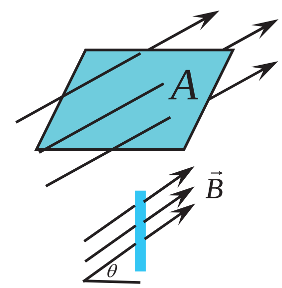
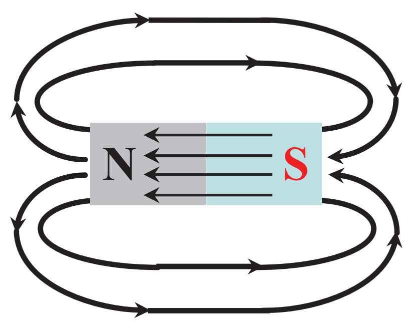
Summary of Lecture 28 – ELECTROMAGNETIC INDUCTION

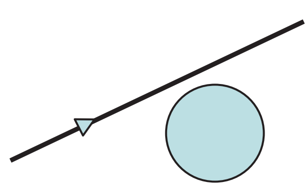
1. Earlier we had defined the flux of any vector field. For a magnetic field, this means that flux of a uniform magnetic field (see figure) is $\Phi = B_{\perp} A = BA\cos\theta$. If the field is not constant over the area then we must add up all the little pieces of flux: $\Phi_B = \int \vec{B}\cdot d\vec{A}$. The dimension of flux is magnetic field $\times$ area, and the unit is called weber, where $1 \text{ weber} = 1 \text{ tesla}\cdot\text{metre}^2$.



2. A fundamental law of magnetism states that the net flux through a closed surface is always zero, $\Phi_B = \int \vec{B}\cdot d\vec{A} = 0$. Note that this is very different from what you learned earlier in electrostatics where the flux is essentially the electric charge. There is no such thing as a magnetic charge! What we call the magnetic north (or south) pole of a magnet are actually due to the particular electronic currents, not magnetic charges. In the bar magnet below, no matter which closed surface you draw, the amount of flux leaving the surface is equal to that entering it.



Example : A sphere of radius R is placed near a long, straight wire that carries a steady current I. The magnetic field generated by the current is B. Find the total magnetic flux passing through the sphere.



Answer: zero, of course!

3. Faraday's Law for Induced EMF: when the magnetic flux changes in a circuit, an electro-

motive force is induced which is proportional to the rate of change of flux. Mathematically, $\mathcal{E} = -\frac{d\Phi_B}{dt}$ where $\mathcal{E}$ is the induced emf. If the coil consists of N turns, then $\mathcal{E} = -N\frac{d\Phi_B}{dt}$.

How does the flux through a coil change? Consider a coil and magnet. We can:

a) move the magnet,

b) change the size and shape of the coil by squeezing it,

c) move the coil.

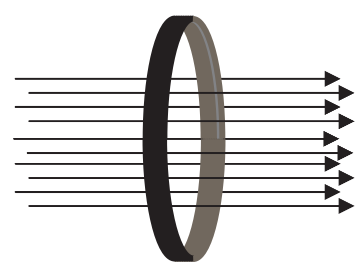
In all cases, the flux through the coil changes and $\frac{d\Phi_B}{dt}$ is non-zero leading to an induced emf.

Example: A flexible loop has a radius of 12cm and is in a magnetic field of strength 0.15T. The loop is grasped at points A and B and stretched until it closes. If it takes 0.20s to close the loop, find the magnitude of the average induced emf in it during this time.

Solution: Here the loop area changes, hence the flux. So the induced emf is:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \approx -\left[\frac{\text{final flux} - \text{initial flux}}{\text{time taken}}\right] = -\left[\frac{0 - \pi(0.12)^2 \times 0.15}{0.2}\right] = 0.034 \text{ Volts.}$$

Example : A wire loop of radius 0.30m lies so that an external magnetic field of +0.30T is perpendicular to the loop. The field changes to -0.20T in 1.5s. Find the magnitude of the average induced EMF in the loop during this time.



