Instanton Physics

A short course

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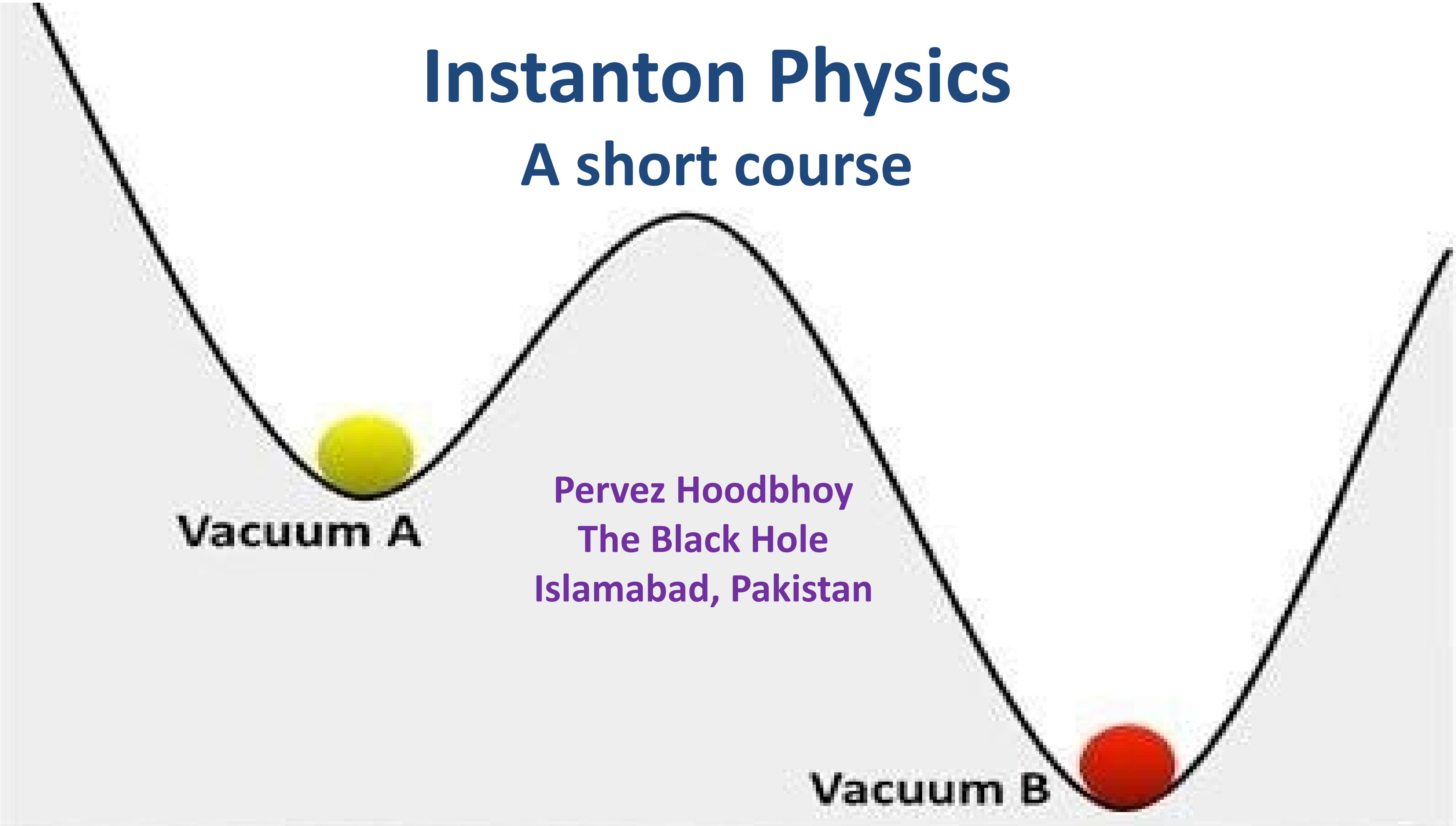


Course Outline

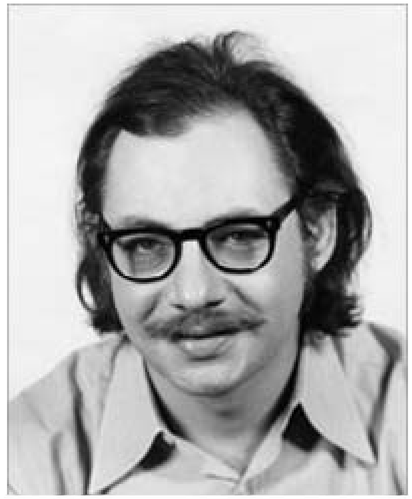
Instantons in particle QM

- Intro to path integral
- Imaginary time
- Symmetric double well
- Decay of metastable states
- The functional determinant

Tunneling of quantum fields

- Basic QFT for a scalar field
- Tunneling of field configurations
- The O(4) instanton
- Gauge fields and multiple vacuua
- Effective action in QFT
- How/when will the universe end?

Textbook: *The Theory and Applications of Instanton Calculations* by Manu Paranjape, Cambridge University Press (2022).



