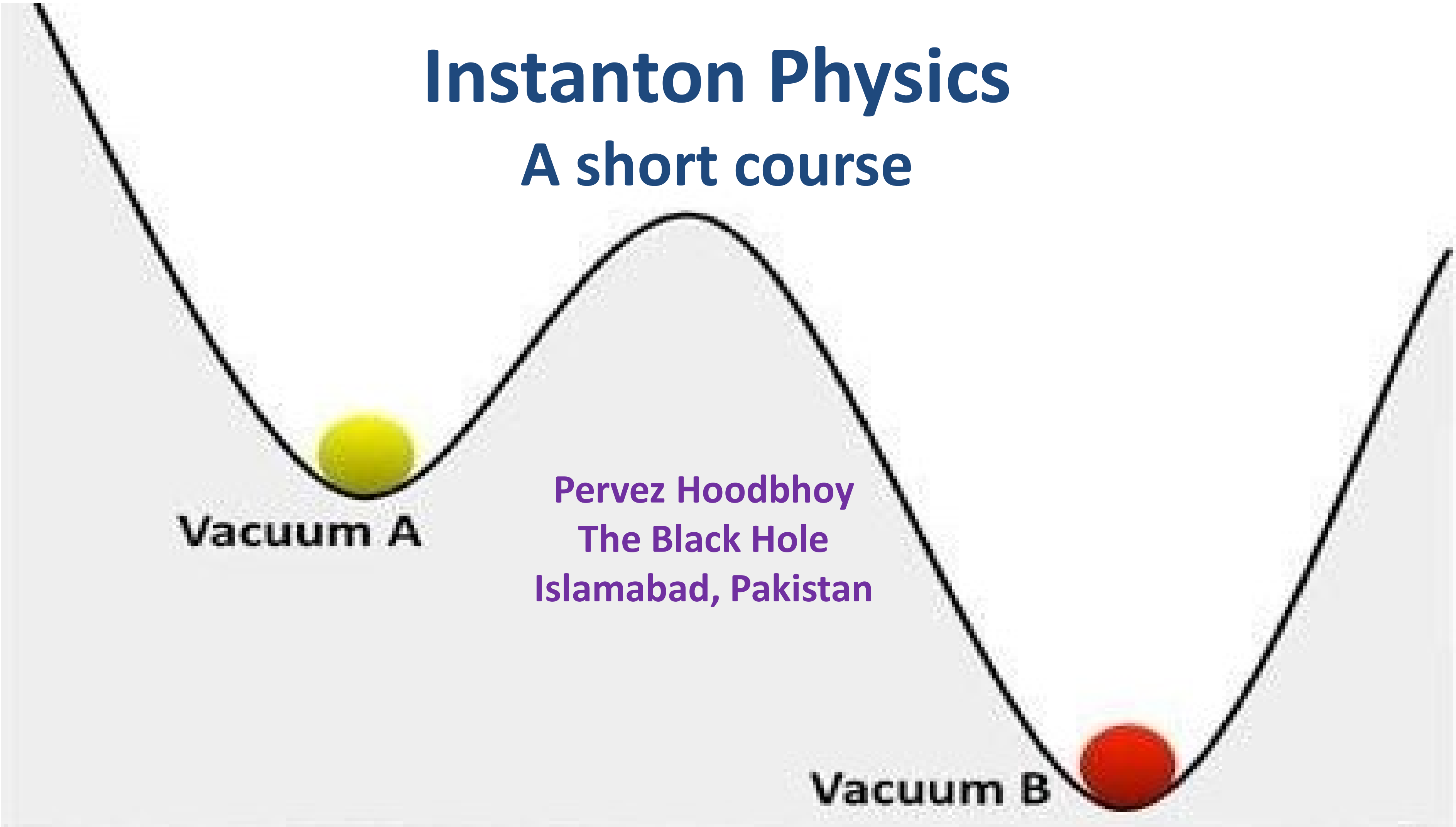
Instanton Physics

A short course



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Course Outline

Instantons in particle QM

- Intro to path integral ✔
- Imaginary time
- Instantons in a symmetric double well
- Decay of metastable states
- The functional determinant

Tunneling of quantum fields

- Basic QFT for a scalar field
- Tunneling of field configurations
- The O(4) instanton
- Gauge fields and tunneling
- Effective action
- How/when will the universe end?

Review of functional analysis

- 

  $$f(x) = x^2$$

  $$F[g] = \int_a^b dx L(x, g(x), g'(x), g''(x), \cdots)$$

- $$\frac{df(x)}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

  $$\frac{\delta F[g(y)]}{\delta g(x)} = \lim_{\epsilon \to 0} \frac{F[g(y) + \epsilon \delta(x-y)] - F[g(y)]}{\epsilon}$$

- $$f(x) = x^2 \to f' = 2x$$

  $$F[g(x)] = g(x)^2 \to F' = 2\delta(x-y)$$

- Usual Taylor expansion: $f(x+\delta x) = f(x) + \delta x f'(x) + \frac{1}{2}\delta x^2 f''(x) + \cdots$

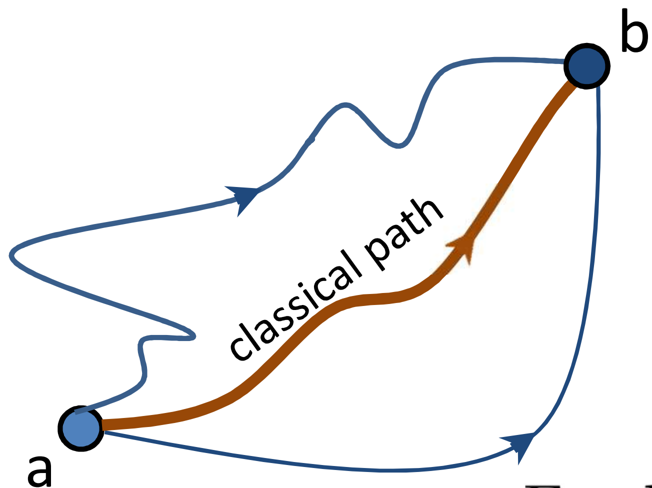
  Similarly there's a Taylor expansion for functionals

  $$F[g(x) + \delta g(x)] = F[g(x)] + \int_a^b dx' \frac{\delta F[g(x)]}{\delta g(x')} \delta g(x')$$

  $$+\frac{1}{2} \int_a^b dx' \int_a^b dx'' \frac{\delta^2 F[g(x)]}{\delta g(x') \delta g(x'')} \delta g(x') \delta g(x'') + \cdots$$

