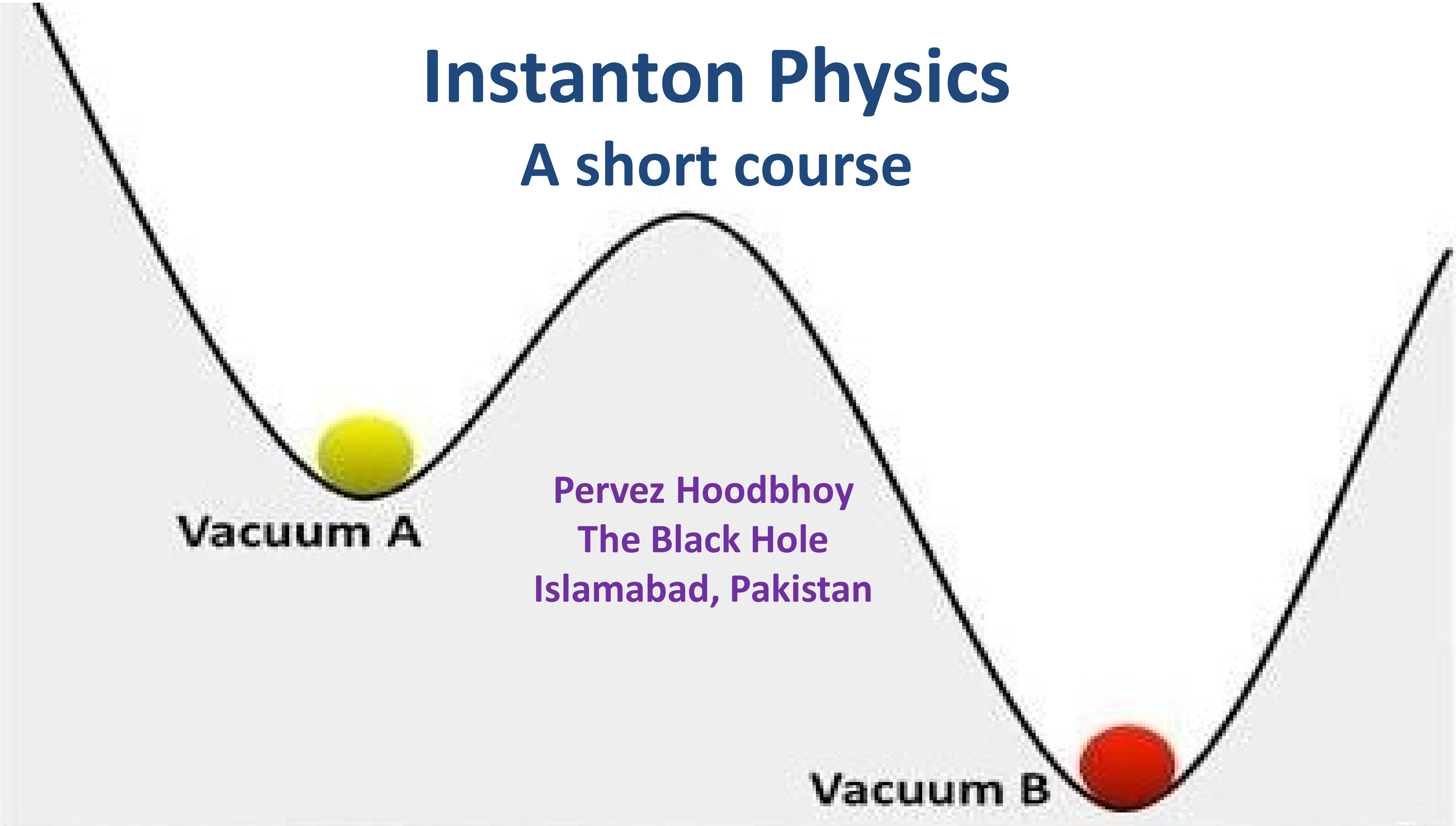
Instanton Physics
A short course



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Course Outline

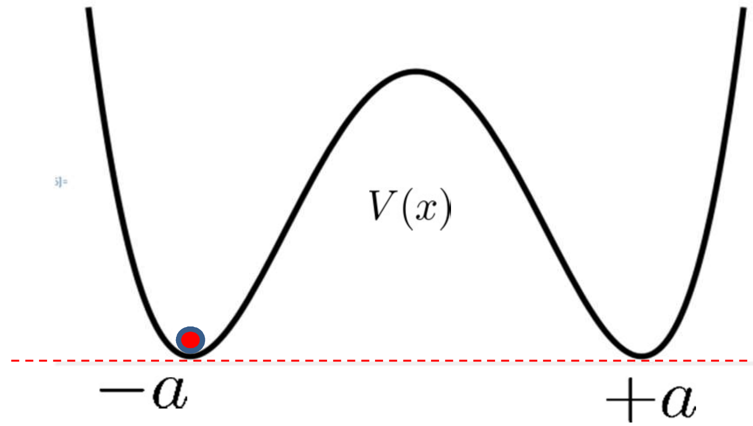
Instantons in particle QM

- Intro to path integral ✔
- Imaginary time ✔
- Instantons in a symmetric double well ✘
- The functional determinant ✘
- Decay of metastable states

Tunneling of quantum fields

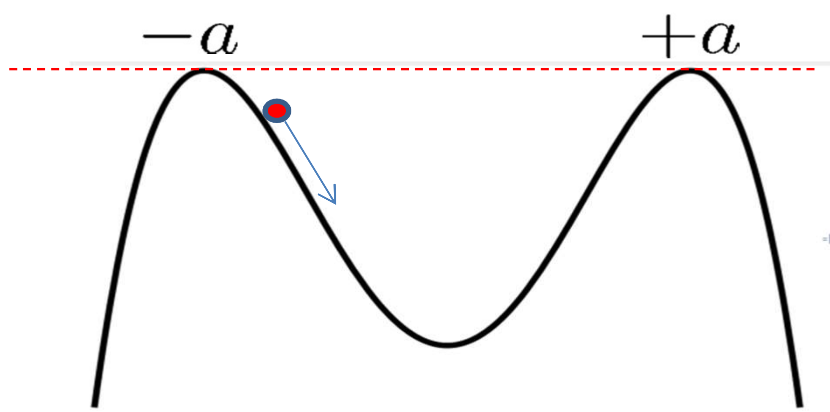
- Basic QFT for a scalar field
- Tunneling of field configurations
- The O(4) instanton
- Gauge fields and tunneling
- Effective action
- How/when will the universe end?