PHYSICAL REVIEW D VOLUME 15, NUMBER 10 15 MAY 1977

Fate of the false vacuum: Semiclassical theory*

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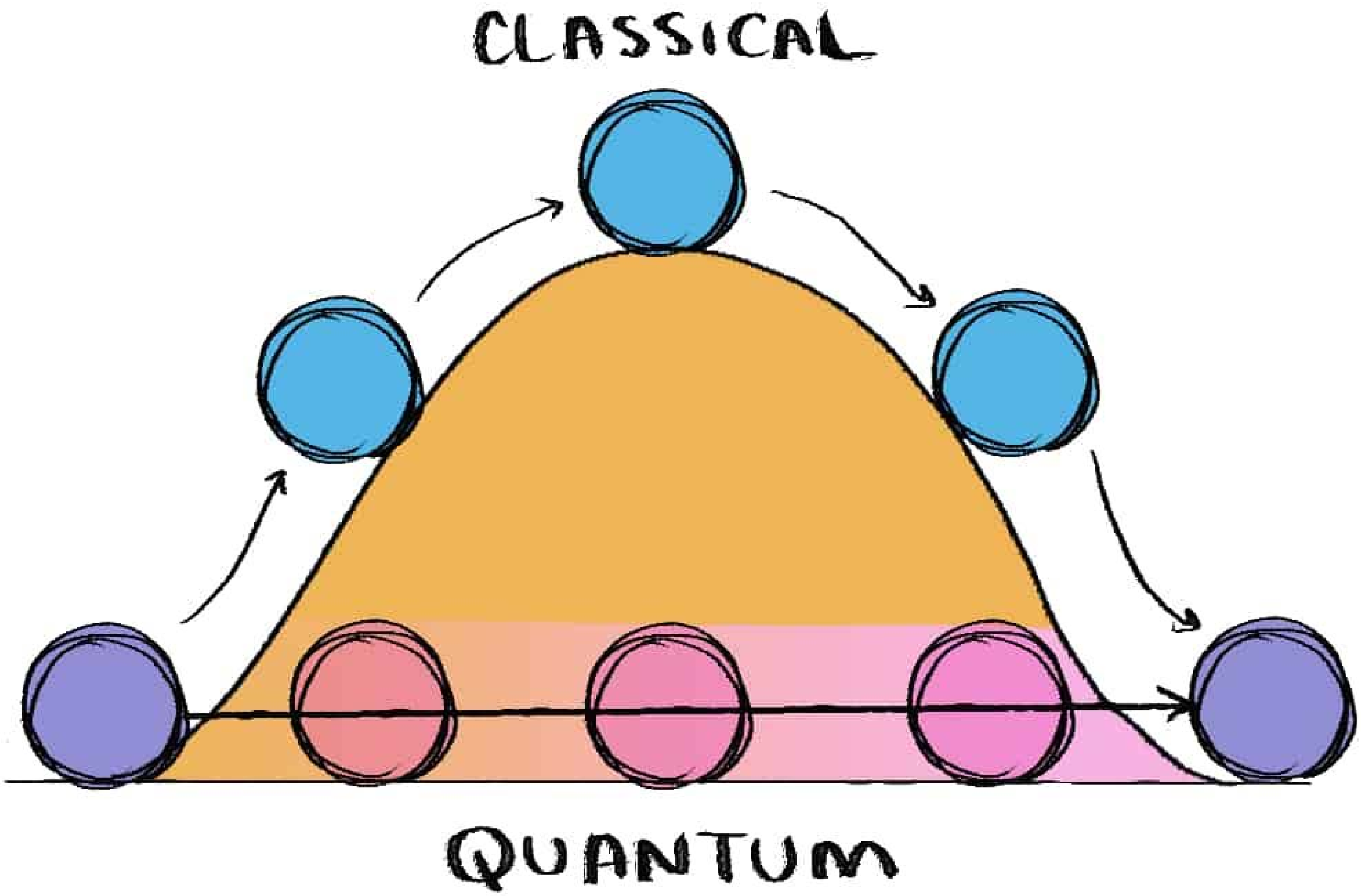
(Received 24 January 1977)

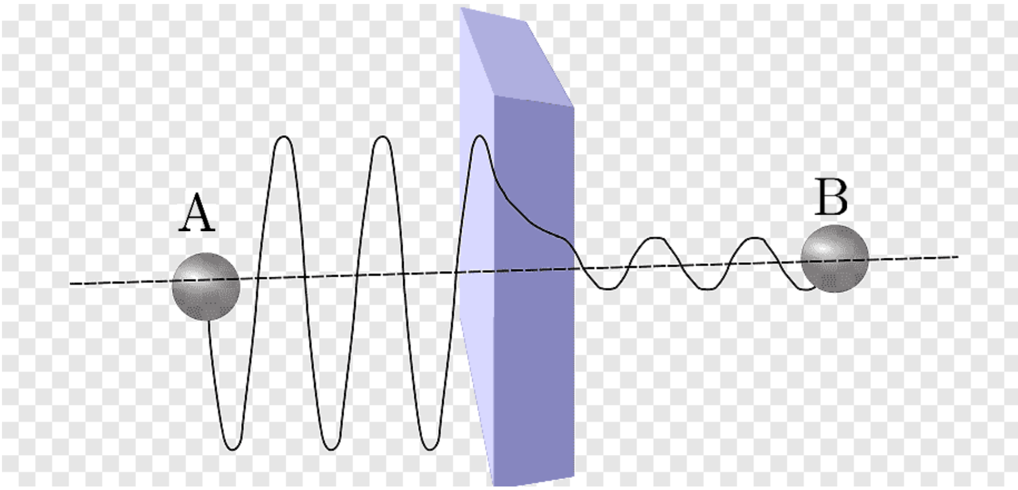
It is possible for a classical field theory to have two homogeneous stable equilibrium states with different energy densities. In the quantum version of the theory, the state of higher energy density becomes unstable through barrier penetration; it is a false vacuum. This is the first of two papers developing the qualitative and quantitative semiclassical theory of the decay of such a false vacuum for theories of a single scalar field with nonderivative interactions. In the limit of vanishing energy density between the two ground states, it is possible to obtain explicit expressions for the relevant quantities to leading order in $\hbar$; in the more general case, the problem can be reduced to solving a single nonlinear ordinary differential equation.