Real Time Feynman Path Integral (derived in lecture 1)

$$\mathscr{A}_{a\to b} = \sum_{\text{paths}} e^{i\frac{S}{\hbar}} \to \int [dx] e^{i\frac{S[x]}{\hbar}}$$



$$[dx] = \mathscr{N} \lim_{n\to\infty} \prod_{i=1}^{n-1} dx_i$$

$$S[x] = \int_0^t dt \mathscr{L}(x,\dot{x})$$

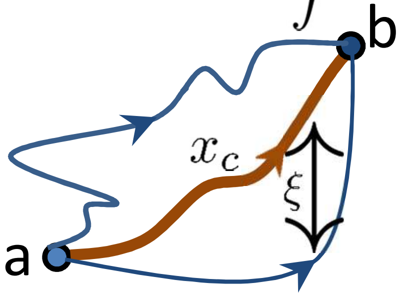
$$\mathscr{L}(x,\dot{x}) = \frac{1}{2}\dot{x}^2 - V(x)$$

For $\hbar \to 0$ the phase oscillates wildly!

Imaginary time path integral - 1

Put $t = -i\tau$ ➡ $\left(\frac{dx}{dt}\right)^2 = -\left(\frac{dx}{d\tau}\right)^2 \equiv -\dot{x}^2$ $\qquad \frac{i}{\hbar}\int dt = \frac{1}{\hbar}\int d\tau$

$$\mathscr{A}_{A\to B} = \int [dx] \exp\left(-\frac{S[x]}{\hbar}\right), \qquad S[x] = \int_{\tau_a}^{\tau_b} d\tau \left(\frac{1}{2}\dot{x}^2 + V(x)\right)$$ Euclidean action



Let $x_c(\tau)$ be the classically determined path from a to b.

General path is $x(\tau) = x_c(\tau) + \xi(\tau)$ with $\xi(\tau_a) = \xi(\tau_b) = 0$.

Expand $S[x]$ around x_c in a functional Taylor series:

$$S[x] = S[x_c] + \int_{\tau_a}^{\tau_b} d\tau(-\ddot{x}_c + V'(x_c))\xi(\tau)$$
$$+ \frac{1}{2}\int_{\tau_a}^{\tau_b} d\tau\, \xi(\tau)\left(-\frac{d^2}{d\tau^2} + V''(x_c)\right)\xi(\tau) + \cdots$$

First order variation vanishes at minimum ➡
$\ddot{x}_c = V'(x_c)$ (Newton's Law)

$$S[x] = S[x_c] + \frac{1}{2}\int_{\tau_a}^{\tau_b} d\tau\, \xi(\tau)\left(-\frac{d^2}{d\tau^2} + V''(x_c)\right)\xi(\tau) \equiv S^{(0)} + S^{(2)}$$

(Action at 2nd order for an arbitrary path $x_c + \xi$)

Imaginary time path integral - 2

$$\mathscr{A}_{a\to b} = \int [dx]\, e^{-\frac{1}{\hbar}S[x]} = e^{-\frac{1}{\hbar}S^{(0)}[x_c]} \int [dx]\, e^{-\frac{1}{\hbar}S^{(2)}[x]} = e^{-\frac{1}{\hbar}S^{(0)}[x_c]} \int [d\xi]\, e^{-\frac{1}{\hbar}S^{(2)}[x_c+\xi]}$$

Note: we are only integrating over fluctuations. Hence, $[dx] = [d\xi]$.

Some low hanging fruit

- $$\frac{dE}{d\tau} \equiv \frac{d}{d\tau}\left(-\frac{1}{2}\dot{x_c}^2 + V(x_c)\right) = -\dot{x}_c\ddot{x}_c + V'(x_c)\dot{x}_c = (-\ddot{x}_c + V'(x_c))\dot{x}_c = 0$$

  Hence the "energy" E is conserved on $x_c(\tau)$ (but not away from it!).

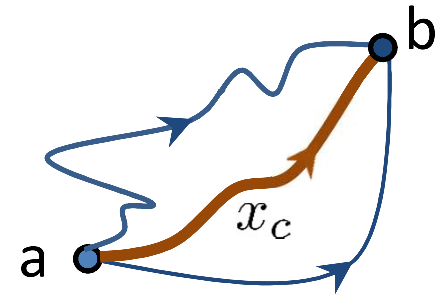
- $x_c(\tau)$ fixes $S_c^{(0)}[x_c] = \int d\tau \left(\frac{1}{2}\dot{x_c}^2 + V(x_c)\right)$ and this fixes $e^{-\frac{1}{\hbar}S_c^{(0)}[x_c]}$

- If $\int [d\xi] e^{-\frac{1}{\hbar}S^{(2)}[\xi]}$ is x_c independent then it's just a normalization constant!

Recapitulation of achievements so far

$$S_{total} = S^{(0)}_{classical} + S^{(2)}_{fluctuation}$$

$$\mathscr{A}_{total} = \mathscr{A}^{(0)}_{classical} \times \mathscr{A}^{(2)}_{fluctuation}$$



$$S^{(0)}_{classical} = \int_{\tau_a}^{\tau_b} d\tau \left(\frac{1}{2} \dot{x_c}^2 + V(x_c) \right)$$

$$S^{(2)}_{fluctuation} = \frac{1}{2} \int_{\tau_a}^{\tau_b} d\tau \; \xi(\tau) \left(-\frac{d^2}{d\tau^2} + V''(x_c) \right) \xi(\tau)$$

Work out $\mathcal{S}_{\text{classical}}$ and $\mathcal{A}_{\text{classical}}$ for free particle and SHO

1. Free Particle: $V = 0$ BC's are $(x, 0) \rightarrow (x', T)$

$$\frac{d^2}{d\tau^2} x_c = 0 \quad \therefore x_c(\tau) \sim c\tau + d \qquad x_c(\tau) = x + \frac{x' - x}{T}\tau$$

$$S_c = \int_0^T d\tau \left(\frac{1}{2}\dot{x_c}^2 + 0 \right) = \frac{(x' - x)^2}{2T} \qquad \therefore \mathcal{A}_c = e^{-\frac{(x'-x)^2}{2\hbar T}}$$

2. SHO: $V(x) = \frac{1}{2}\omega^2 x^2$ with same BC's as for free particle

$$\frac{d^2}{d\tau^2} x_c = \omega^2 x_c \quad \therefore \ x_c(\tau) \sim c e^{\omega\tau} + d e^{-\omega\tau} \qquad \text{(Fix } c, d \text{ by BC's)}$$

$$S_c = \int_0^T d\tau \left(\frac{1}{2}\dot{x_c}^2 + \frac{1}{2}\omega^2 x^2 \right)$$

$$\mathcal{A}_c = \exp\left[-\frac{\omega}{2\hbar \sinh \omega T} [(x'^2 + x^2)\cosh \omega T - 2x'x] \right]$$

Imaginary time path integral - 3

How to compute $S^{(2)} = \frac{1}{2}\int_{\tau_a}^{\tau_b} d\tau\, \xi(\tau)\Big(-\frac{d^2}{d\tau^2} + V''(x_c)\Big)\xi(\tau)$? Let's expand $\xi(\tau)$ in an orthonormal basis $\{\phi_n(\tau)\}$, $\Big(-\frac{d^2}{d\tau^2}+V''(x_c)\Big)\phi_n(\tau) = \lambda_n\phi_n(\tau)$ with $\phi_n(\tau_a) = \phi_n(\tau_b) = 0$.

