Instanton Physics

A short course

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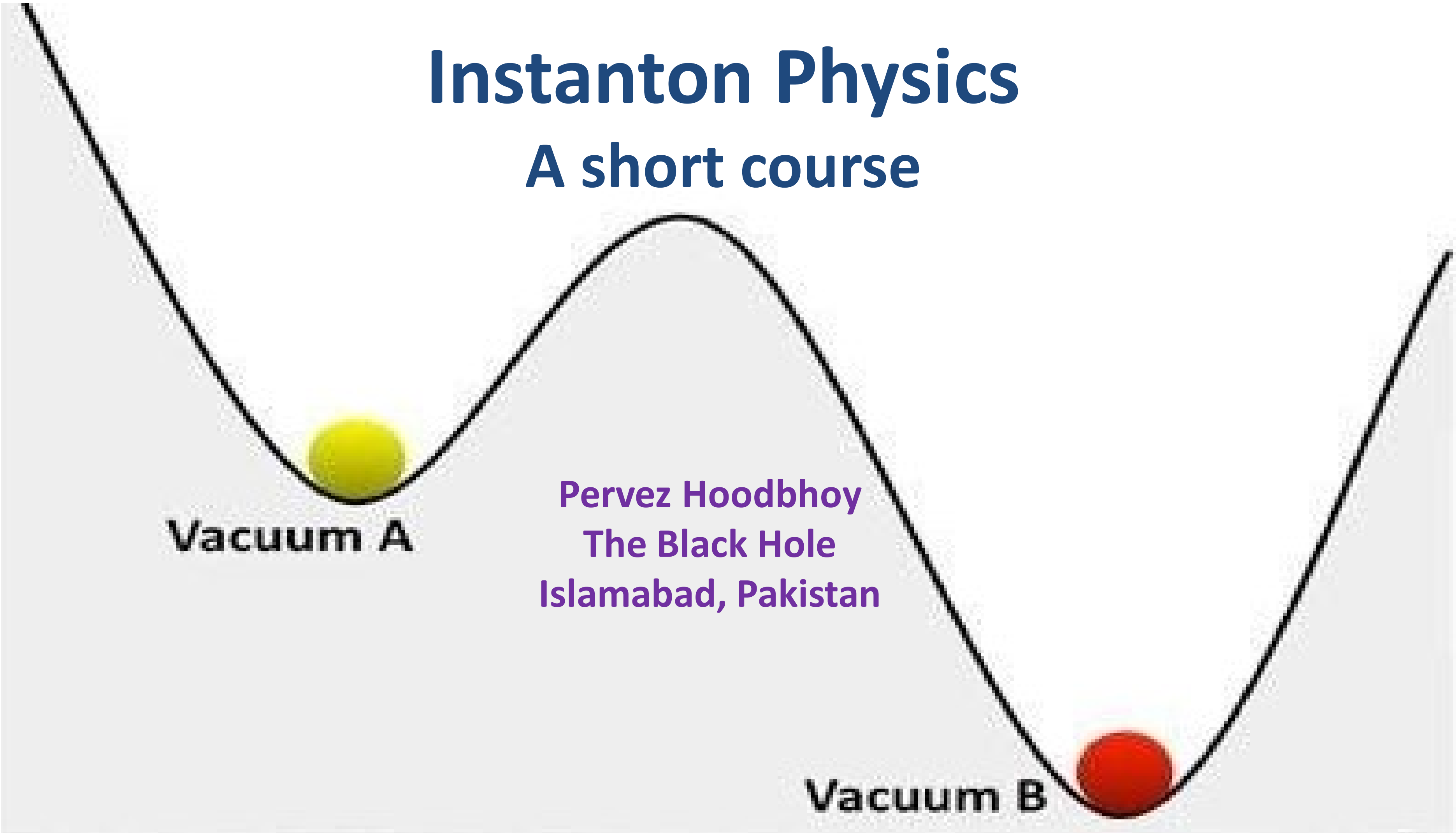


Course Outline

Instantons in particle QM

- Intro to path integral ✓
- Imaginary time ✓
- Instantons in a symmetric double well ✓
- Decay of metastable states ✗
- The functional determinant ✓

Tunneling of quantum fields

- Basic QFT for a scalar field
- Tunneling of field configurations
- The O(4) instanton
- Gauge fields and tunneling
- Effective action
- How/when will the universe end?