“Vacuum decay is the ultimate ecological catastrophe; in a new vacuum there are new constants of nature; after vacuum decay, not only is life as we know it impossible, so is chemistry as we know it. However, one could always draw stoic comfort from the possibility that perhaps in the course of time the new vacuum would sustain, if not life as we know it, at least some creatures capable of knowing joy. This possibility has now been eliminated.”

Sidney Coleman, Frank De Luccia (1980)


CLASSICAL
QUANTUM

Undergraduate quantum mechanics

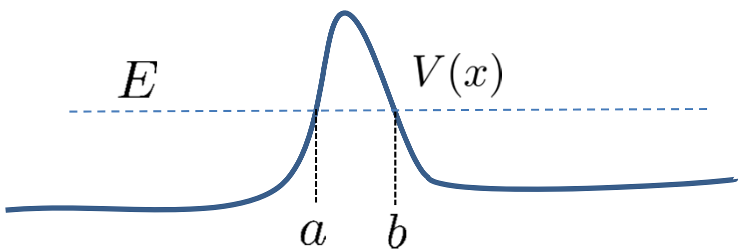


WKB approximation for tunneling amplitude

$$\mathcal{A}_{A\to B} \sim e^{-\frac{2}{\hbar}\int_a^b dx\sqrt{V(x)-E}}$$

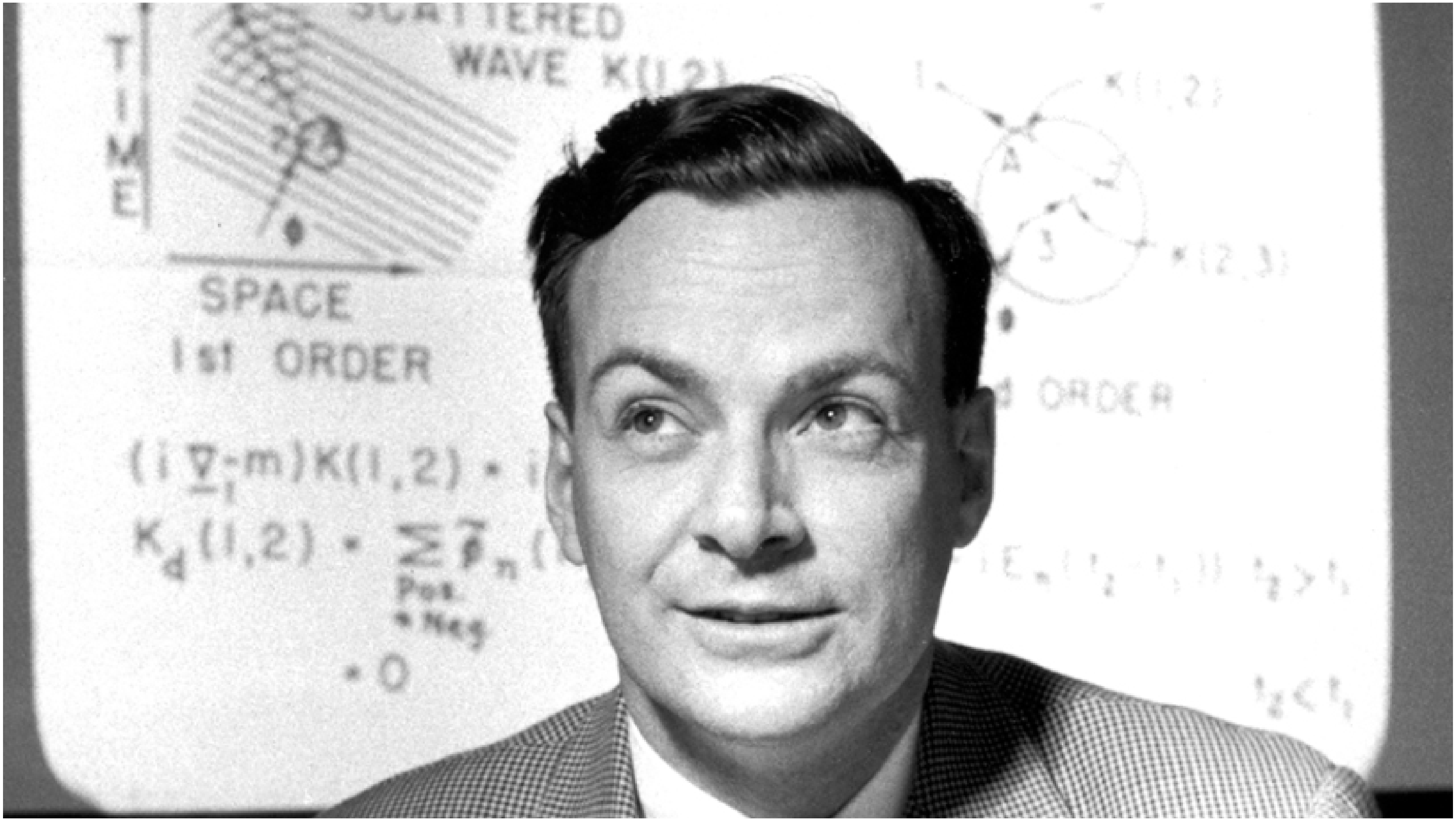
a, b are classical turning points

$$V(a) = V(b) = E$$

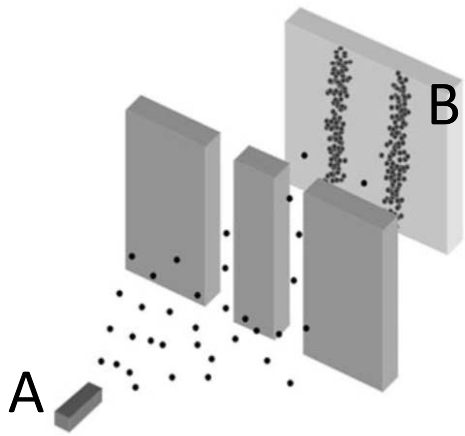


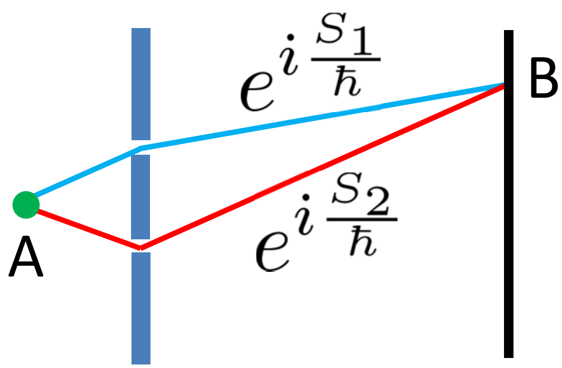
- Fundamentally non-perturbative
- Difficult to improve systematically

An alternative to WKB is provided by the Feynman path integral.