- $x(\tau) = x_c(\tau) + \xi(\tau) = x_c(\tau) + \sum_{n=0}^{\infty} c_n\phi_n(\tau)$ ⇒ $S^{(2)} = \frac{1}{2}\sum_{n=0}^{\infty}\lambda_n c_n^2$
- We need: $\int [d\xi] e^{-\frac{1}{\hbar}S^{(2)}(x_c+\xi)}$ where $[dx] = [d\xi] = \mathcal{N}'\prod_{n=0}^{\infty}\frac{1}{\sqrt{2\pi\hbar}}dc_n$

  Note: Integrating over paths is the same as integrating over c_n
- $\int [dx] e^{-\frac{1}{\hbar}S^{(2)}} = \mathcal{N}'\prod_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{dc_n}{\sqrt{2\pi\hbar}}e^{-\frac{1}{2\hbar}\lambda_n c_n^2} = \mathcal{N}'\prod_{n=0}^{\infty}\frac{1}{\sqrt{\lambda_n}}$

$= \frac{\mathcal{N}'}{\sqrt{\det S''}}$ where, $S'' = -\frac{d^2}{d\tau^2} + V''(x_c)$

$$\mathscr{A}_{a\to b} = e^{-\frac{1}{\hbar}S^{(0)}[x_c]}\frac{\mathscr{N}'}{\sqrt{\det S''}}$$

Work out $\mathcal{A}_{\text{fluctuation}}$ for free particle

1. Free Particle: $V = 0$ BC's are $(x, 0) \to (x', T)$

EV problem: $\left(-\frac{d^2}{d\tau^2} + V''(x_c) \right) \phi_n(\tau) = \lambda_n \phi_n(\tau)$ with $\phi_n(0) = \phi_n(T) = 0$.

($V''(x_c) \to 0$)

Normalized solution: $\phi_n(\tau) = \sqrt{\frac{2}{T}} \sin \frac{n\pi}{T} \tau, \quad \lambda_n = \frac{n^2 \pi^2}{T^2}$

$$\therefore \mathcal{A}_{fluct} = \int [dx] e^{-\frac{1}{\hbar} S^{(2)}} = \mathcal{N}' \prod_{i=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} = \underbrace{\mathcal{N}' \prod_{n=1}^{\infty} \left(\frac{T}{\pi n} \right)}_{\mathcal{N} = \sqrt{\frac{1}{2\pi\hbar T}}}$$

The first factor is infinite and the second is zero. How to make sense out of this total mathematical nonsense?

Answer: we know the product because we have earlier on calculated the normalization for a free particle!

Work out $\mathcal{A}_{\text{fluctuation}}$ for SHO

1. SHO: $V(x) = \frac{1}{2}\omega^2 x^2,\ V''(x) = \omega^2$ and BC's are as in free particle case.

EV problem: $\left(-\frac{d^2}{d\tau^2} + \omega^2\right)\phi_n(\tau) = \lambda_n \phi_n(\tau)$ with $\phi_n(0) = \phi_n(T) = 0$.

Solution: $\phi_n(\tau) = \sqrt{\frac{2}{T}} \sin\frac{n\pi}{T}\tau, \quad \lambda_n = \frac{n^2\pi^2}{T^2} + \omega^2 = \frac{n^2\pi^2}{T^2}\left(1 + \frac{\omega^2 T^2}{n^2\pi^2}\right)$

$$\therefore \mathcal{A}_{fluct} = \int [dx] e^{-\frac{1}{\hbar}S^{(2)}} = \mathcal{N}' \prod_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}}$$

$$= \underbrace{\mathcal{N}' \prod_{n=1}^{\infty} \left(\frac{T}{n\pi}\right)}_{\sqrt{\frac{1}{2\pi\hbar T}}} \times \underbrace{\sqrt{\prod_{n=1}^{\infty} \frac{1}{1 + \frac{\omega^2 T^2}{n^2\pi^2}}}}_{\sqrt{\frac{\omega T}{\sinh \omega T}}} = \sqrt{\frac{\omega}{2\pi\hbar \sinh \omega T}} \quad ✔$$

$$G(x', T; x, 0) = \langle x' | e^{-\frac{1}{\hbar}\hat{H}T} | x \rangle$$ Knowing G buys us a lot!

Let $\hat{H}|n\rangle = \epsilon_n |n\rangle$. These form a complete set, $\sum_n |n\rangle\langle n| = \hat{1}$

Consider: $G(x', T; x, 0) = \langle x' | e^{-\frac{1}{\hbar}\hat{H}T} | x \rangle = \langle x' | \left(\sum_n |n\langle n| \right) e^{-\frac{1}{\hbar}\hat{H}T} | x \rangle$

$$= \sum_n e^{-\frac{1}{\hbar}\epsilon_n T} \phi_n(x') \phi_n^*(x) = e^{-\frac{1}{\hbar}\epsilon_0 T} \phi_0(x') \phi_0^*(x) + \cdots$$

As $T \to \infty$ only the ground state survives and can be extracted. Let's check:

$$\lim_{T\to\infty} \sqrt{\frac{\omega}{2\pi\hbar \sinh \omega T}} \exp\left[-\frac{\omega}{2\hbar \sinh \omega T} [(x'^2 + x^2)\cosh \omega T - 2x'x] \right]$$