Solution: Again, we will find the initial and final fluxes first and then divide by the time taken for the change. Use $\Phi = BA = B\pi r^2$ to calculate the flux.

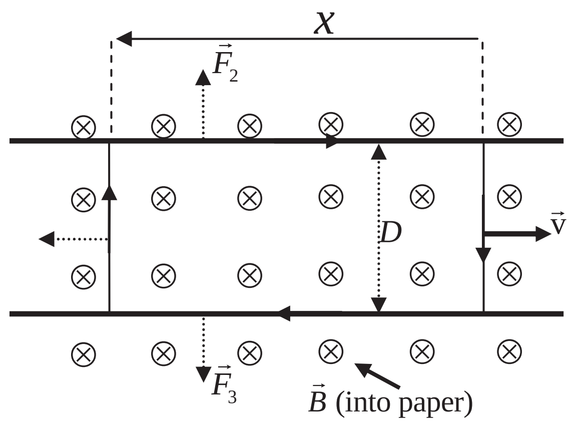
$$\Phi_i = \quad 0.30 \times \pi \times (0.30)^2 = 0.085 \text{ Tm}^2$$

$$\Phi_f = -0.20 \times \pi \times (0.30)^2 = -0.057 \text{ Tm}^2$$

$$\varepsilon = \frac{\Delta\Phi}{\Delta t} = \frac{\Phi_f - \Phi_i}{\Delta t} = \frac{0.085 - 0.057}{1.5} = 0.095 \text{ V}$$

4. Remember that the electromotive force is not really a force but the difference in electric potentials between two points. In going around a circular wire, where the electric field is constant as a function of angle, the emf is $\varepsilon = E(2\pi r)$. More generally, for any size or shape of a closed circuit, $\varepsilon = \int \vec{E} \cdot d\vec{s}$. So Faraday's Law reads: $\int \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$.

Example : A conducting wire rests upon two parallel rails and is pulled towards the right with speed v. A magnetic field B is perpendicular to the plain of the rails as shown below. Find the current that flows in the circuit.



Solution: As the wire is pulled to the right, the area of the circuit increases and so the flux increases. Measure x as above so that $x = \text{v}t,$ i.e. x keeps increasing as we pull. The flux at any value of x is $\Phi_B = BDx,$ and so,

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d\left(BDx\right)}{dt} = -BD\frac{dx}{dt} = BD\text{v}.$$

To calculate the current, we simply use Ohm's Law: $I = \frac{\varepsilon}{R} = \frac{BD\text{v}}{R}$. Here R is the resistance of the circuit.

Example : Find the power dissipated in the above circuit using $I^2R,$ and then by directly calculating the work you do by pulling the wire.

Solution: Clearly $I^2R = \frac{B^2D^2\text{v}^2}{R}$ is the power dissipated, as per usual formula. Now let us calculate the force acting upon the piece of wire (of length D) that you are pulling. From the formula for the force on a wire, $\vec{F} = I\,\vec{L} \times \vec{B},$ the magnitude is $F = IBD = \frac{B^2D^2\text{v}}{R}$. So, the power is $P = F\text{v} = \frac{B^2D^2\text{v}^2}{R}$. This is exactly the value calculated above!

5. **Lenz's Law :** The direction of any magnetic induction effect is such as to oppose the cause of the effect. Imagine a coil wound with wire of finite resistance. If the magnetic field decreases, the induced EMF is positive. This produces a positive current. The magnetic field produced by the current opposes the decrease in flux. Of course, because of finite resistance in loop, the induced current cannot completely oppose the change in flux.

