Summary of Lecture 22 – ELECTROSTATICS I

1. Like charges repel, unlike charges attract. But by how much? Coulomb's Law says that this depends both upon the strength of the two charges and the distance between them. In mathematical terms, $F \propto \frac{q_1 q_2}{r^2}$ which can be converted into an equality, $F = k\frac{q_1 q_2}{r^2}$. The constant of proportionality will take different values depending upon the units we choose. In the MKS system, charge is measured in Coulombs (C) and $k = \frac{1}{4\pi\varepsilon_0}$ with $\varepsilon_0 = 8.85\times 10^{-12} C^2/Nm^2$ and hence $k = 8.99\times 10^9 Nm^2/C^2$.

2. The situation is quite similar to that of gravity, except that electric charges and not masses are the source of force. In vector form, $\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}$ is the force exerted by 2 on 1, where the unit vector is $\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$. On the other hand, $\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{21}$ is the force exerted by 1 on 2. By Newton's Third Law, $\vec{F}_{12} = -\vec{F}_{21}$. For many charges, the force on charge 1 is given by, $\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \cdots$

3. Charge is quantized. This means that charge comes in certain units only. So the size of a charge can only be 0, $\pm e, \pm 2e, \pm 3e, \cdots$ where $e = 1.602\times 10^{-19} C$ is the value of the charge present on a proton. By definition we call the charge on a proton positive. This makes the charge on an electron negative.

4. Charge is conserved. This means that charge is never created or destroyed. Equivalently, in any possible situation, the total charge at an earlier time is equal to the charge at a later time. For example, in any of the reactions below the initial charge = final charge:

   $e^- + e^+ \to \gamma + \gamma$ (electron and positron annihilate into neutral photons)

   $\pi^0 \to \gamma + \gamma$ (neutral pion annihilates into neutral photons)

   $H^2 + H^2 \to H^3 + p$ (two deuterons turn into tritium and proton)

5. **Field :** this a quantity that has a definite value at any point in space and at any time. The simplest example is that of a scalar field, which is a *single number* for any value of x, y, z, t. Examples: temperature inside a room $T(x,y,z,t)$, density in a blowing wind $\rho(x,y,z,t)$, $\cdots$ There are also *vector fields,* which comprise of three numbers at each value of x, y, z, t. Examples: the velocity of wind, the pressure inside a fluid, or even a sugarcane field. In

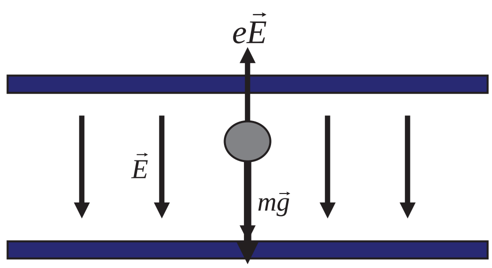
every case, there are 3 numbers: $\vec{V}(x,y,z,t)=\{V_1(x,y,z,t),V_2(x,y,z,t),V_3(x,y,z,t)\}$.

6. The electric field is also an example of a vector field, and will be the most important for our purpose. It is defined as the force on a unit charge. Or, since we don't want the charge to disturb the field it is placed in, we should properly define it as the force on a "test" charge q_0, $E \equiv \dfrac{F}{q_0}$. Here q_0 is very very small. The electric field due to a point charge can be calculated by considering two charges. The force between them is $F=\dfrac{1}{4\pi\varepsilon_0}\dfrac{qq_0}{r^2}$ and so $E=\dfrac{F}{q_0}=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}$. A way to visualize E fields is to think of lines starting on positive charges and ending on negative charges. The number of lines leaving/entering gives the amount of charge.

7. Typical values for the maginitude of the electric field E:

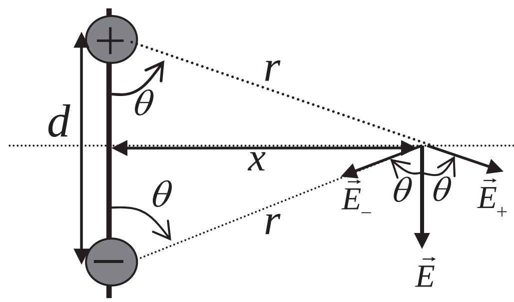
Inside an atom-	10^{11} N/C
Inside TV tube-	10^{5} N/C
In atmosphere-	10^{2} N/C
Inside a wire-	10^{-2} N/C

8. Measuring charge. One way to do this is to balance the gravitational force pulling a charged particle with mass m with the force exerted on it by a known electric field (see below). For equilibrium, the two forces must be equal and so $mg=qE$. The unknown charge q can then be found from $q=\dfrac{mg}{E}$.