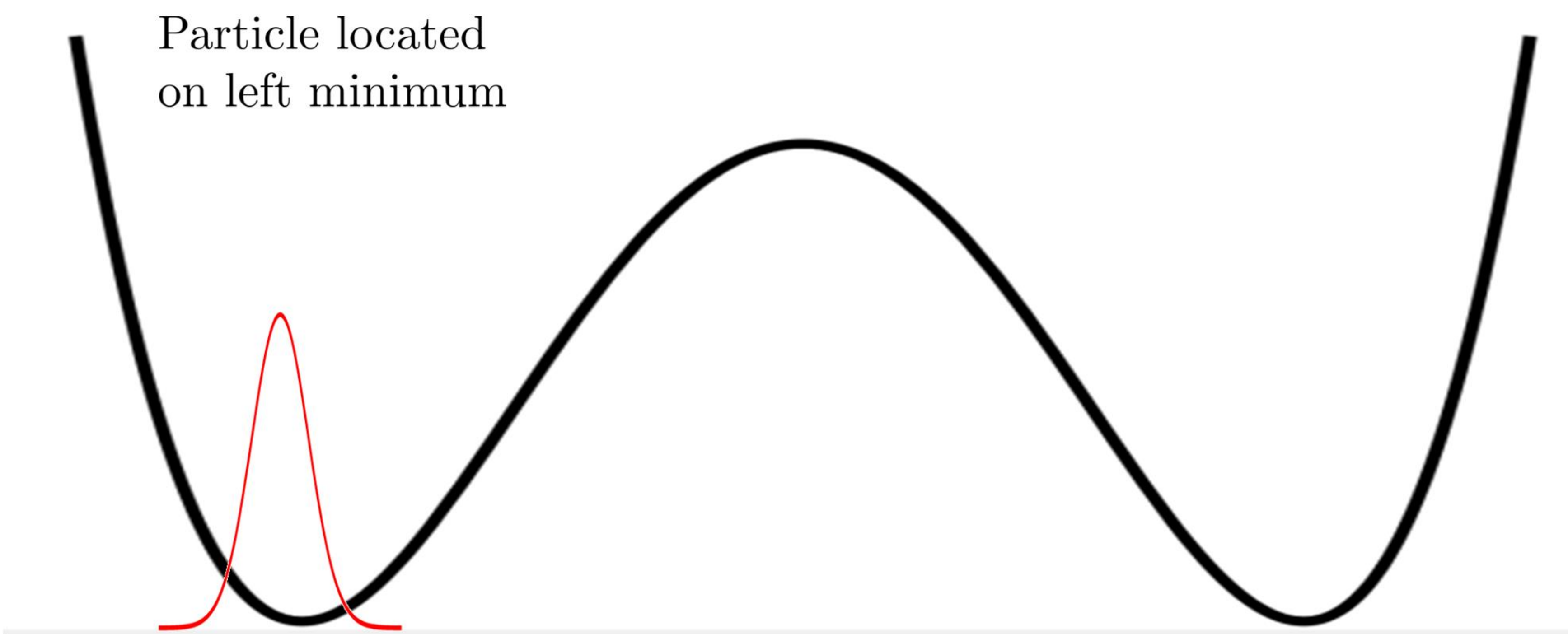
$$= e^{-\frac{1}{2}\omega T} \left(\frac{\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{\omega}{2\hbar} x'^2} \left(\frac{\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{\omega}{2\hbar} x^2}$$

Correct SHO energies and wavefunctions!

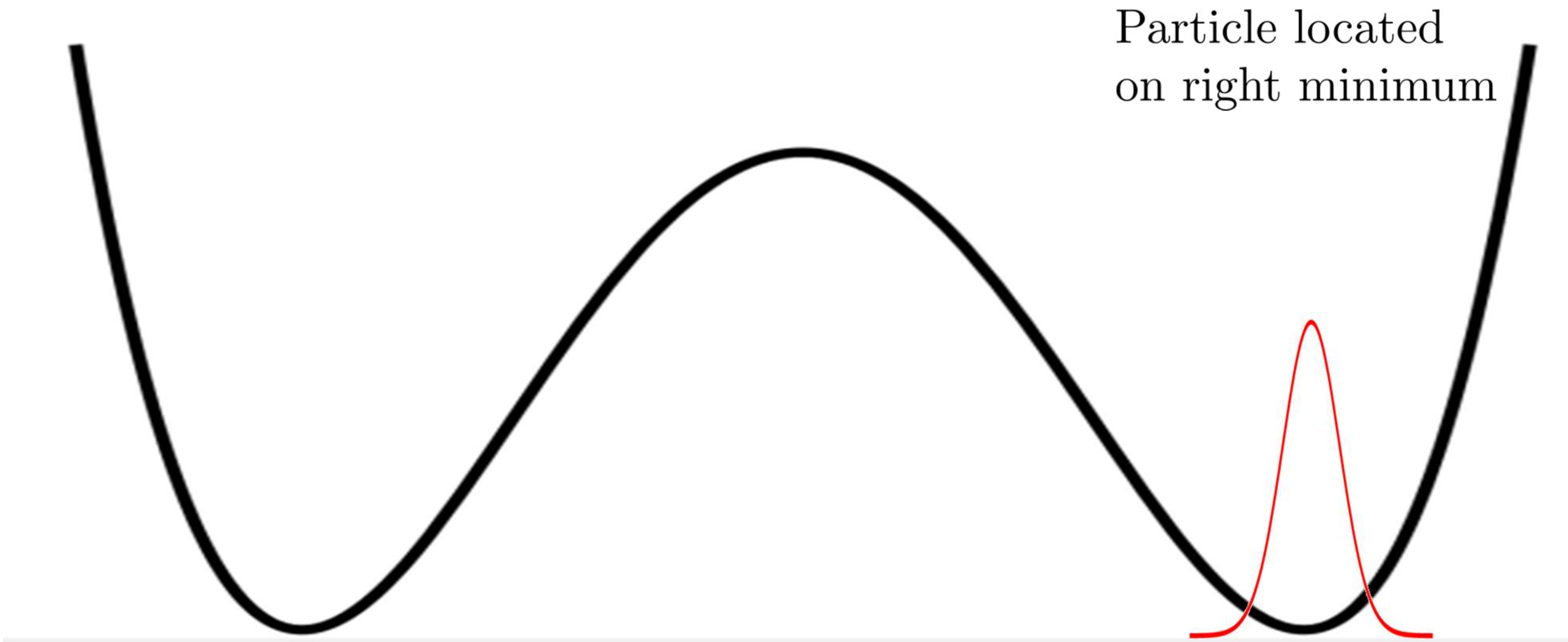
The Double Well

(as in undergrad QM)