QUESTIONS AND EXERCISES – 28

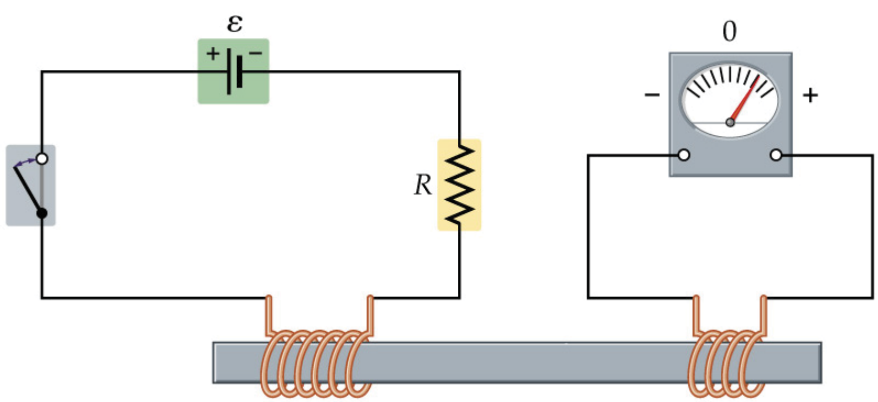
Q.1 a) Give an argument why field lines cannot cross.

b) If magnetic charges did exist, what would be the form of Gauss's Law for magnetism?

Q.2 You have a circular coil and a uniform magnetic field between the poles of a magnet. How will you spin the coil so that the maximum emf is induced in it?

Q.3 Land mines made of metal can be detected by sending in a pulse of magnetic field but plastic land mines cannot be detected in this way. Why?

Q.4 In the figure below the switch is closed in the primary coil and a current is induced in the secondary coil.



Answer the following:

a) What is the purpose of the iron bar? Will there be current induced in the secondary if the bar is removed and a wooden stick inserted instead?

b) If the value of R is increased/decreased, what will be the effect on the induced current?

c) To induce the maximum current in the secondary, should there be fewer/greater turns in the primary compared ti the secondary?

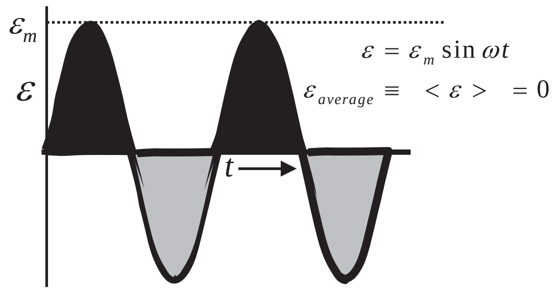
d) By what means is energy being transfered from the battery to the secondary?

e) What does Lenz's Law have to say about the primary current?

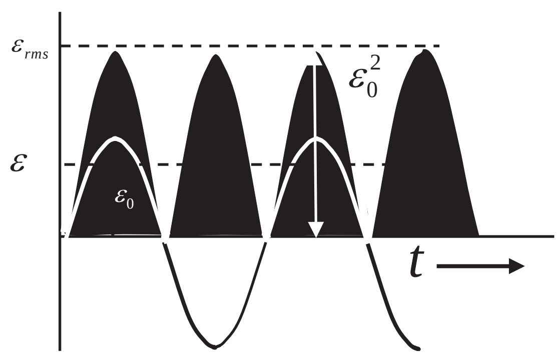
f) What is the value of the final current in the primary and secondary coils?

Summary of Lecture 29 – ALTERNATING CURRENT

1. Alternating current (AC) is current that flows first in one direction along a wire, and then in the reverse direction. The most common AC is sinuisoidal in which the current (and voltage) follow a sine function, as in the graph below. The average value is zero because the current flows for the same time in one direction as in the other.



However, the square of any AC wave is always positive. Thus its average is not zero. As we shall see, upon averaging the square we get half the square of the peak value.



