

7.13 ***Orbits in general relativity*** In the theory of general relativity, the path equation for a planet moving in the gravitational field of the Sun is, in the standard notation,

$$\frac{d^2u}{d\theta^2} + u = \frac{MG}{L^2} + \left(\frac{3MG}{c^2}\right)u^2,$$

where c is the speed of light. Find an approximation to the 'annual' advance of the perihelion of a planet moving in a nearly circular orbit of radius a.

Scattering

7.14 A uniform flux of particles is incident upon a fixed hard sphere of radius a. The particles that strike the sphere are reflected elastically. Find the differential scattering cross section.

7.15 A uniform flux of particles, each of mass m and speed V, is incident upon a fixed scatterer that exerts the repulsive radial force $\boldsymbol{F} = (m\gamma^2/r^3)\,\widehat{\boldsymbol{r}}$. Find the impact parameter p as a function of the scattering angle θ, and deduce the differential scattering cross section. Find the total back-scattering cross-section.

Assorted inverse square problems

Some useful **data**:

The radius R of the Earth is 6380 km. To obtain the value of MG, where M is the mass of the Earth, use the formula $MG = R^2 g$, where $g = 9.80\ \mathrm{m\,s^{-2}}$.

1 AU per year is 4.74 km per second. In astronomical units, $G = 4\pi^2$.

7.16 In Yuri Gagarin's first manned space flight in 1961, the perigee and apogee were 181 km and 327 km above the Earth. Find the period of his orbit and his maximum speed in the orbit.

7.17 An Earth satellite has a speed of 8.60 km per second at its perigee 200 km above the Earth's surface. Find the apogee distance above the Earth, its speed at the apogee, and the period of its orbit.

7.18 A spacecraft is orbiting the Earth in a circular orbit of radius c when the motors are fired so as to multiply the speed of the spacecraft by a factor k ($k > 1$), its direction of motion being unaffected. [You may neglect the time taken for this operation.] Find the range of k for which the spacecraft will escape from the Earth, and the eccentricity of the escape orbit.

7.19 A spacecraft travelling with speed V approaches a planet of mass M along a straight line whose perpendicular distance from the centre of the planet is p. When the spacecraft is at a distance c from the planet, it fires its engines so as to multiply its current speed by a factor k ($0 < k < 1$), its direction of motion being unaffected. [You may neglect the time taken for this operation.] Find the condition that the spacecraft should go into orbit around the planet.

7.20 A body moving in an inverse square attractive field traverses an elliptical orbit with major axis $2a$. Show that the time average of the potential energy $V = -\gamma/r$ is $-\gamma/a$. [Transform the time integral to an integral with repect to the eccentric angle ψ.]

Deduce the time average of the kinetic energy in the same orbit.

7.21 A body moving in an inverse square attractive field traverses an elliptical orbit with eccentricity e and major axis $2a$. Show that the time average of the distance r of the body from the centre of force is $a(1 + \frac{1}{2}e^2)$. [Transform the time integral to an integral with respect to the eccentric angle ψ.]

7.22 A spacecraft is 'parked' in a circular orbit 200 km above the Earth's surface. The spacecraft is to be sent to the Moon's orbit by Hohmann transfer. Find the velocity changes Δv^E and Δv^M that are required at the Earth and Moon respectively. How long does the journey take? [The radius of the Moon's orbit is 384,000 km. Neglect the gravitation of the Moon.]

7.23* A spacecraft is 'parked' in an *elliptic* orbit around the Earth. What is the most fuel efficient method of escaping from the Earth by using a single impulse?

7.24 A satellite already in the Earth's heliocentric orbit can fire its engines only once. What is the most fuel efficient method of sending the satellite on a 'flyby' visit to another planet? The satellite can visit either Mars or Venus. Which trip would use less fuel? Which trip would take the shorter time? [The orbits of Mars and Venus have radii 1.524 AU and 0.723 AU respectively.]

7.25 A satellite is 'parked' in a circular orbit 250 km above the Earth's surface. What is the most fuel efficient method of transferring the satellite to an (elliptical) synchronous orbit by using a single impulse? [A synchronous orbit has a period of 23 hr 56 m.] Find the value of $\Delta \boldsymbol{v}$ and apogee distance.

Effect of resistance

7.26 A satellite of mass m moves under the attractive inverse square field $-(m\gamma/r^2)\widehat{\boldsymbol{r}}$ and is also subject to the linear resistance force $-mK\boldsymbol{v}$, where K is a positive constant. Show that the governing equations of motion can be reduced to the form

$$\ddot{r} + K\dot{r} + \frac{\gamma}{r^2} - \frac{L_0^2\, e^{-2Kt}}{r^3} = 0, \qquad r^2\dot{\theta} = L_0\, e^{-Kt},$$

where L_0 is a constant which will be assumed to be positive.

Suppose now that the effect of resistance is slight and that the satellite is executing a 'circular' orbit of slowly changing radius. By neglecting the terms in $\dot{r}$ and $\ddot{r}$, find an approximate solution for the time variation of r and θ in such an orbit. Deduce that small resistance causes the circular orbit to contract slowly, but that the satellite speeds up!

7.27 Repeat the last problem for the case in which the particle moves under the simple harmonic attractive field $-(m\Omega^2 r)\widehat{\boldsymbol{r}}$ with the same law of resistance. Show that, in this case, the body slows down as the orbit contracts. [This problem can be solved exactly in Cartesian coordinates, but do not do it this way.]

Computer assisted problems

7.28 ***See the advance of the perihelion of Mercury*** It is possible to 'see' the advance of the perihelion of Mercury predicted by general relativity by direct numerical solution. Take Einstein's path equation (see Problem 7.13) in the dimensionless form

$$\frac{d^2\upsilon}{d\theta^2} + \upsilon = \frac{1}{1-e^2} + \eta\upsilon^2,$$

where $\upsilon = au$. Here a and e are the semi-major axis and eccentricity of the non-relativistic elliptic orbit and $\eta = 3MG/ac^2$ is a small dimensionless parameter. For the orbit of Mercury, $\eta = 2.3 \times 10^{-7}$ approximately.

Solve this equation numerically with the initial conditions $r = a(1+e)$ and $\dot{r} = 0$ when $\theta = 0$; this makes $\theta = 0$ an aphelion of the orbit. To make the precession easy to see, use a fairly eccentric ellipse and take η to be about 0.005, which speeds up the precession by a factor of more than 10^4!

7.29 ***Orbit with linear resistance*** Confirm the approximate solution for small resistance obtained in Problem 7.26 by numerical solution of the governing simultaneous ODEs. First write the governing equations in dimensionless form. Suppose that, in the absence of resistance, a circular orbit with $r = a$ and $\dot{\theta} = \Omega$ is possible; then $\gamma = a^3\Omega$ and $L_0 = a^2\Omega$. On taking dimensionless variables ρ, τ defined by $\rho = r/a$ and $\tau = \Omega t$, and taking $L_0 = a^2\Omega$, the governing equations become

$$\frac{d^2\rho}{d\tau^2} + \epsilon\,\frac{d\rho}{d\tau} + \frac{1}{\rho^2} - \frac{e^{-2\epsilon\tau}}{\rho^3} = 0, \qquad \rho^2\frac{d\theta}{d\tau} = e^{-2\epsilon\tau},$$

where $\epsilon = K/\Omega$ is the dimensionless resistance parameter. Solve these equations with the initial conditions $\rho = 1$, $d\rho/d\tau = 0$ and $\theta = 0$ when $\tau = 0$. Choose some small value for ϵ and plot a polar graph of the path.