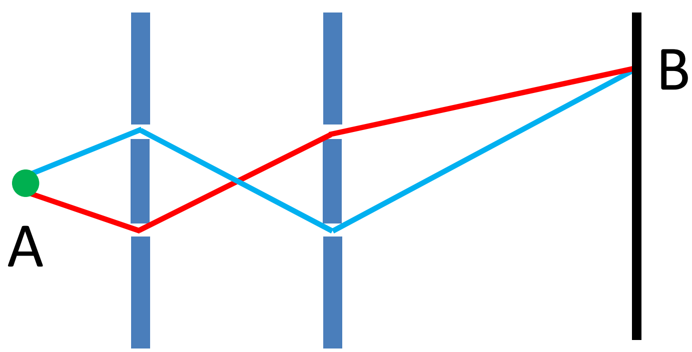

SCATTERED
WAVE
TIME
SPACE
1st ORDER
ORDER

Double Slit Experiment



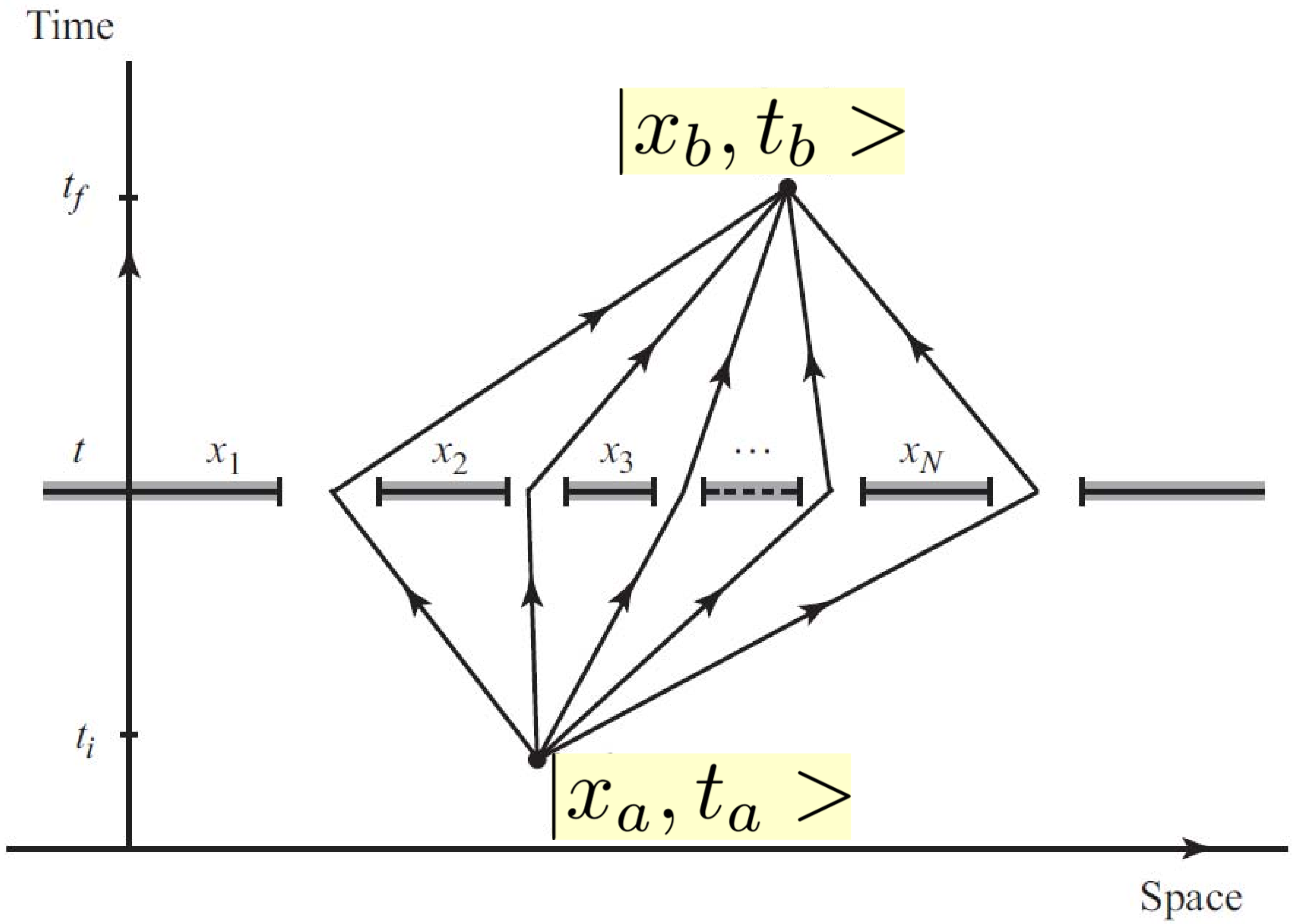


$$\mathscr{A}_{A\to B} = e^{i\frac{S_1}{\hbar}} + e^{i\frac{S_2}{\hbar}}$$

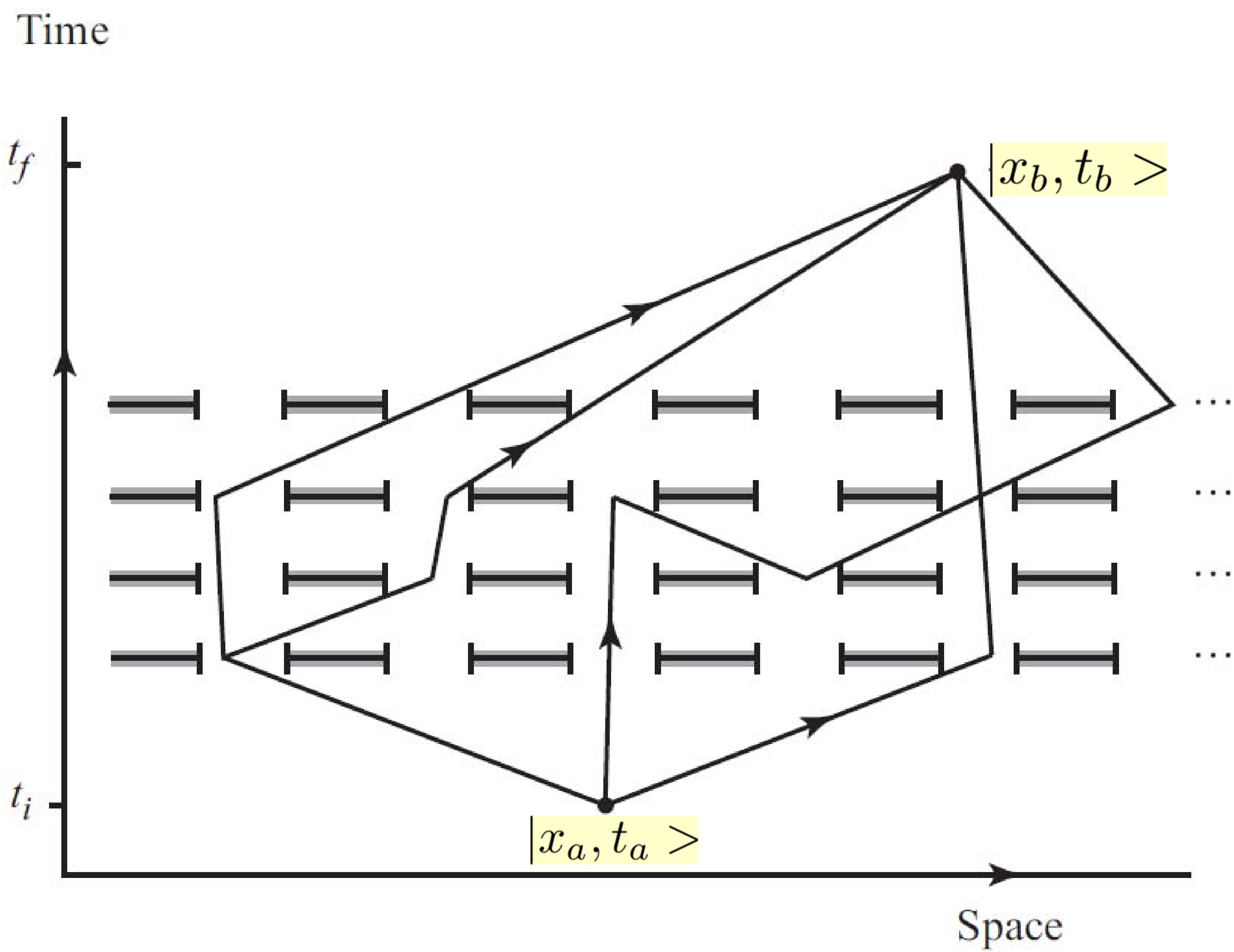