9. Given several charges, one can find the total electric field at any point as the sum of the fields produced by the charges at that point individually, $\vec{E}=\vec{E}_1+\vec{E}_2+\vec{E}_3+\cdots\cdots$ or $\vec{E}=\sum_i \vec{E}_i = k\sum_i \dfrac{q_i}{r_i^2}\hat{r}_i \quad (i=1,2,3,\cdots)$. Here $\hat{r}_i$ is the unit vector pointing from the charge to the point of observation.

10. Let us apply the principle we have just learned to a system of two charges $+q$ and $-q$ which are separated by a distance d (see diagram). Then, $\vec{E}=\vec{E}_{+}+\vec{E}_{-}$. Just to make things easier (not necessary; one can do it for any point) I have taken a point that lies on the x-axis. The magnitudes of the electric field due to the two charges are equal; $E_{+}=E_{-}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}=\frac{1}{4\pi\varepsilon_0}\frac{q}{x^2+(d/2)^2}$. The vertical components cancel out, and the net electric field is directed downwards with magnitude, $E=E_{+}\cos\theta+E_{-}\cos\theta=2E_{+}\cos\theta$. From the diagram below you can see that $\cos\theta=\frac{d/2}{\sqrt{x^2+(d/2)^2}}$. Substituting this, we find: $E=2\frac{1}{4\pi\varepsilon_0}\frac{q}{x^2+(d/2)^2}\frac{d/2}{\sqrt{x^2+(d/2)^2}}=\frac{1}{4\pi\varepsilon_0}\frac{qd}{\left[x^2+(d/2)^2\right]^{3/2}}$.



11. The result above is so important that we need to discuss it further. In particular, what happens if we are very far away from the dipole, meaning $x>>d$? Let us first define the *dipole moment* as the product of the charge $\times$ the separation between them $p=qd$. Then, $E=\frac{1}{4\pi\varepsilon_0}\frac{p}{x^3}\frac{1}{\left[1+(d/2x)^2\right]^{3/2}}=\frac{1}{4\pi\varepsilon_0}\frac{p}{x^3}\left[1+\left(\frac{d}{2x}\right)^2\right]^{-3/2}=\frac{1}{4\pi\varepsilon_0}\frac{p}{x^3}$. In the above, $d/2x$ has been neglected in comparison to 1. So finally, we have found that for $x>>d$, $E=\frac{1}{4\pi\varepsilon_0}\frac{p}{x^3}$.

12. It is easy to find the torque experienced by an electric dipole that is placed in a uniform electric field: The magnitude is $\tau=F\frac{d}{2}\sin\theta+F\frac{d}{2}\sin\theta=Fd\sin\theta,$ and the direction is perpendicular and into the plane. Here θ is the angle between the dipole and the electric field. So $\tau=(qE)d\sin\theta=pE\sin\theta$.

QUESTIONS AND EXERCISES – 22

1. What will be the *direction* of the electric field in each of the following circumstances :
   (a) In the middle of a square with four equal positive charges at the corners?
   (b) In the middle of a square with 3 equally positive and one negative charge at the corners?
   (c) At the centre of a hollow sphere with charges distributed uniformly on the surface?
   (d) At the centre of a hollow sphere with the charge on one hemisphere is positive, and the charge on the other hemisphere is negative.

2. How could you experimentally investigate (as a matter of principle) where the lines of electric force are? Why can two lines of electric force never cross each other?

3. (a) The electric field of a dipole does not fall off as $\frac{1}{r^2}$. Why?
   (b) Instead of calculating the electric field of the dipole, use the figure on the previous page to calculate the electric potential on the x-axis.