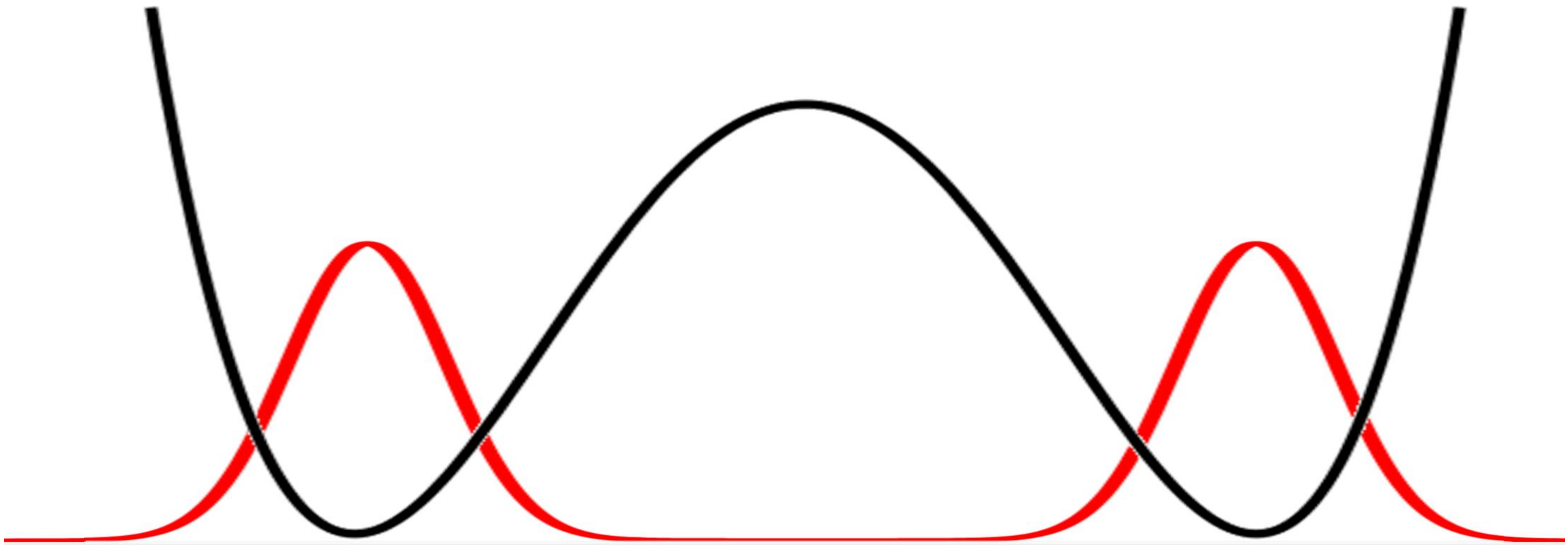
This is not an eigenstate of the Hamiltonian



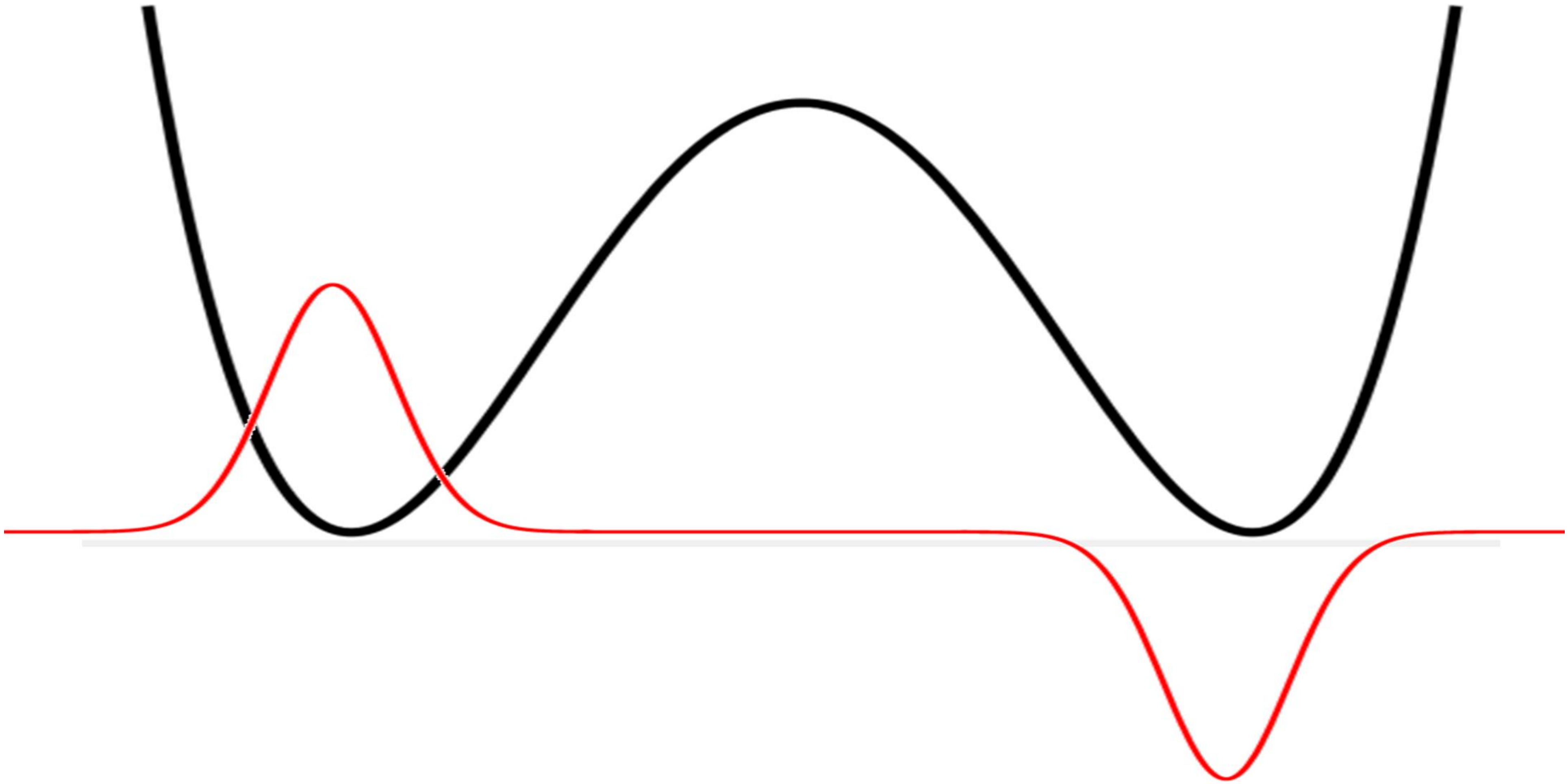
This is not an eigenstate of the Hamiltonian