$$\mathscr{A}_{A\to B} = e^{i\frac{S_1}{\hbar}} + e^{i\frac{S_2}{\hbar}} + e^{i\frac{S_3}{\hbar}} + e^{i\frac{S_4}{\hbar}}$$

$$= \sum_{\text{paths}} e^{i\frac{S}{\hbar}} \to \int [dx] e^{i\frac{S}{\hbar}}$$



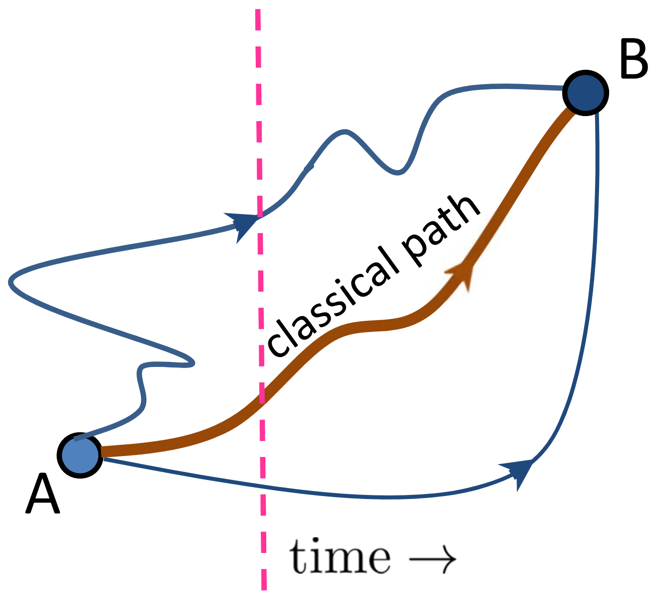
Time
t_f
t
t_i
x_1
x_2
x_3
···
x_N
$|x_b, t_b>$
$|x_a, t_a>$
Space