4. Work is done by an electric field acting upon a dipole because the dipole is turned through a certain angle. So, $W = \int dW = \int_{\theta_0}^{\theta} \vec{\tau}.d\vec{\theta} = \int_{\theta_0}^{\theta} -\tau d\theta.$
   (a) Why is there a negative sign in the last equality?
   (b) Show that $W = pE\left(\cos\theta - \cos\theta_0\right)$
   (c) Show that the change in potential energy is $\Delta U = U\left(\theta\right) - U\left(\theta_0\right) = -W$
   (d) Show that the potential energy can also be written as $U = -\vec{p}.\vec{E}$

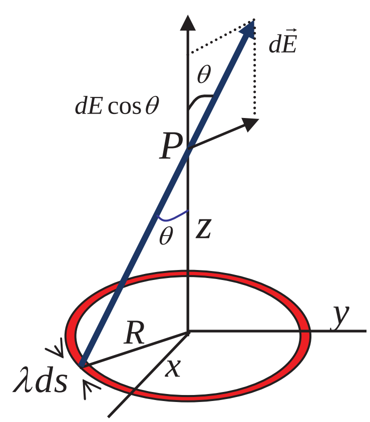
Summary of Lecture 23 – ELECTROSTATICS II

1. In the last lecture we learned how to calculate the electric field if there are any number of point charges. But how to calculate this when charges are continuously distributed over some region of space? For this, we need to break up the region into little pieces so that each piece is small enough to be like a point charge. So, $\vec{E} = \Delta\vec{E}_1 + \Delta\vec{E}_2 + \Delta\vec{E}_3 + \cdots$, or $\vec{E} = \sum \Delta\vec{E}_i$ is the total electric field. Remember that $\vec{E}$ is a vector that can be resolved into components, $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$. In the limit where the pieces are small enough, we can write it as an integral, $\vec{E} = \int d\vec{E}$ (or $E_x = \int dE_x$, $E_y = \int dE_y$, $E_z = \int dE_z$)

2. Charge Density: when the charges are continuously distributed over a region - a line, the surface of a material, or inside a sphere - we must specify the *charge density*. Depending upon how many dimensions the region has, we define:
   - (a) For linear charge distribution: $dq = \lambda ds$
   - (b) For surface charge distribution: $dq = \sigma dA$
   - (c) For volume charge distribution: $dq = \rho dV$

   The dimensions of λ, σ, ρ are determined from the above definitions.

3. As an example of how we work out the electric field coming from a continuous charge distribution, let us work out the electric field from a uniform ring of charge at the point P.



The small amount of charge λds gives rise to an electric field whose magnitude is

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{r^2} = \frac{\lambda ds}{4\pi\varepsilon_0\left(z^2 + R^2\right)}$$

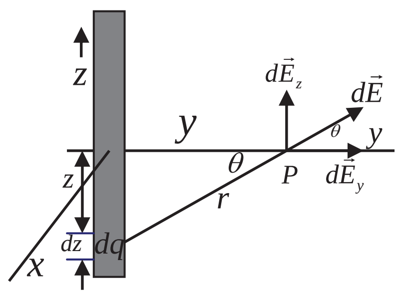
The component in the z direction is $dE_z = dE\cos\theta$ with $\cos\theta = \frac{z}{r} = \frac{z}{\left(z^2 + R^2\right)^{1/2}}$.

So $dE_z = \frac{z\lambda ds}{4\pi\varepsilon_0 \left(z^2 + R^2\right)^{3/2}}$. Since s, which is the arc length, does not depend upon z or R,

$$E_z = \frac{z\lambda}{4\pi\varepsilon_0 \left(z^2 + R^2\right)^{3/2}} \int ds = \frac{z\lambda(2\pi R)}{4\pi\varepsilon_0 \left(z^2 + R^2\right)^{3/2}} = \frac{qz}{4\pi\varepsilon_0 \left(z^2 + R^2\right)^{3/2}}.$$ Answer!!

Note that if you are very far away, the ring looks like a point: $E_z = \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2}, \quad (z >> R).$

4. As another example, consider a continuous distribution of charges along a wire that lies along the z-axis, as shown below. We want to know the electric field at a distance x from the wire. By symmetry, the only non-cancelling component lies along the y-axis.