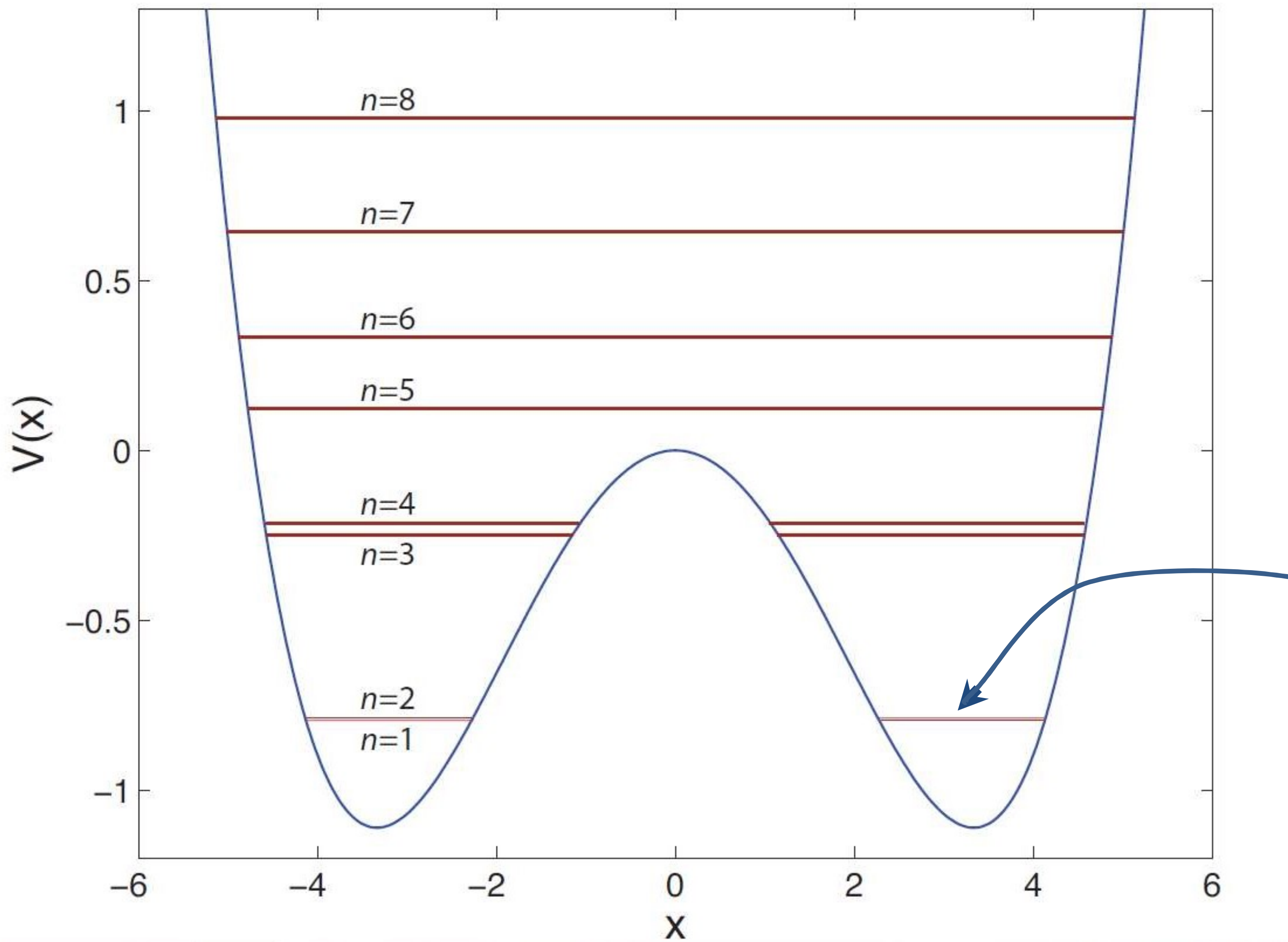


Symmetric eigenstate of the Hamiltonian



Anti-symmetric eigenstate of the Hamiltonian

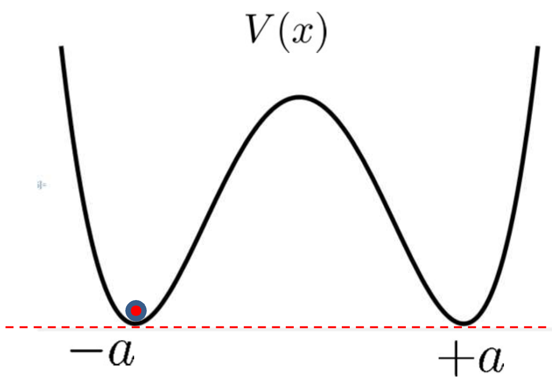


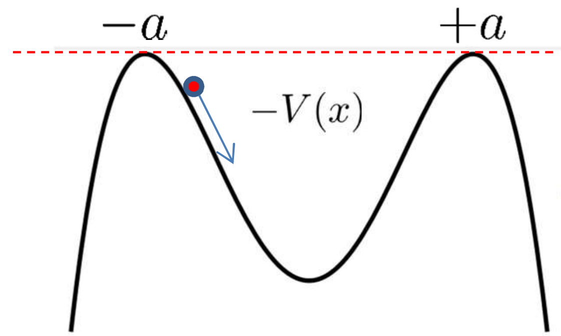


Numerical solution of Schrodinger equation

Very closely spaced levels

Double well instantons – a first look (purely classical)



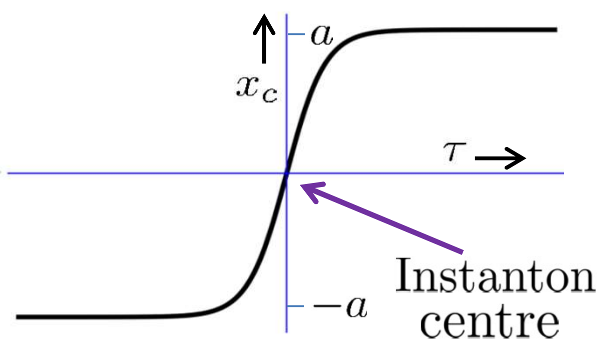


$$\frac{d^2}{d\tau^2} x_c = V'(x_c)$$

$$-\frac{1}{2}\dot{x_c}^2 + V(x_c) = E$$

$$V(-x) = V(x)$$

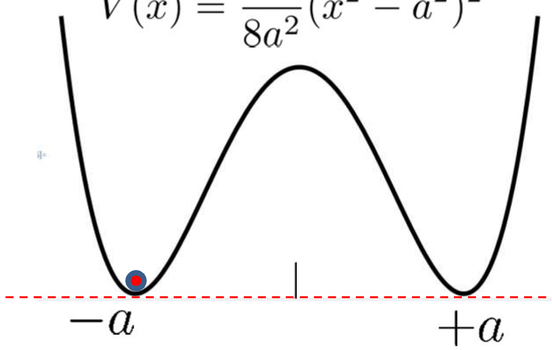
- Stay-at-home solution with zero energy, i.e. by choice $V(0) = 0, E = 0$
  $x_c(-T/2) = x_c(T/2) = -a$ has $S[x_c] = 0$ and $\mathcal{A} = e^{-\frac{1}{\hbar}0} = 1$
- Left-to-Right: $x_c(-T/2) = -a$, $x_c(+T/2) = +a$ (We're interested in $T \to \infty$)