Time
t_f
$|x_b, t_b>$
t_i
$|x_a, t_a>$
Space

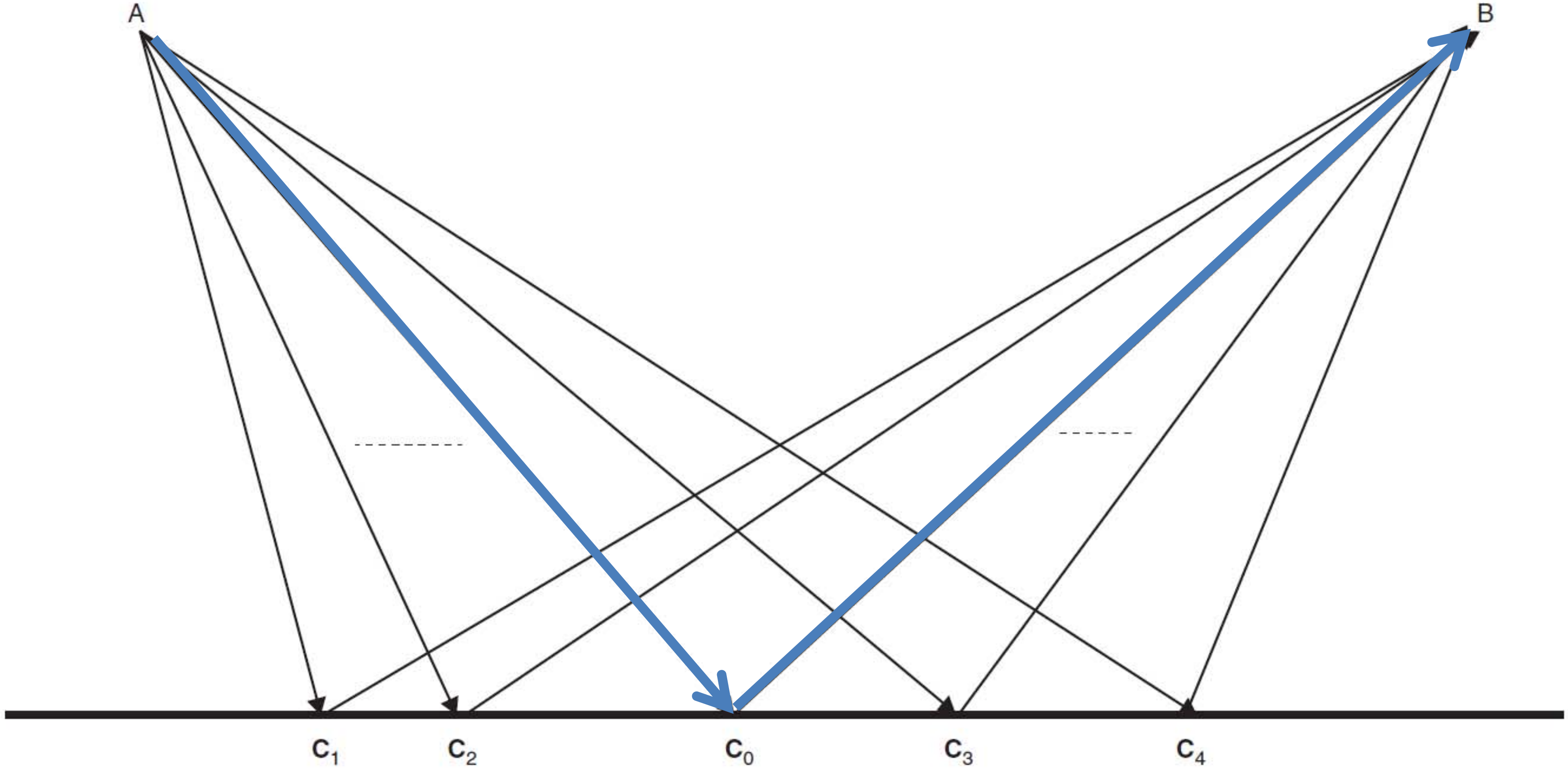
Feynman Path Integral

$$\mathscr{A}_{A\to B} = \sum_{\text{paths}} e^{i\frac{S}{\hbar}} \to \int [dx] e^{i\frac{S}{\hbar}}$$



- Which paths are allowed?
- How many paths?
- How can the particle travel *simultaneously* on all paths?
- Why do all paths have exactly the same weight?
- Why do we see only one single (classical) path?

Reflected light also follows the classical path



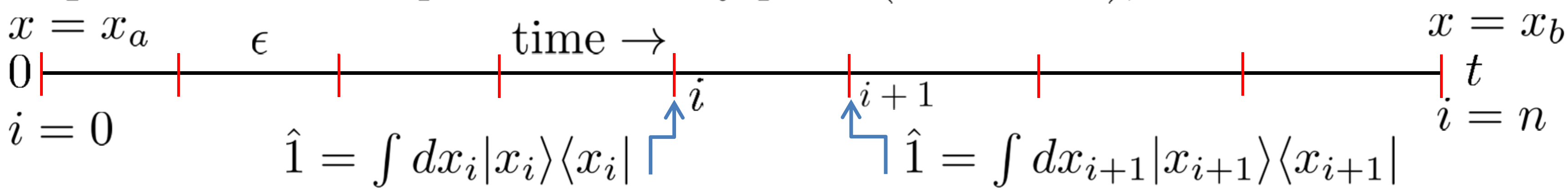
Deriving the Feynman path integral - I

$$\mathscr{A}_{a\to b} = \langle x_b | e^{-\frac{i}{\hbar}\hat{H}(t_b - t_a)} | x_a \rangle \qquad \hat{H} = \frac{\hat{p}^2}{2} + V(\hat{x}) \qquad [\hat{x}, \hat{p}] = i\hbar$$

Step 1: Take $t_a = 0, t_b = t$ and slice time $[0, t]$ into n pieces, $t = n\epsilon$.