Applying Coulomb's law to the small amount of charge λdz along the z axis gives,

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dz}{y^2 + z^2}$$

the component along the y direction is $dE_y = dE\cos\theta.$ Integrating this gives,

$$E_y = \int dE = \int_{z=-\infty}^{z=\infty} \cos\theta dE = 2\int_{z=0}^{z=\infty} \cos\theta dE = \frac{\lambda}{2\pi\varepsilon_0}\int_{z=0}^{z=\infty} \cos\theta \frac{dz}{y^2 + z^2}.$$

The rest is just technical: to solve the integral, put $z = y\tan\theta \Rightarrow dz = y\sec^2\theta d\theta$. And so,

$E = \frac{\lambda}{2\pi\varepsilon_0 y}\int_{\theta=0}^{\theta=\pi/2} \cos\theta d\theta = \frac{\lambda}{2\pi\varepsilon_0 y}$. Now, we could have equally well taken the x axis. The only thing that matters is the distance from the wire, and so the answer is better written as:

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

5. The **flux** of any vector field is a particularly important concept. It is the measure of the "flow" or penetration of the field vectors through an imaginary fixed surface. So, if there is a uniform electric field that is normal to a surface of area A, the flux is $\Phi = EA$. More generally, for any surface, we divide the surface up into little pieces and take the

component of the electric field normal to each little piece, $\Phi_E = \sum \vec{E}_i \cdot \Delta \vec{A}_i$. If the pieces are made small enough, then in this limit, $\Phi = \int \vec{E} \cdot d\vec{A}$.

5. Let us apply the above concept of flux to calculate the flux leaving a sphere which has a charge at its centre. The electric field at any point on the sphere has magnitude equal to $\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$ and it is directed radially outwards. Let us now divide up the surface of the sphere into small areas. Then $\Phi = \sum E\Delta A = E\sum \Delta A = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left(4\pi r^2\right)$. So we end up with the important result that the flux leaving this closed surface is $\Phi = \frac{q}{\varepsilon_0}$.

6. **Gauss's Law**: the total electric flux leaving a closed surface is equal to the charge enclosed by the surface divided by ε_0. We can express this directly in terms of the mathematics we have learned, $\Phi \equiv \int \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}$. Actually, we have already seen why this law is equivalent to Coulomb's Law in point 5 above, but let's see it again. So, applying Gauss's Law to a sphere containing charge, $\varepsilon_0 \int \vec{E} \cdot d\vec{A} = \varepsilon_0 \int E\, dA = q_{enclosed}$. If the surface is a sphere, then E is constant on the surface and $\varepsilon_0 E \int dA = q$ and from this $\varepsilon_0 E\left(4\pi r^2\right) = q \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$. This is Coulomb's Law again, but the power of Gauss's law is that it holds for any shape of the (closed) surface and for any distribution of charge.

7. Let us apply Gauss's Law to a hollow sphere that has charges only on the surface. At any distance r from the centre, Gauss's Law is $\varepsilon_0 E\left(4\pi r^2\right) = q_{enclosed}$. Now, if we are inside the sphere then $q_{enclosed} = 0$ and there is no electric field. But if we are outside, then the total charge is $q_{enclosed} = q$ and $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$, which is as if all the charge was concentrated at the centre.

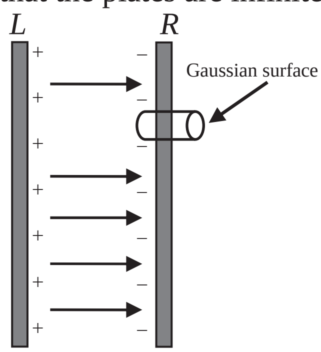
8. Unfortunately, it will not be possible for me to prove Gauss's Law in the short amount of time and space available but the general method can be outlined as follows: take any volume and divide it up into little cubes. Each little cube may contain some small amount of charge. Then show that for each little cube, Gauss's Law follows from Coulomb's Law. Finally, add up the results. For details, consult any good book on electromagnetism.

QUESTIONS AND EXERCISES – 23

Q.1 As we all know, everything is made up of atoms and the only charges present inside matter reside on atoms or electrons. So everything is ultimately "bumpy". When then is the concept of a continuous distribution of charge used in this lecture? Under what conditions would this become inadequate, or even wrong?