Reminder from basic quantum mechanics

$\hat{H}$ is a hermitian operator if $\hat{H} = \hat{H}^\dagger$. Therefore its eigenvalues are real and states preserve their norm in time. Check: ✓

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle \quad \therefore \langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|e^{\frac{i}{\hbar}(\hat{H}^\dagger - \hat{H})t}|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle$$

But even simple Hamiltonians like $\dfrac{\hat{p}^2}{2m}$ can fail to be hermitian! Let's see why:

$$\langle\psi|\hat{H}^\dagger|\psi\rangle = \frac{1}{2m}\int dx\ (\hat{p}^2\psi^*)\psi = -\frac{\hbar^2}{2m}\int dx\ (\frac{d^2}{dx^2}\psi^*)\psi$$

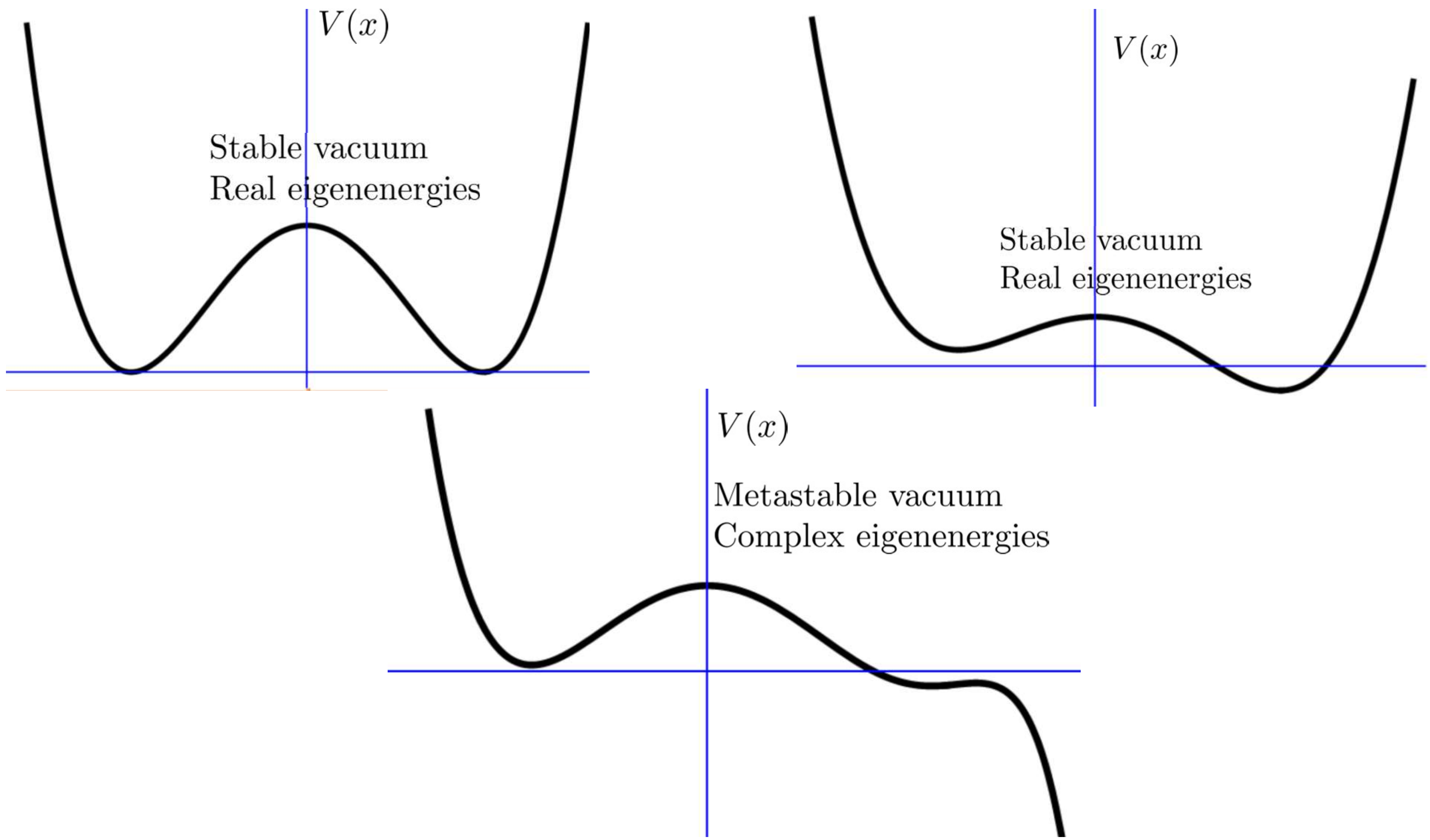
$$= -\frac{\hbar^2}{2m}\int dx\ \overbrace{\frac{d}{dx}\left\{\frac{d\psi^*}{dx}\psi - \psi^*\frac{d\psi}{dx}\right\}}^{\text{normally zero} \,\to\, 0} - \frac{\hbar^2}{2m}\int dx\ \psi^*\frac{d^2}{dx^2}\psi$$

$$= \frac{1}{2m}\int dx\ \psi^*\hat{p}^2\psi \quad = \langle\psi|\hat{H}|\psi\rangle \ ✓$$

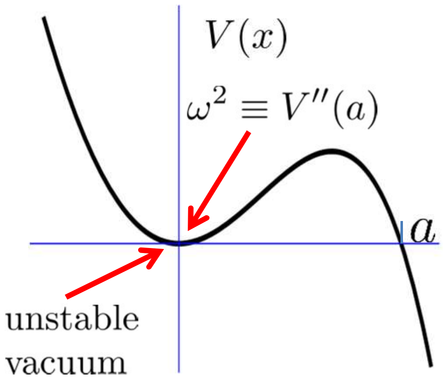
True only if the boundary terms vanish i.e. no current leaks out.