A quick recapitulation



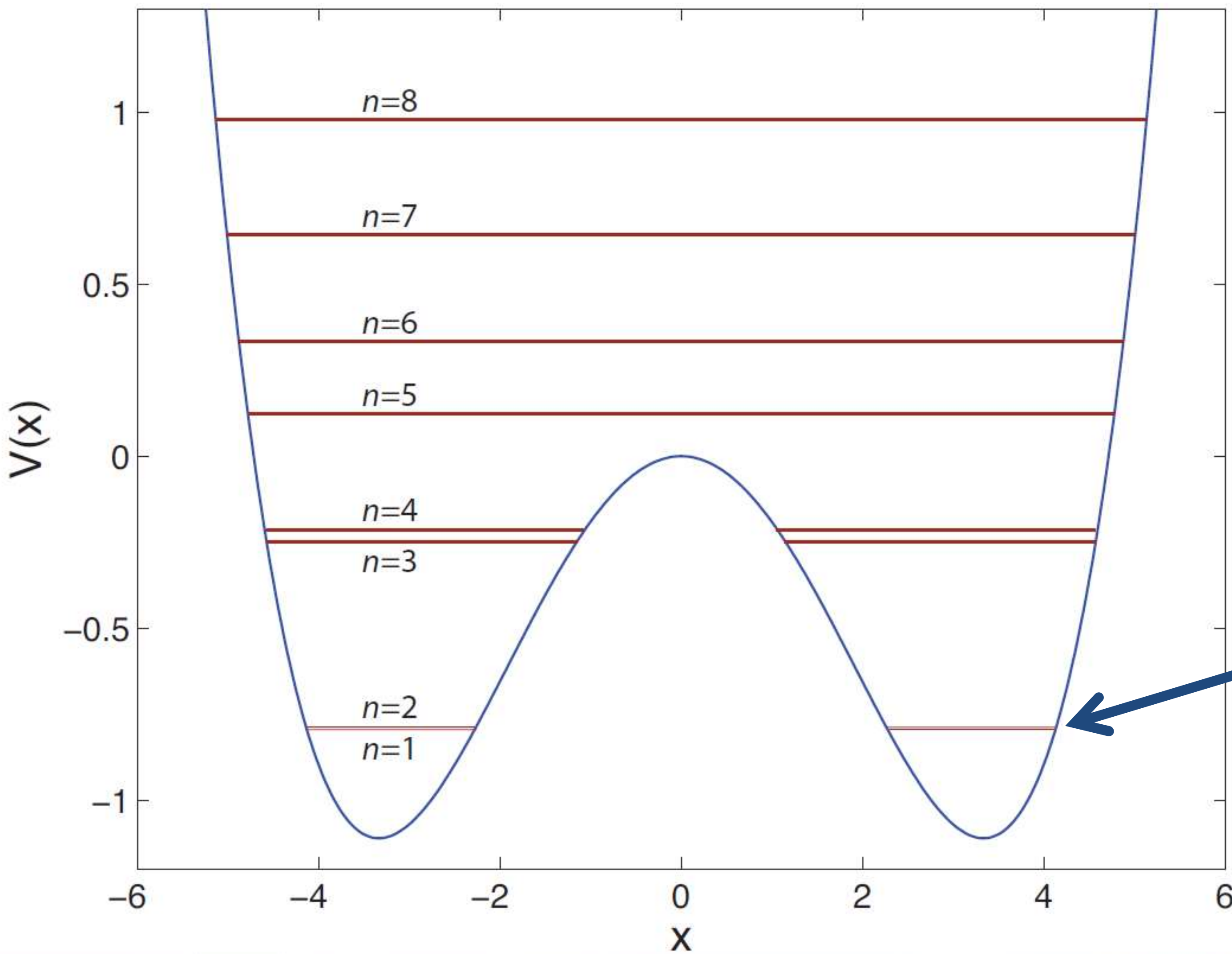
Symmetric potential

$$V(-x) = V(x)$$



$$\frac{d^2}{d\tau^2}x_c = V'(x_c)$$

$$-\frac{1}{2}\dot{x_c}^2 + V(x_c) = E$$

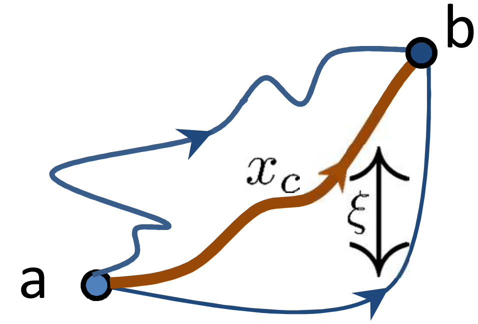


We will find the energy difference between these two eigenstates.

Our goal in Lecture#3 is to calculate the fluctuation part

$$\mathscr{A}_{total} = \mathscr{A}^{(0)}_{classical} \times \mathscr{A}^{(2)}_{fluctuation}$$

$$S^{(2)}_{fluctuation} = \frac{1}{2}\int_{-T/2}^{T/2} d\tau\, \xi(\tau)\left(-\frac{d^2}{d\tau^2} + V''(x_c)\right)\xi(\tau)$$



- In Lecture #2 we expanded $\xi(\tau)$ in an orthonormal, complete basis $\phi_n(\tau)$, $\xi(\tau) = \sum_{n=0}^{\infty} c_n \phi_n(\tau)$ such that $\left(-\frac{d^2}{d\tau^2} + V''(x_c)\right)\phi_n = \lambda_n \phi_n$ and the BC's, $\phi_n(-T/2) = \phi_n(T/2) = 0$. This gave us: $S^{(2)} = \frac{1}{2}\sum_{n=0}^{\infty} \lambda_n c_n^2$
- With $[d\xi] = \mathcal{N}\prod_{n=0}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} dc_n$ we found: $\int [d\xi] e^{-\frac{1}{\hbar}S^{(2)}} = \mathcal{N}\prod_{n=0}^{\infty}\frac{1}{\sqrt{\lambda_n}}$

  But what if some eigenvalue λ_n vanishes? TROUBLE !! $\quad = \frac{\mathcal{N}}{\sqrt{\det S''}}$

The zero mode – a delicate matter to be handled stepwise

- **Step 1:** Suppose $x_c(\tau)$ is a solution of the EOM $\frac{d^2}{d\tau^2} x_c(\tau) = V'(x_c(\tau))$.

  Then $x_c(\tau - \tau')$ is an equally good solution (time translational invariance).

- **Step 2:** $-\frac{d^2}{d\tau^2} + V''(x_c) \equiv S''$ has $\dot{x}_c(\tau)$ as eigenfunction with zero eigenvalue.