Q.2 Give the dimensions of λ, σ, ρ using their definitions. In 1,2,3 dimensions write the expression for the total charge.

Q.3 In the diagram below you see two parallel plates with surface charge density $+\sigma$ and -σ respectively. We would like to calculate the electric field everywhere using Gauss's theorem, and the assumption that the plates are infinitely long in the vertical direction.



a) Show that the electric field vanishes everywhere except in between the two plates. Use Gauss's Law to show this. [Hint: draw a suitable gaussian surface that extends both to the left and right, and use the fact that the total enclosed charge is zero.]

b) Look at the gaussian surface that has been drawn in the diagram and evaluate each of terms defining the flux, $\Phi = \oint \vec{E} \cdot d\vec{A} = \int_{\substack{\text{outer}\\\text{cap}}} \vec{E} \cdot d\vec{A} + \int_{\substack{\text{inner}\\\text{cap}}} \vec{E} \cdot d\vec{A} + \int_{\substack{\text{side}\\\text{walls}}} \vec{E} \cdot d\vec{A}$

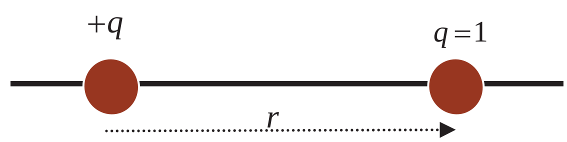
c) Show that the field between the plates is given by $E = \dfrac{\sigma}{\varepsilon_0}$.

d) Suppose that one plate is totally removed and that the charge density is $+\sigma$ on the remaining plate. What is the electric field now?

Summary of Lecture 24 – ELECTRIC POTENTIAL ENERGY

1. You are already familiar with the concept of gravitational potential energy. When you lift a weight, you have to do work against the downwards pull of the Earth. That work is stored as potential energy. Suppose a force $\vec{F}$ acts on something and displaces it by $d\vec{s}$. Then the work done is $\vec{F}\cdot d\vec{s}$. The work done in going from point a to point b (call it W_{ab}) is then got by adding together the little bits of work, $W_{ab} = \int_a^b \vec{F}\cdot d\vec{s}$. The change in potential energy is defined as $\Delta U = U_b - U_a = -W_{ab}$. Remember always that we can only define the potential U at a point if the force is conservative.

2. The electrostatic force is conservative and can be represented by a potential. Let us see how to calculate the potential. So consider two charges separated by a distance as below.



Let us take the point a very far from the fixed charge q, and the unit charge at the point b to be at a distance R from q. Then the work you did in bringing the unit charge from infinity to R is, $W_{ab} = \int_\infty^R (-qE)dr = -\frac{1}{4\pi\varepsilon_0}\int_\infty^R dr\frac{q}{r^2} = -\frac{1}{4\pi\varepsilon_0}q\left(\frac{1}{R}-\frac{1}{\infty}\right) = -\frac{q}{4\pi\varepsilon_0}\frac{1}{R}$. Since the charges repel each other, it is clear that you had to do work in pushing the two charges closer together. So where did the negative sign come from? Answer: the force you exert on the unit charge is directed *towards* the charge q, i.e. is in the negative direction. This is why $\vec{F}\cdot d\vec{s} = (-qE)dr$. Now $\Delta U = U(R) - U(\infty) = \frac{q}{4\pi\varepsilon_0}\frac{1}{R}$. If we take the potential at ∞ to be zero, then the electric potential due to a charge q at the point r is $U(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}$.

Remember that we know how to calculate the force given the potential: $F = -\frac{dU}{dr}$. Apply this here and you see that $F = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}$, (which also has the correct repulsive sign).

3. From the above, it is quite obvious that the potential energy of two charges q_1, q_2 is, $U(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$. Compare this with the formula for gravitational energy, $U(r) = -G\frac{m_1 m_2}{r}$. What is the difference? From here, you can see that the gravitational force is always negative (which means attractive), whereas the electrostatic force can be both attractive

or repulsive because we have both + and - charges in nature.