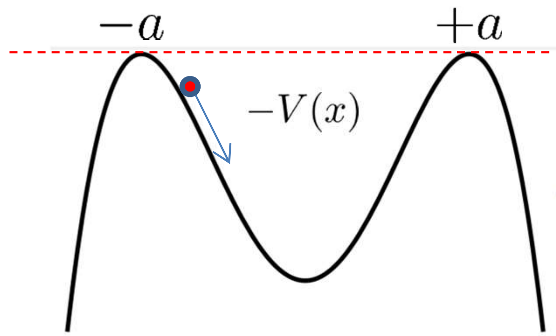


- $x_c(-\tau) = -x_c(\tau)$ from EOM $\to x_c(0) = 0$
- $E = 0$ so $\dot{x}_c = \sqrt{2V(x)}$ is large only near $x = 0$
- x_c "suddenly" jumps from $-a$ to $+a$ and so is called an "instanton". But how sudden is sudden?
- When/where does the jump happen?

Double well instantons – a solvable example (purely classical)

$$V(x) = \frac{\omega^2}{8a^2}(x^2 - a^2)^2$$



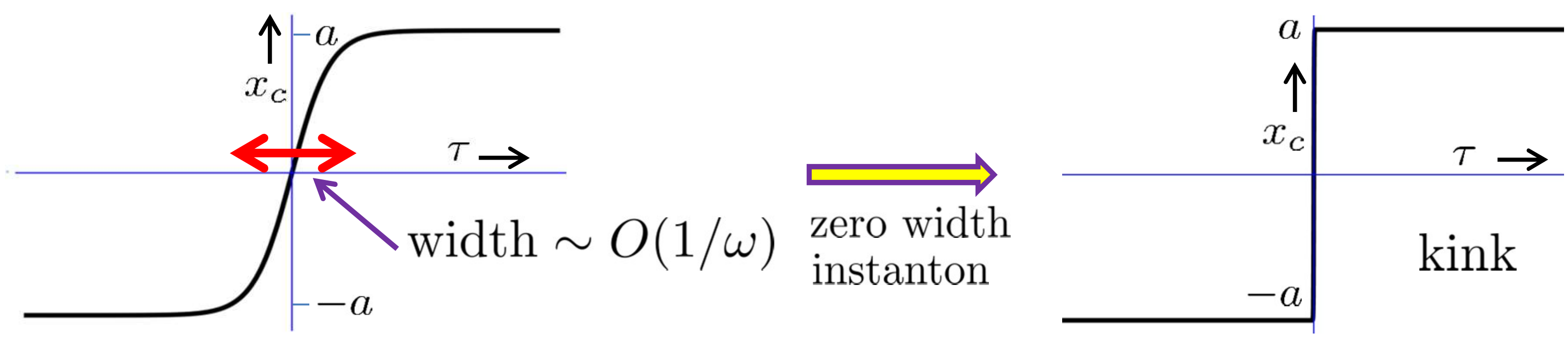


$$\frac{d^2}{d\tau^2}x_c = V'(x_c)$$

$$V'(x) = -\frac{1}{2}x\omega^2\left(1 - \frac{x^2}{a^2}\right)$$

Claimed solution of EOM: $x_c(\tau) = a\tanh\left(\frac{\omega\tau}{2}\right) \quad x_c(-\infty) = -a, \quad x_c(+\infty) = +a$

Check: $\frac{d^2}{d\tau^2}x_c = -\frac{1}{2}a\omega^2\tanh\left(\frac{\tau\omega}{2}\right)\left(1 - \tanh^2\left(\frac{\tau\omega}{2}\right)\right) = -\frac{1}{2}x\omega^2\left(1 - \frac{x^2}{a^2}\right) = V'(x_c)$ ✓



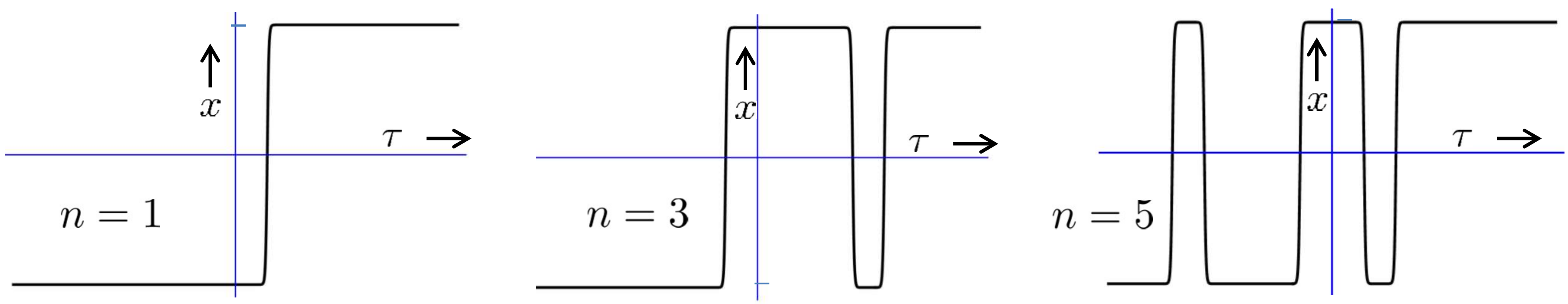
Reversing time for the instanton

- Recall EOM: $\frac{d^2 x_c(\tau)}{d\tau^2} = V'(x_c(\tau)),\quad V(-x(\tau)) = V(x(\tau)) \quad \therefore V'(-x(\tau)) = -V(x(\tau))$

  Let $\tau \to -\tau$ then $\frac{d^2 x_c(-\tau)}{d\tau^2} = V'(x_c(-\tau)) = -V'(-x_c(-\tau))$ $\quad \therefore y(\tau) = -x(-\tau)$ is also a solution