  Proof: $\left(-\frac{d^2}{d\tau^2} + V''(x_c)\right)\frac{dx_c}{d\tau} = \frac{d}{d\tau}\left(-\frac{d^2 x_c}{d\tau^2} + V'(x_c)\right) = 0$ using the EOM.

  Note that $\dot{x}_c(\tau)$ obeys the correct BC's, i.e. $\dot{x}_c(-T/2) = \dot{x}_c(+T/2) = 0$.

- **Step 3:** The zero mode wavefunction can be properly normalized:

  $\phi_0(\tau) \equiv C\frac{dx_c}{d\tau}$ where $C = \left[\int_{-T/2}^{T/2} d\tau \left(\frac{dx_c}{d\tau}\right)^2\right]^{-\frac{1}{2}} = \left[\int_{-a}^{+a} dx\, \sqrt{2V(x_c)}\right]^{-\frac{1}{2}} = S_1^{-\frac{1}{2}}$

  Here S_1 is the single instanton action.

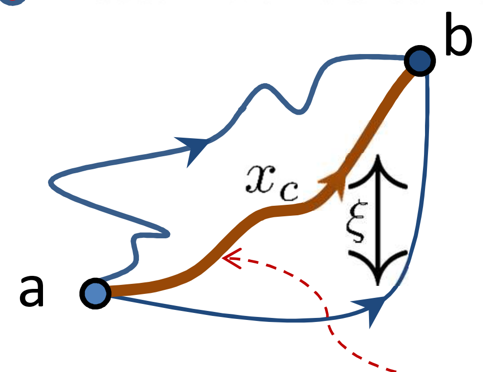
How to assign the proper integration measure to the zero mode - 1

Let's go back to the source of the problem (from lecture #2):

$$\int [dx] e^{-\frac{1}{\hbar} S^{(2)}} = \mathcal{N} \prod_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dc_n}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar}\lambda_n c_n^2} = \mathcal{N} \prod_{n=0}^{\infty} \frac{1}{\sqrt{\lambda_n}}$$

$n = 0, 1, 2 \cdots$ is no problem unless $\lambda_0 = 0$

- But if $\lambda_0 = 0$ how to handle this: $\int_{-\infty}^{\infty} \frac{dc_0}{\sqrt{2\pi\hbar}} e^{-0} = \infty$? The infinity owes to a symmetry!
- Recall we had earlier expanded about: $x(\tau) = x_c(\tau) + \xi(\tau) = x_c(\tau) + \sum_{n=0}^{\infty} c_n \phi_n(\tau)$



Sliding instanton on this path needs no effort

But we can choose the integration variables differently:

$$x(\tau - \tau') = x_c(\tau) + \xi(\tau) = x_c(\tau) + \sum_{n=0}^{\infty} c_n \phi_n(\tau)$$

Assigning the proper integration measure to the zero mode - 2

Now use this integration variable: $x(\tau-\tau') = x_c(\tau) + \sum_{n=0}^{\infty} c_n\phi_n(\tau)$ and operate with

$$\int_{-T/2}^{T/2} d\tau\phi_0(\tau) \Rightarrow \int_{-T/2}^{T/2} d\tau\phi_0(\tau)x(\tau-\tau') = \int_{-T/2}^{T/2} d\tau\phi_0(\tau)x_c(\tau) + c_0$$

Use, $\phi_0(\tau) = C\dot{x}(\tau)$

Note: $\int_{-T/2}^{T/2} d\tau\phi_0(\tau)x_c(\tau) = C\int_{-T/2}^{T/2} d\tau\, \dot{x}_c(\tau)x_c(\tau) = C\int_{-T/2}^{T/2} d\tau \frac{1}{2}\frac{d}{d\tau}x_c^2 = 0$

This gives c_0 for different choices of path, $c_0(\tau') = C\int_{-T/2}^{T/2} d\tau\dot{x}_c(\tau)x(\tau-\tau')$

$$\frac{dc_0(\tau')}{d\tau'} = C\int_{-T/2}^{T/2} d\tau\dot{x}_c(\tau)\frac{d}{d\tau'}x(\tau-\tau') = -C\int_{-T/2}^{T/2} d\tau\dot{x}_c(\tau)\frac{d}{d\tau}x(\tau-\tau')$$

$$= -C\int_{-T/2}^{T/2} d\tau\dot{x}_c(\tau)\dot{x}_c(\tau) + 0 = -S_1^{\frac{1}{2}}$$