$$e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{\hbar}\hat{H}\epsilon} \cdots e^{-\frac{i}{\hbar}\hat{H}\epsilon} = e^{-\frac{i}{\hbar}(\frac{\hat{p}^2}{2} + V(\hat{x}))\epsilon} \cdots e^{-\frac{i}{\hbar}(\frac{\hat{p}^2}{2} + V(\hat{x}))\epsilon}$$

Step 2: Insert unit operator at every point (here $n = 8$),



$$\mathscr{A}_{a\to b} = \int dx_1 \cdots \int dx_{n-1} \langle x_b | e^{-\frac{i}{\hbar}\hat{H}\epsilon} | x_{n-1}\rangle \cdots \langle x_{i+1} | e^{-\frac{i}{\hbar}\hat{H}\epsilon} | x_i \rangle \cdots \langle x_1 | e^{-\frac{i}{\hbar}\hat{H}\epsilon} | x_a \rangle$$

$$= \mathscr{A}_{a\to 1}\mathscr{A}_{1\to 2} \cdots \mathscr{A}_{i\to i+1} \cdots \mathscr{A}_{n-1\to b}$$

Step 3: Find an expression for $\mathscr{A}_{i\to i+1} \equiv \langle x_{i+1} | e^{-\frac{i}{\hbar}\hat{H}\epsilon} | x_i \rangle$

$$\langle x_{i+1}|e^{-\frac{i}{\hbar}\hat{H}\epsilon}|x_i\rangle = \langle x_{i+1}|e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2}\epsilon}\, e^{-\frac{i}{\hbar}V(\hat{x})\epsilon}|x_i\rangle$$