The calculation follows: if there is a sine wave of amplitude (height) equal to one and frequency ω, ($\omega = 2\pi / T$, T=time period), then the squared amplitude is $\sin^2 \omega t$ and its average is:

$$<\sin^2 \omega t> \equiv \frac{1}{T}\int_0^T dt \sin^2 \omega t = \frac{1}{T}\int_0^T dt (\frac{1-\cos 2\omega t}{2}) = \frac{1}{T}\int_0^T dt \frac{1}{2} = \frac{1}{2}$$

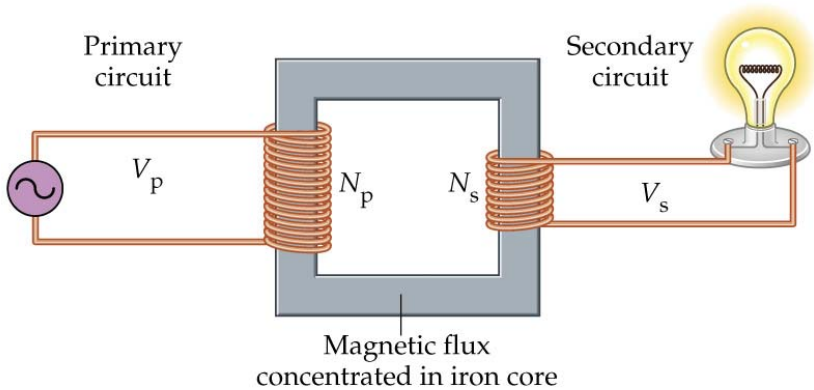
Taking the square root gives the root mean square value as $1/\sqrt{2}$ of the maximum value. Of course, it does not matter whether this is of the voltage or current:

$$\varepsilon_{rms} = \sqrt{\frac{\varepsilon_m^2}{2}} = \frac{\varepsilon_m}{\sqrt{2}} = 0.707\varepsilon_m, \text{ and } I_{rms} = \sqrt{\frac{I_m^2}{2}} = \frac{I_m}{\sqrt{2}} = 0.707 I_m.$$

Exactly the same results are obtained for cosine waves. This is what one expects since the difference between sine and cosine is only that one starts earlier than the other.

2. AC is generated by a coil rotating in a magnetic field. We know from Faraday's Law, $\mathcal{E} = -\frac{d\Phi_B}{dt}$, that a changing magnetic flux gives rise to an emf. Imagine a magnetic field and a coil of area A, rotating with frequency ω so that the flux through the coil at any instant of time is $\Phi_B = BA\cos\omega t$. Then the induced emf is $\varepsilon = BA\omega\sin\omega t$.

3. AC is particular useful because *transformers* make it possible to step up or step down voltages. The basic transformer consists of two coils - the primary and secondary - both wrapped around a core (typically iron) that enhances the magnetic field.



Suppose that the flux in the core is Φ and that its rate of change is $\frac{\partial\Phi}{\partial t}$. Call the number of turns in the primary and secondary N_p and N_s respectively. Then, from Faraday's law, the primary emf is $\varepsilon_p = -N_p\frac{\partial\Phi}{\partial t}$ and the secondary emf is $\varepsilon_s = -N_s\frac{\partial\Phi}{\partial t}$. The ratio is, $\frac{\varepsilon_p}{\varepsilon_s} = \frac{N_p}{N_s}$. So, the secondary emf is $\varepsilon_s = \frac{N_s}{N_p}\varepsilon_p$. If $\frac{N_s}{N_p}$ is less than one, then it is called a step-down transformer because the secondary voltage is less than the input voltage. Else, it is a step-up transformer. Both types are used.

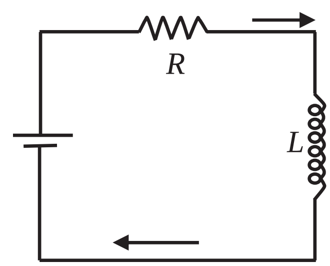
4. If this is a lossless transformer (and good transformers are 99% lossless), then the input power must equal the output power, $\varepsilon_p I_p = \varepsilon_s I_s$. From above, this shows that the ratio of currents is $I_s = \frac{N_p}{N_s}I_p$.