3. The electric potential (or simply potential) is the energy of a unit charge in an electric field. So, in our MKS units, the unit of potential is $1\ \dfrac{\text{Joule}}{\text{Coulomb}} = 1$ Volt. Another useful unit is "electron volt" or eV. The definition is:

$$\textit{One electron - volt = energy gained by moving one electron charge through one Volt}$$
$$= \left(1.6\times10^{-19}C\right)\times 1V = 1.6\times10^{-19}J$$

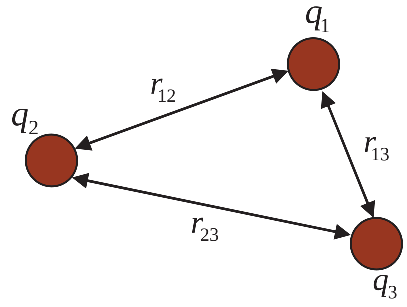
It is useful to note that 1 Kev $= 10^3 eV$ (kilo-electron-volt)
1 Mev = $10^6 eV$ (million-electron-volt)
1 Gev $= 10^9 eV$ (giga-electron-volt)
1 Tev $= 10^{12} eV$ (tera-electron-volt)

4. Every system seeks to minimize its potential energy (that is why a stone falls down!). So, positive charges accelerate toward regions of lower potential, but negative charges accelerate toward regions of higher potential. Note that only the potential difference matters - even if a charge is placed in a region where there is a high potential, it will not want to move unless there is some other place where the potential is higher/lower.

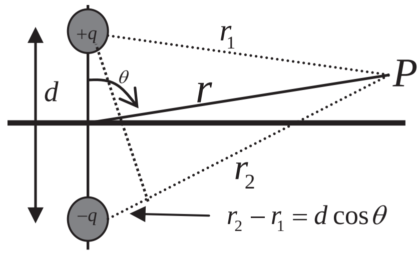
5. Given a system of charges, we can always compute the force - and hence the potential - that arises from them. Here are some important general statements:
a)Potentials are more positive in regions which have more positive charge.
b)The electric potential is a scalar quantity (a scalar field, actually).
c)The electric potential determines the force through $F = -\dfrac{dU}{dr}$, and hence the electric field because $F = qE$.
d)The electric potential exists only because the electrostatic force is conservative.

6. To compute the potential at a point, the potentials arising from charges 1, 2, $\cdots$ N must be added up:
$V = V_1 + V_2 + \cdots V_N = \sum_{i=1}^{N} V_i = \dfrac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\dfrac{q_i}{r_i}$. Here r_i is the distance of the i'th charge from the point where the potential is being calculated or measured. As an example, the potential from the three charges is:

$$\mathrm{V}(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{12}} + \frac{1}{4\pi\varepsilon_0}\frac{q_1q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2q_3}{r_{23}}.$$

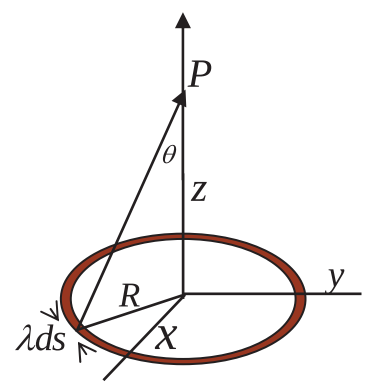


7. Let us apply these concepts to the dipole system considered earlier. With two charges, $V_P = V_1 + V_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} + \frac{-q}{r_2}\right) = \frac{q}{4\pi\varepsilon_0}\frac{r_2 - r_1}{r_1 r_2}$. We are particularly interested in the situation where $r >> d$. from the diagram you can see that $r_2 - r_1 \approx d\cos\theta$ and that $r_1 r_2 \approx r^2$. Hence, $V \approx \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$. So we have calculated the potential at any θ with such little difficulty. Note that $V = 0$ at $\theta = \frac{\pi}{2}$.