This, finally, tells us how to replace the integration over $c_0(\tau)$ with integration over all instanton times from $-T/2$ to $T/2$

$$\frac{dc_0}{\sqrt{2\pi\hbar}} = \sqrt{\frac{S_1}{2\pi\hbar}}d\tau$$

Now go back and correct all earlier "mistakes"

Including fluctuations, and after integrating over all instanton positions between $-T/2$ and $+T/2$, the correct single instanton amplitude is:

$$\mathcal{A}_{n=1} = \int [dx] e^{-\frac{1}{\hbar}(S^{(0)}+S^{(2)})} = \mathcal{N} e^{-\frac{1}{\hbar}S_1} \sqrt{\frac{S_1}{2\pi\hbar}}\, T \prod_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}}$$

from zero mode

$$\prod_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} \rightarrow \frac{1}{\sqrt{\det' S''}}$$

$K \sim$ ratio of the SHO determinant to that of the actual determinant with the zero mode removed

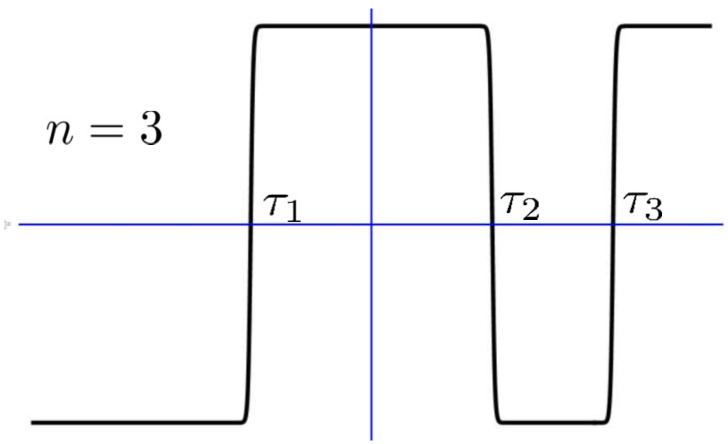
$$K \equiv \sqrt{\frac{S_1}{2\pi\hbar}} \frac{\det\left[-\frac{d^2}{d\tau^2}+\omega^2\right]^{\frac{1}{2}}}{\det'\left[-\frac{d^2}{d\tau^2}+V''(x_c)\right]^{\frac{1}{2}}}$$

$$\therefore \mathcal{A}_{n=1} = \frac{\mathcal{N}}{\det\left[-\frac{d^2}{d\tau^2}+\omega^2\right]^{\frac{1}{2}}} \times e^{-\frac{1}{\hbar}S_1} KT = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega T}{2}} \times e^{-\frac{1}{\hbar}S_1} KT$$

$$\frac{\mathcal{N}}{\det\left[-\frac{d^2}{d\tau^2}+\omega^2\right]^{\frac{1}{2}}} = \sqrt{\frac{\omega}{2\pi\hbar \sinh \omega T}} \xrightarrow[T\to\infty]{} \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega T}{2}}$$

This second part here will exponentiate for $n > 1$

Refresher from Lecture#2



$$\mathcal{A}^c_{n=3} = \int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 \int_{\tau_2}^{T/2} d\tau_3 \; e^{-\frac{1}{\hbar}3S_1}$$

$$= e^{-\frac{1}{\hbar}3S_1} \int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 (\frac{T}{2} - \tau_2)$$

$$= e^{-\frac{1}{\hbar}3S_1} \frac{1}{2} \int_{-T/2}^{T/2} d\tau_1 (\frac{T}{2} - \tau_1)^2$$

$$= \frac{1}{3!} T^3 e^{-\frac{1}{\hbar}3S_1}$$

We can readily generalize to any n, $\quad \mathcal{A}^c_n = \frac{T^n}{n!}[e^{-\frac{1}{\hbar}S_1}]^n \quad n = 1, 3, 5 \cdots$

Define: $Z^c_{LR} = \sum_{n \text{ odd}}^{\infty} \frac{1}{n!}[Te^{-\frac{1}{\hbar}S_1}]^n = \sinh\left(\exp[-\frac{1}{\hbar}S_1]\; T\right)$

Summary of main results so far

After summing over all instanton numbers from 0 to ∞,

$$Z_{LR} \equiv \mathcal{A}_{(-\frac{T}{2},-a)\to(\frac{T}{2},a)} = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega T}{2}} \sinh\left(e^{-\frac{1}{\hbar}S_1} KT\right)$$

$$Z_{LL} \equiv \mathcal{A}_{(-\frac{T}{2},-a)\to(\frac{T}{2},-a)} = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega T}{2}} \cosh\left(e^{-\frac{1}{\hbar}S_1} KT\right)$$