  Example: $a \tanh\left(\frac{\omega\tau}{2}\right) \xrightarrow[\tau \to -\tau]{} -a \tanh\left(\frac{\omega\tau}{2}\right)$ (called anti-instanton)

- Suppose we put here $\tau - \tau_l$ in place of τ. Nothing changes! This means that $x_c(\tau - \tau_l)$ is an equally good solution. So we can have many instantons moving between $-T/2$ and $+T/2$. (Remember that $T \to \infty$)



Action for one instanton or anti-instanton

Reminder 1: the one instanton action is $S[x_c] = \int d\tau \left(\frac{1}{2}\dot{x_c}^2 + V(x_c) \right)$

Reminder 2: energy conservation says $-\frac{1}{2}\dot{x_c}^2 + V(x_c) = E \; = 0$ by our own choosing

- $S_{n=1}[x_c] = \int_{-\infty}^{\infty} d\tau \; \dot{x}_c^2 \approx \int_{-\delta}^{\delta} d\tau \; \dot{x}_c(\tau)^2$ that's because the derivative is non-zero only around $x = 0$, i.e. the crossing point

  We can also write $\int d\tau \; \dot{x}_c(\tau)^2 = \int d\tau \frac{dx_c}{d\tau} \dot{x}_c = \int dx \sqrt{2V(x_c)}$

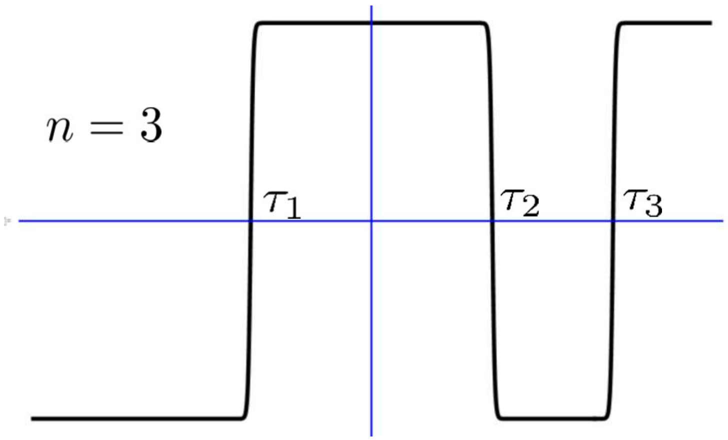
- For the special case $V(x) = \frac{\omega^2}{8a^2}(x^2 - a^2)^2$ the action is calculated as $S_{n=1} = \frac{2}{3}\omega a^2$

- For n non-overlapping instantons, $S_n = nS_{n=1} \equiv nS_1$ (often called instanton gas)

- A single instanton's centre can be located anywhere between $-T/2$ and $+T/2$ Thus at the classical level (no fluctuations included yet), contributions from all values of T must be summed over, $\int_{-T/2}^{+T/2} d\tau e^{\frac{1}{\hbar}S_1} = Te^{\frac{1}{\hbar}S_1}$

Now look at the case for many instantons and anti-instantons

$n = 3$

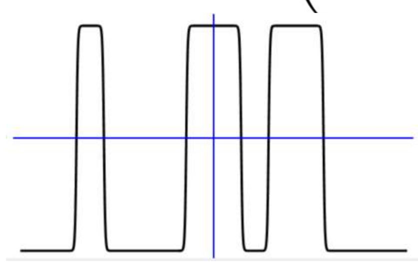


$$\mathcal{A}^c_{n=3} = \int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 \int_{\tau_2}^{T/2} d\tau_3 \; e^{-\frac{1}{\hbar}3S_1}$$
$$= e^{-\frac{1}{\hbar}3S_1} \int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 (\frac{T}{2} - \tau_2)$$
$$= e^{-\frac{1}{\hbar}3S_1} \frac{1}{2} \int_{-T/2}^{T/2} d\tau_1 (\frac{T}{2} - \tau_1)^2$$
$$= \frac{1}{3!} T^3 e^{-\frac{1}{\hbar}3S_1}$$

We can readily generalize to any n, $\quad \mathcal{A}^c_n = \frac{T^n}{n!}[e^{-\frac{1}{\hbar}S_1}]^n \quad n = 1, 3, 5 \cdots$

Define: $Z^c_{LR} = \sum_{n \text{ odd}}^{\infty} \frac{1}{n!}[T e^{-\frac{1}{\hbar}S_1}]^n = \sinh\Big(\exp[-\frac{1}{\hbar}S_1]\, T\Big)$

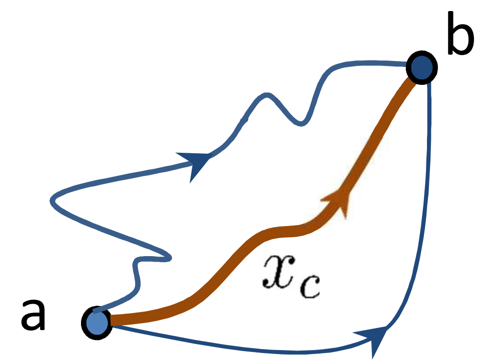
For the solutions which return to the left sum only on even n



$$Z^c_{LL} = \cosh\Big(\exp[-\frac{1}{\hbar}S_1]\, T\Big)$$

Conclusions of Lecture#2

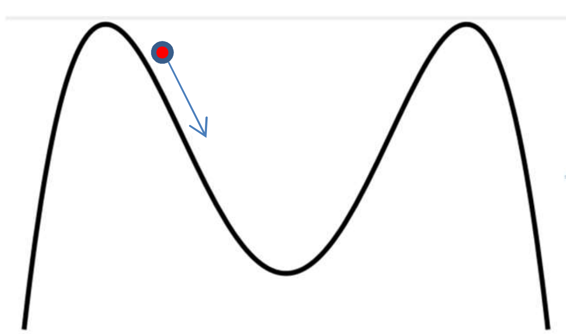
$$S_{total} = S^{(0)}_{classical} + S^{(2)}_{fluctuation}$$



$$\mathscr{A}_{total} = \mathscr{A}^{(0)}_{classical} \times \mathscr{A}^{(2)}_{fluctuation}$$

$$Z^c_{LR} = \sinh\left(\exp[-\frac{1}{\hbar}S_1]\;T\right)$$

$$Z^c_{LL} = \cosh\left(\exp[-\frac{1}{\hbar}S_1]\;T\right)$$



S_1 is the single instanton action.
Start time is $-\frac{T}{2}$ and the end time is $+\frac{T}{2}$

References

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