8. Now let us calculate the potential which comes from charges that are uniformly spread over a ring. This is the same problem as in the previous lecture, but simpler. Give the small amount of potential coming from the small amount of charge $dq = \lambda ds$ some name, $dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$. Then obviously

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{R^2 + z^2}}.$$



QUESTIONS AND EXERCISES – 24

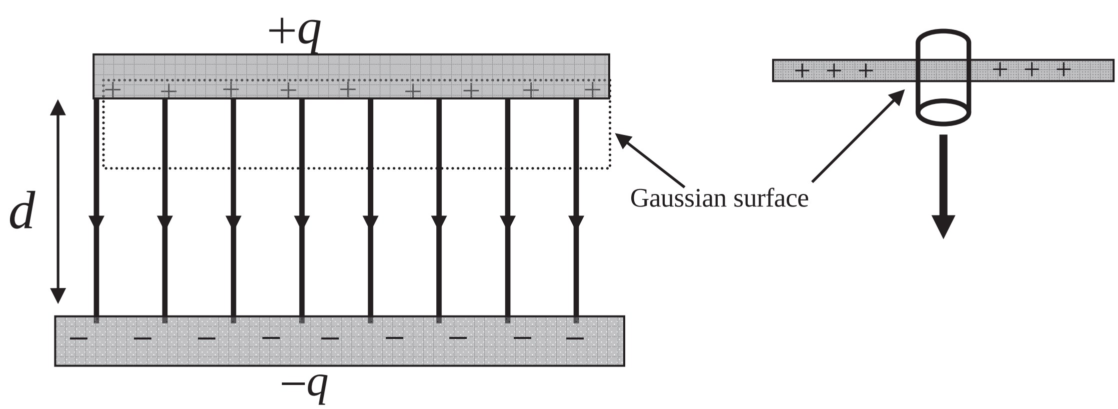
Q.1 How much work (in joules) is done by a force that moves a charge of 2 coulombs through a distance of 10cm in a constant electric field of 7 V/cm?

Q.2 It is common to call the potential of the earth as zero. Is this necessary? Is it okay to do so, and why?

Q.3 There are two hollow metal spheres each with one coulomb of charge upon it. One has radius 4cm and the other 14cm. Find the ratio of the electric fields on the two surfaces. Repeat if instead they have the same charge density σ on the surfaces.

Summary of Lecture 25 – CAPACITORS AND CURRENTS

1. Two conductors isolated from one another and from their surroundings, form a *capacitor*. These conductors may be of any shape and size, and at any distance from each other. If a potential difference is created between the conductors (say, by connecting the terminals of a battery to them), then there is an electric field in the space between them. The electric field comes from the charges that have been pushed to the plates by the battery. The amount of charge pushed on to the conductors is proportional to the potential difference between the battery terminals (which is the same as between the capacitor plates). Hence, $Q \propto V$. To convert this into an equality, we write $Q = CV$. This provides the definition of capacitance, $C = \frac{Q}{V}$.
2. Using the above definition, let us calculate the capacitance of two parallel plates separated by a distance d as in the figure below.

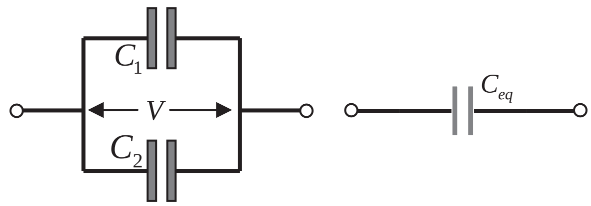


Recall Gauss's Law: $\Phi \equiv \int \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}$. Draw any Gaussian surface. Since the electric field is zero above the top plate, the flux through the area A of the plate is $\Phi = EA = \frac{Q}{\varepsilon_0}$, where Q is the total charge on the plate. Thus, $E = \frac{Q}{\varepsilon_0 A}$ is the electric field in the gap between the plates. The potential difference is $V = \frac{E}{d}$, and so $C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}$. You can see that the capacitance will be large if the plates are close to each other, and if the plates have a large area. We have simplified the calculation here by assuming that the electric field is strictly directed downwards. This is only true if the plates are infinitely long. But we can usually neglect the side effects. Note that any arrangement with two plates forms a capacitor: plane, cylindrical, spherical, etc. The capacitance depends upon the geometry, the size of plates and the gap between them.