In the dilute gas approximation only the single instanton action enters.

$$S_1 = \int d\tau \left(\frac{1}{2}\dot{x_c}^2 + V(x_c)\right) = \int_{-a}^{+a} dx \, \sqrt{2V(x)}$$

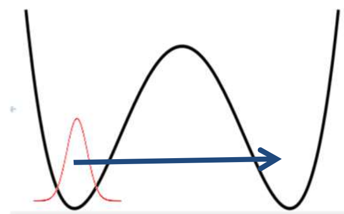
$$\frac{\det\left[-\frac{d^2}{d\tau^2} + \omega^2\right]^{\frac{1}{2}}}{\det'\left[-\frac{d^2}{d\tau^2} + V''(x_c)\right]^{\frac{1}{2}}} = ?$$

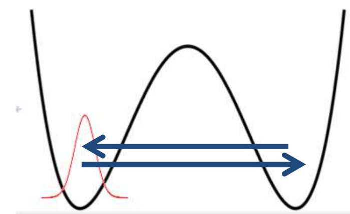
This is the trickiest part of the calculation. Read Coleman's paper or the prescribed textbook for a proper treatment. We'll return to it in a moment.

Calculation of the energy splitting between lowest two levels - I

We now have expressions for $Z_{LR}(= Z_{RL})$ and $Z_{LL}(= Z_{RR})$.

But all energy eigenstates must have either positive or negative parity:
$H|0\rangle = E_0|0\rangle$ and $H|1\rangle = E_1|1\rangle$:
$\langle -a|0\rangle = \langle a|0\rangle,\ \langle -a|1\rangle = -\langle a|1\rangle.$





$$Z_{LR} = \langle a|e^{-\frac{HT}{\hbar}}|-a\rangle = \langle a|e^{-\frac{HT}{\hbar}}\left(\sum |n\rangle\langle n|\right)|-a\rangle$$
$$= e^{-\frac{E_0 T}{\hbar}}\langle a|0\rangle\langle 0|-a\rangle + e^{-\frac{E_1 T}{\hbar}}\langle a|1\rangle\langle 1|-a\rangle + \cdots$$
$$= e^{-\frac{E_0 T}{\hbar}}|\langle a|0\rangle|^2 - e^{-\frac{E_1 T}{\hbar}}|\langle a|1\rangle|^2 = |\langle a|0\rangle|^2(e^{-\frac{E_0 T}{\hbar}} - e^{-\frac{E_1 T}{\hbar}})$$

Similarly, $Z_{LL} = \langle -a|e^{-\frac{HT}{\hbar}}|-a\rangle = |\langle a|0\rangle|^2(e^{-\frac{E_0 T}{\hbar}} + e^{-\frac{E_1 T}{\hbar}})$

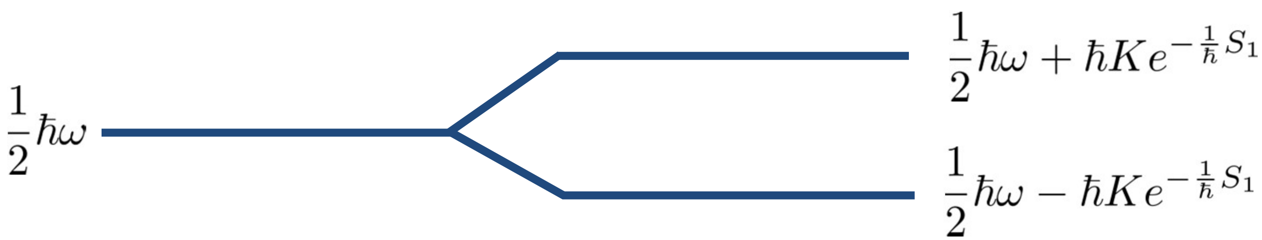
$$\therefore \frac{Z_{LL} - Z_{LR}}{Z_{LL} + Z_{LR}} = \exp[-\frac{(E_1 - E_0)T}{\hbar}] \rightarrow \boxed{E_1 - E_0 = -\frac{\hbar}{T}\log\frac{Z_{LL} - Z_{LR}}{Z_{LL} + Z_{LR}}}$$ as $T \rightarrow \infty$