Insert $\hat{1} = \int dp|p\rangle\langle p|$ here

$$e^{-\frac{i}{\hbar}V(\hat{x})\epsilon}|x_i\rangle = e^{-\frac{i}{\hbar}V(x_i)\epsilon}|x_i\rangle$$

$$\langle x_{i+1}|e^{-\frac{i}{\hbar}\hat{H}\epsilon}|x_i\rangle = \int_{-\infty}^{\infty} dp\; e^{-\frac{i}{\hbar}\frac{p^2}{2}\epsilon}\langle x_{i+1}|p\rangle\langle p|x_i\rangle e^{-\frac{i}{\hbar}V(x)\epsilon}$$

$$\langle x_{i+1}|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}px_{i+1}} \qquad \langle p|x_i\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}px_i}$$

$$= e^{-\frac{i}{\hbar}V(x)\epsilon}\int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\; e^{-\frac{i}{\hbar}\frac{p^2}{2}\epsilon}e^{\frac{i}{\hbar}p(x_{i+1}-x_i)}$$

Use: $\int_{-\infty}^{\infty} dx e^{-iax^2} = \sqrt{\frac{\pi}{ia}}$

$$= e^{\frac{i}{2\hbar}\left(\frac{x_{i+1}-x_i}{\epsilon}\right)^2\epsilon}\; e^{-\frac{i}{\hbar}V(x)\epsilon}\int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\; e^{-\frac{i}{2\hbar}\left(p-\frac{x_{i+1}-x_i}{\epsilon}\right)^2\epsilon}$$

$n \to \infty$

$$e^{i\frac{\epsilon}{\hbar}\left(\frac{\dot{x}^2}{2}-V(x)\right)}\int_{-\infty}^{\infty}\frac{ds}{2\pi\hbar}\; e^{-\frac{i\epsilon}{2\hbar}s^2} = \sqrt{\frac{1}{2\pi\hbar i\epsilon}}e^{\frac{1}{\hbar}\epsilon\mathcal{L}}$$

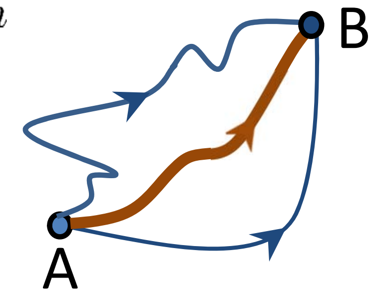
Deriving the Feynman path integral - II

$$\mathscr{A}_{a\to b} = \int dx_1 \cdots \int dx_{n-1} \langle x_b | e^{-\frac{i}{\hbar}\hat{H}\epsilon} | x_{n-1}\rangle \cdots \langle x_{i+1} | e^{-\frac{i}{\hbar}\hat{H}\epsilon} | x_i\rangle \cdots \langle x_1 | e^{-\frac{i}{\hbar}\hat{H}\epsilon} | x_a\rangle$$

$$= \mathscr{A}_{a\to 1}\mathscr{A}_{1\to 2}\cdots\mathscr{A}_{i\to i+1}\cdots\mathscr{A}_{n-1\to b}$$

$$= \left[\frac{1}{2\pi i\hbar\epsilon}\right]^{\frac{n}{2}} \int dx_1 \cdots \int dx_{n-1} \underbrace{e^{\frac{i}{\hbar}\epsilon\mathscr{L}} \ldots \; e^{\frac{i}{\hbar}\epsilon\mathscr{L}} \ldots}_{n}$$

$$\mathscr{A}_{a\to b} = \int [dx] \exp\left(\frac{i}{\hbar}S\right)$$



where, $[dx] = \lim_{n\to\infty}\left(\frac{n}{2\pi i\hbar t}\right)^{\frac{n}{2}} \prod_{i=1}^{n-1} dx_i$ and, $S[x] = \int_0^t dt\, \mathscr{L}(x(t), \dot{x}(t))$

Computation in simple cases of: $\mathscr{A}_{a\to b} = \int [dx] \exp\left(\frac{i}{\hbar} S\right)$

Do Gaussian integrals one-by-one using $\int_{-\infty}^{\infty} dx e^{-iax^2} = \sqrt{\frac{\pi}{ia}}$

Free Particle: $V(x) = 0$

$$\mathscr{A}_{a\to b} = \mathscr{N} \exp\left[\frac{i}{2\hbar} \frac{(x_b - x_a)^2}{t_b - t_a}\right] \qquad \mathscr{N} = \sqrt{\frac{1}{2\pi\hbar i(t_b - t_a)}}$$

Simple Harmonic Oscillator: $V(x) = \frac{1}{2}\omega^2 x^2$

$$\mathscr{A}_{a\to b} = \mathscr{N} \exp\left[\frac{i\,\omega}{2\hbar \sin \omega(t_b - t_a)} [(x_b^2 + x_a^2) \cos \omega t - 2x_a x_b]\right]$$

$\mathscr{A}_{a\to b} =$ is also called the Green's function $G(x_b, t_b; x_a, t_a)$