3. One can take two (or more) capacitors in various ways and thus change the amount of charge they can contain. Consider first two capacitors connected in parallel with each other. The same voltage exists across both. For each capacitor, $q_1 = C_1 V$, $q_2 = C_2 V$ where V is the potential between terminals *a* and *b*. The total charge is:

$$Q = q_1 + q_2 = C_1 V + C_2 V = (C_1 + C_2)V$$

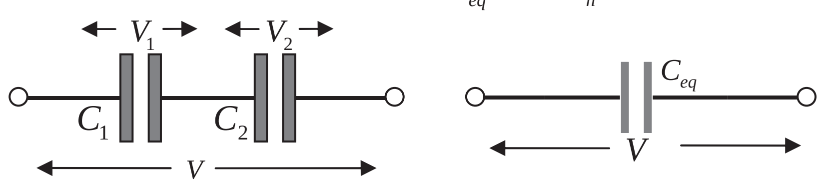
Now, let us define an "effective" or "equivalent" capacitance as $C_{eq} = \frac{Q}{V}$. Then we can immediately see that for 2 capacitors $C_{eq} = C_1 + C_2$, and $C_{eq} = \sum C_n$ (for *n* capacitors).



4. We can repeat the analysis above when the capacitors are put in series. Here the difference is that now we must start with $V = V_1 + V_2$, where V_1 and V_2 are the voltages across the two. Clearly the same charge had to cross both the capacitors. Hence,

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = Q(\frac{1}{C_1} + \frac{1}{C_2}).$$

From our definition, $C_{eq} = \frac{Q}{V}$, it follows that $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$. The total capacitance is now less than if they were in parallel. In general, $\frac{1}{C_{eq}} = \sum \frac{1}{C_n}$ (for *n* capacitors).



5. When a battery is connected to a capacitor, positive and negative charges appear on the opposite plates. Some energy has been transferred from the battery to the capacitor, and now been stored in it. When the capacitor is discharged, the energy is recovered. Now let us calculate the energy required to charge a capacitor from zero to V volts.
   Begin: the amount of energy required to transfer a small charge dq to the plates is $dU = \text{v}dq$, where v is the voltage at a time when the charge is $q = C\text{v}$. As time goes on, the total charge increases until it reaches the final charge Q (at which point the voltage becomes V). So,

$$dU = \text{v}dq = \frac{q}{C}dq \quad \Rightarrow \quad U = \int dU = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C} = \frac{1}{2}CV^2.$$

But where in the capacitor is the energy stored? Answer, it is present in the electric field in the volume between the two plates. We can calculate the energy density:

$$u = \frac{\text{energy stored in capacitor}}{\text{volume of capacitor}} = \frac{U}{Ad} = \frac{\frac{1}{2}CV^2}{Ad} = \frac{\varepsilon_0}{2}\left(\frac{V}{d}\right)^2 = \frac{1}{2}\varepsilon_0 E^2.$$

In the above we have used $C = \frac{\varepsilon_0 A}{d}$, derived earlier. The important result here is that $u \propto E^2$. Turning it around, wherever there is an electric field, there is energy available.

6. **Dielectrics.** Consider a free charge $+Q$. Around it is an electric field, $E = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{Q}{r^2}$.

Now suppose this charge is placed among water molecules. These molecules will polarise, i.e. the centre of positive charge and centre of negative charge will be slightly displaced. The negative part of the water molecule will be attracted toward the positive charge $+Q$. So, in effect, the electric field is weakened by $\frac{1}{\varepsilon_r}$ and becomes, $\frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{Q}{r^2}$. Here I have introduced a new quantity ε_r called "dielectric constant". This is a number that is usually bigger than one and measures the strength of the polarization induced in the material. For air, $\varepsilon_r = 1.0003$ while $\varepsilon_r \approx 80$ for pure water. The effect of a dielectric is to increase the capacitance of a capacitor: if the air between the plates of a capacitor is replaced by a dielectric, $C = \frac{\varepsilon_0 A}{d} \rightarrow \varepsilon_r \frac{\varepsilon_0 A}{d}$.

QUESTIONS AND EXERCISES – 25

Q.1 Suppose a dielectric material is inserted between the plates of a parallel plate capacitor. What will happen to the electric field in the gap between plates? To the total stored energy in the capacitor?

Q.2 Two capacitors of 10 μF and 15 μF are joined together in parallel. A third capacitor of 15 μF is placed in series with these two. What is the total capacitance?

Q.3 Two metal spheres of radius R_1 (small) and R_2 (large) have the same charge Q initially. When connected by a wire, how much charge will flow in it?