Calculation of the energy splitting between lowest two levels - II

$$\Delta E = E_1 - E_0 = -\frac{\hbar}{T}\log\frac{Z_{LL} - Z_{LR}}{Z_{LL} + Z_{LR}} \qquad \text{Put,}\ \ \Omega = e^{-\frac{1}{\hbar}S_1}K$$

$$\text{then}\ \ \Delta E = -\frac{\hbar}{T}\log\frac{\cosh\Omega T - \sinh\Omega T}{\cosh\Omega T + \sinh\Omega T}$$

$$\xrightarrow[T\to\infty]{} \ -\frac{\hbar}{T}\log\frac{e^{-\Omega T}}{e^{\Omega T}} = 2\hbar\Omega = 2\hbar e^{-\frac{1}{\hbar}S_1}K$$

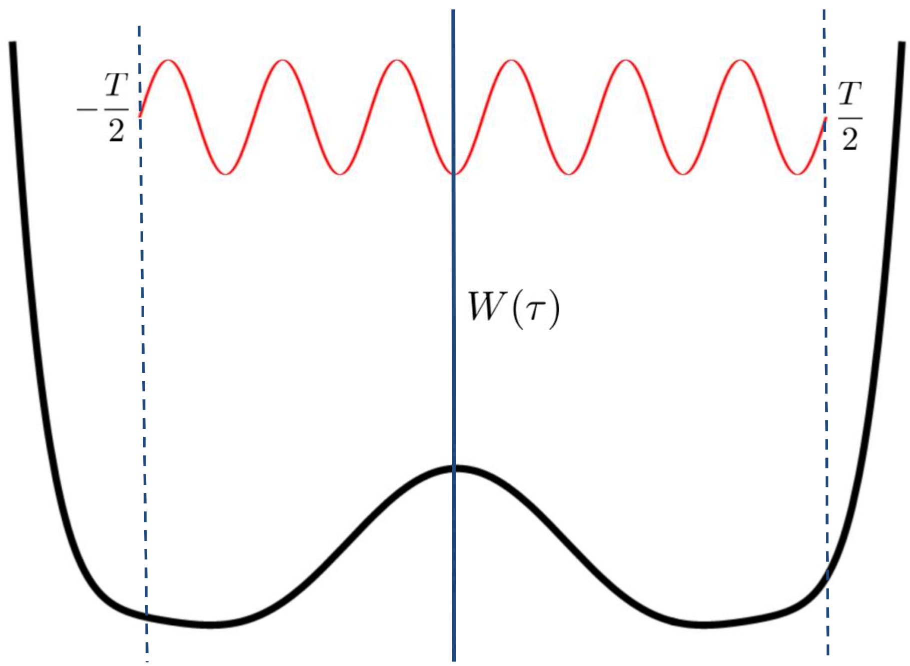


So if we can calculate K the calculation will be complete.

Remarks on evaluation of K

Suppose $W(\tau)$ is an arbitrary potential. The eigenvalue problem is,

$$\left(-\frac{d^2}{d\tau^2} + W\right)\phi = \lambda\phi \quad \text{and} \quad \phi(-T/2) = \phi(T/2) = 0$$

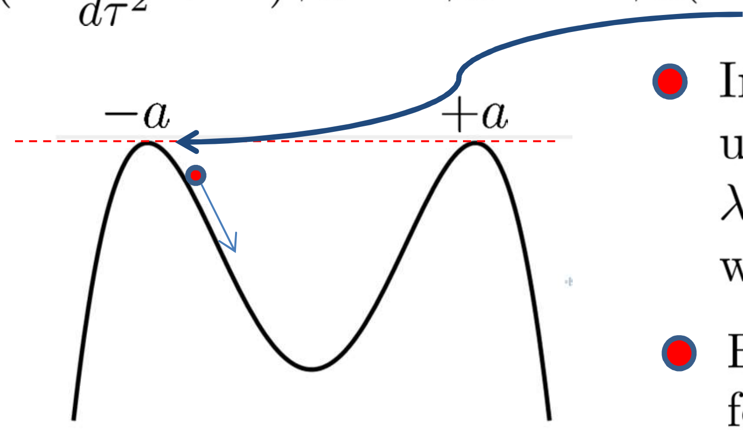


- As λ becomes large W becomes less relevant. This means for two potentials W_1, W_2 the $\lambda_n' s$ will approach each other,
  $$\lim_{n\to\infty} (\lambda_n^1 - \lambda_n^2)) = 0$$
- $F(\lambda) = \prod_{n=1}^{\infty} \frac{\lambda_n^1 - \lambda}{\lambda_n^2 - \lambda}$ is possibly convergent and well behaved.
- $$\lim_{\lambda\to 0} F(\lambda) = \frac{\det_1}{\det_2}$$
  $$\lim_{\lambda\to\infty} F(\lambda) = 1$$