5. Whatever the shape or size of a current carrying loop, the magnetic flux that passes through it is proportional to the current, $\Phi \propto I$. The inductance L (called the self-inductance if

there is only one coil) is the constant of proportionality in the relation $\Phi = LI$. The unit of inductance is called Henry, 1 Henry= 1 Tesla metre^2 / Ampere. Note that inductance, like capacitance, is purely geometrical and depends only upon the shape and sizes of wires. It does not depend on the current.

6. Let us calculate the inductance of a long coil wound with n turns per unit length. As calculated earlier, the B field is $B = \mu_0 nI$, and the flux passing through N turns of the coil of length l and area A is $N\Phi_B = (nl)(BA) = \mu_0 n^2 IAl$. From the definition, we find:
$L = \frac{N\Phi_B}{I} = \frac{\mu_0 n^2 lIA}{I} = \mu_0 n^2 lA$ (inductance of long solenoid).
Example: Find the inductance of a coil with 3500 turns, length 10 cm, and radius 5cm.
Solution: $L = 4\pi \cdot 10^{-7} \frac{T \cdot m}{A} \cdot 3500^2 \cdot \frac{\pi(0.05m)^2}{0.10m} = 1.21 \frac{T \cdot m^2}{A} = 1.21H$.

7. When the current changes through an inductor, by Faraday's Law it induces an emf equal to $-\frac{d\Phi_B}{dt} = -\frac{d(LI)}{dt} = -L\frac{dI}{dt}$. Let us use this to calculate the current through the circuit shown here, where a resistor is present. Then, by using Kirchoff's Law, $\varepsilon = IR + L\frac{dI}{dt}$ or $\varepsilon I = I^2 R + LI\frac{dI}{dt}$. In words: the power expended by the battery equals energy dissipated in the resistor + work done on inductor.



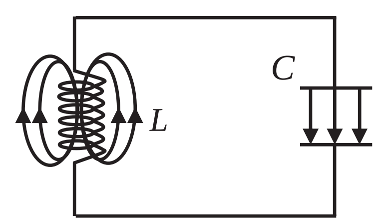
8. Before we go on to the AC case, suppose that we suddenly connect a battery to an inductor so that the voltage suddenly increases from zero to ε across the circuit. Then, $L\frac{dI}{dt} + IR = \varepsilon$ has solution: $I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$ where $\tau = \frac{L}{R}$. You can easily see that this true using $\frac{dI}{dt} = \frac{\varepsilon}{R}\frac{1}{\tau}e^{-t/\tau}$. The solution shows that the current increases from zero to a maximum of $\frac{\varepsilon}{R}$, i.e. that given by Ohm's Law. We need to pay a little attention to the "time constant" τ, which is the time after which the constant approaches 63% of its final value. Units: $[\tau] = \frac{[L]}{[R]} = \frac{\text{henry}}{\text{ohm}} = \frac{\text{volt.second / ampere}}{\text{ohm}} = \left(\frac{\text{volt}}{\text{ampere.ohm}}\right)\text{second} = \text{second}.$

Note that when t=τ, then $I = \frac{\varepsilon}{R}\left(1-e^{-1}\right) = \frac{\varepsilon}{R}\left(1-0.37\right) = \frac{\varepsilon}{R}0.63$

9. When we pass current through an inductor a changing magnetic field is produced. This, by Faraday's Law, induces an emf across the coil. So work has to be done to force the current through. How much work? The power, or rate of doing work, is emf $\times$ current. Let U_B be the work done in passing current I. Then, $\frac{dU_B}{dt} = \left(L\frac{dI}{dt}\right)I = LI\frac{dI}{dt}$. Let us integrate $dU_B = LI\,dI$. Then, $\int_0^{U_B} dU_B = \int_0^I LIdI \quad \Rightarrow \; U_B = \frac{1}{2}LI^2$. This is an important result. It tells us that an inductor L carrying current I requires work $\frac{1}{2}LI^2$. By conservation of energy, this is also the energy stored in the inductor. Compare this result with the result $U_E = \frac{1}{2}CV^2$ for a capacitor. Notice that $C \leftrightarrow L$ and $V \leftrightarrow I$.

10. Let us use the result derived earlier for the inductance of a solenoid and the magnetic field in it, $L = \mu_0 n^2 lA$ and $B = \mu_0 nI$. Putting this into $U_B = \frac{1}{2}LI^2$ gives $U_B = \frac{1}{2}(\mu_0 n^2 lA)(B/\mu_0 n)^2$. Divide the energy by the volume of the solenoid, $\frac{U_B}{\text{volume}} = \frac{B^2}{2\mu_0}$. This directly gives the energy density (energy per unit volume) contained in a magnetic field.

11. **Electromagnetic oscillations in an LC circuit.**
We know that energy can be stored in a capacitor as well as in an inductor. What happens when we connect them up together and put some charge on the capacitor? As it discharges, it creates a current that transfers energy to the inductor. The total energy remains constant, of course.



This means that the sum $U = U_B + U_E = \frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C}$ is constant. Differentiate this:

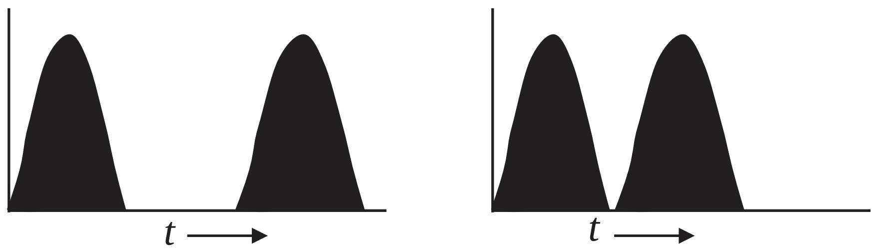
$$0 = \frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C}\right). \text{ Since } I = \frac{dq}{dt}, \text{ and } \frac{dI}{dt} = \frac{d^2q}{dt^2}, \; \therefore \quad \frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

To make this look nicer, put $\omega^2 \equiv \frac{1}{LC} \Rightarrow \frac{d^2q}{dt^2} + \omega^2 q = 0$. We have seen this equation many times earlier. The solution is: $q = q_m \cos\omega t$. The important result here is that the charge, current, and voltage will oscillate with frequency $\omega = \sqrt{\frac{1}{LC}}$. This oscillation will go on for forever if there is no resistance in the circuit.

QUESTIONS AND EXERCISES – 29

Q.1 a) What is the average value of the half-wave sine currents (maximum value=1) below?

b) Repeat the above for the square of the currents below.



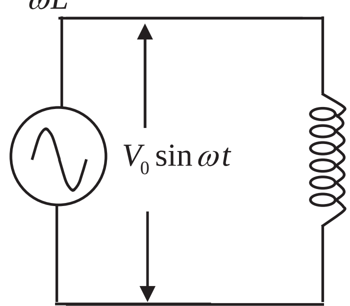
Q.2 Suppose you have a wire that you have to wind into a small space. How should you wind it so that a)the inductance is maximum, b)the inductance is minimum?

Q.3 a) Two coils with inductance L_1 and L_2 are connected in series. Calculate the total L.

b) The two coils are now connected in parallel. Calculate the total L.

c) In both the above parts, what assumption did you have to make? Would your answers be correct if the two coils were close to each other?

Q.4 In the circuit below, first show that Kirchoff's Law gives $L\frac{dI}{dt} = V_0 \sin \omega t$. Then show that the solution is: $I = -\frac{V_0}{\omega L}\cos \omega t = \frac{V_0}{\omega L}\sin(\omega t - \frac{\pi}{2})$.



Q.5 In point no. 11 above, the charge on the capacitor was calculated to be $q = q_m \cos \omega t$.

a) What is meaning of q_m? Relate this to the maximum current that flows in the circuit.

b) What is the maximum value of the energy stored in the capacitor? In the inductor?

c) At what value of the time will there be equal energy in both parts?