More remarks on evaluation of K

Given any $W(\tau)$, for any λ the Cauchy problem is always solvable. Eg:

$$(-\frac{d^2}{d\tau^2}+W)\phi_\lambda = \lambda\phi_\lambda \text{ and } \phi_\lambda(-T/2)=0 \text{ with initial speed } \dot{\phi}_\lambda(-T/2)=1.$$



- In general the ball will overshoot or undershoot. Only when λ equals some λ_n will it reach the other hill top in which case $\phi_\lambda(T/2)$ will be zero.
- Every potential will give the same value for $\phi_\lambda(T/2)$ for large enough λ. So you might as well put $W_1 = constant = \omega^2$.
- $F(\lambda)=\prod_{n=1}^{\infty}\frac{\lambda_n^1-\lambda}{\lambda_n^2-\lambda}$ and $G(\lambda)=\frac{\phi_\lambda^1(T/2)}{\phi_\lambda^2(T/2)}$ have poles and zeros at exactly the same discrete values of λ. They also agree for large $|\lambda|$. Hence they agree everywhere. (F, G are holomorphic functions. Liouville's theorem applies.)

Final remarks on the evaluation of K

Any 2^{nd} order ode like $\left(-\frac{d^2}{d\tau^2}+W\right)\phi=0$ has to have 2 linearly independent solutions. We found one of them, $\phi_0(\tau)\sim\dot{x}(\tau)$. What's the other one?

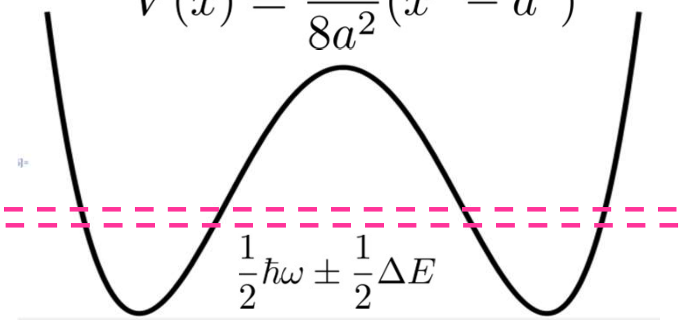
- If $f(\tau), g(\tau)$ are two independent solutions of a 2^{nd} order ode then the Wronskian, $W=fg'-f'g$ is a constant and non-zero.
- By taking a linear combination $c_1 f(\tau)+c_2 g(\tau)$ we can satisfy the ode as well as the initial conditions, $\phi(-T/2)=0, \dot{\phi}(-T/2)=1$.
- After a long song and dance (see references) we find K in terms of V :

$$K=\frac{\hat{a}\omega^{\frac{3}{2}}}{\sqrt{2\pi\hbar}} \quad \text{where } \hat{a}=a\exp\left(\int_0^a dx\left[\frac{\omega}{\sqrt{2V(x)}}-\frac{1}{a-x}\right]\right)$$

$$\omega^2=\left.\frac{d^2V(x)}{dx^2}\right|_{x=\pm a}$$

Completing the model calculation

$$V(x) = \frac{\omega^2}{8a^2}(x^2 - a^2)^2$$



$$\omega^2 = \left.\frac{d^2V(x)}{dx^2}\right|_{x=\pm a}$$

$$S_1 = \int_{-a}^{+a} dx\, \sqrt{2V(x)} \;= \frac{2}{3}a^2\omega$$

$$\hat{a} = a \exp\left(\int_0^a dx \left[\frac{\omega}{\sqrt{2V(x)}} - \frac{1}{a-x}\right]\right)$$
$$= 2a + O(\sqrt{\hbar})$$

Plug into $\Delta E = 2\hbar^{\frac{1}{2}} e^{-\frac{1}{\hbar}S_1} K$ with $K = \frac{\hat{a}\omega^{\frac{3}{2}}}{\sqrt{\pi\hbar}}$

$$\Delta E = 4\sqrt{\frac{\hbar m}{\pi}} a\omega^{\frac{3}{2}} e^{-\frac{2ma^2\omega}{3\hbar}}$$

(m restored here through dimensional reasoning)

References

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