Else $E = \epsilon_R - i\Gamma/2$ and so $\psi(t) = \psi(0)e^{-\frac{i}{\hbar}\epsilon_R t}e^{-\frac{\Gamma}{2\hbar}t}$ is a decaying state.

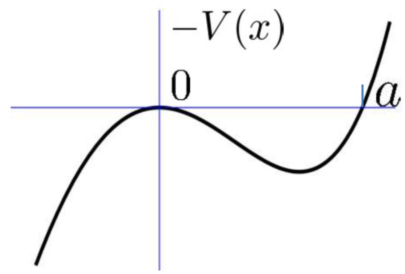
Decay of a metastable state



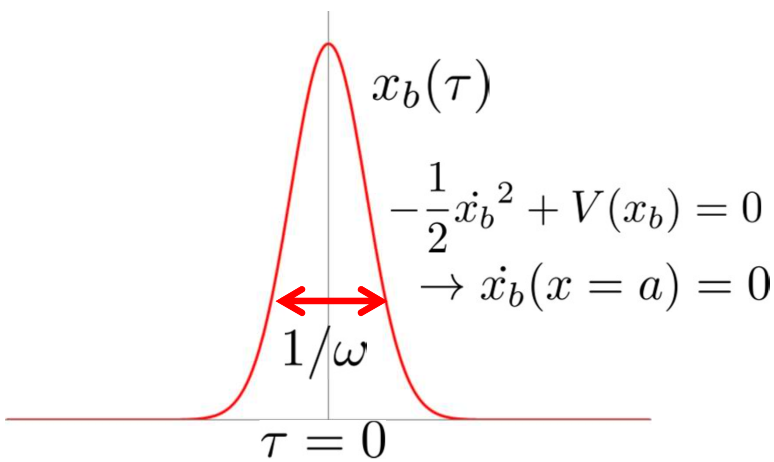
Decay of a metastable state

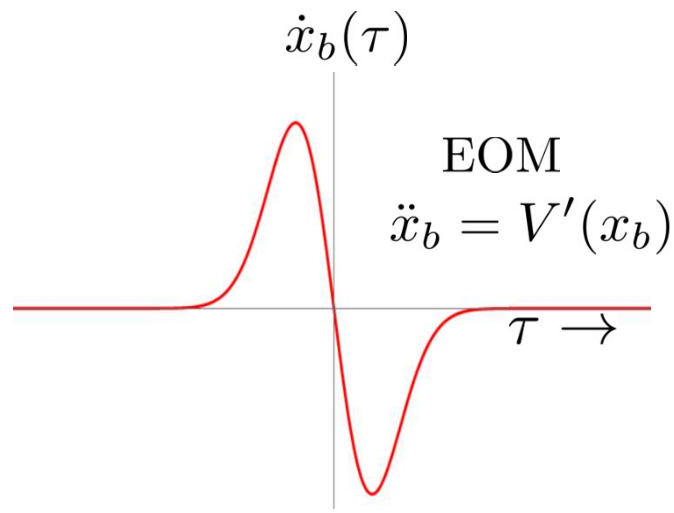


inverted potential
seen by instantons



- Instanton travels from $x = 0$ to $x = a$.
- Departure time from 0 is $\tau = -T/2$
- Arrival time at a is $\tau = 0$.
- Instanton returns to $x = 0$ at $\tau = T/2$.
- The bounce around $\tau = 0$ happens very fast. Most time is spent in transit. Why?



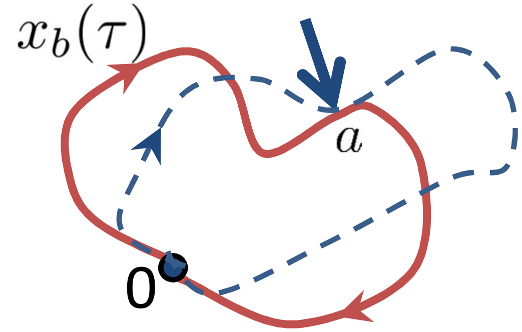


Formally the expanded action is as it was in lecture#2

$$S_{total} = S^{(0)}_{classical} + S^{(2)}_{fluctuation}$$

$$S^{(0)}_{classical} \equiv S^{(0)}_{b} = \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left(\frac{1}{2}\dot{x_b}^2 + V(x_b)\right)$$

$$S^{(2)}_{fluctuation} = \frac{1}{2}\int_{\tau_a}^{\tau_b} d\tau\ \xi(\tau)\left(-\frac{d^2}{d\tau^2} + V''(x_b)\right)\xi(\tau)$$



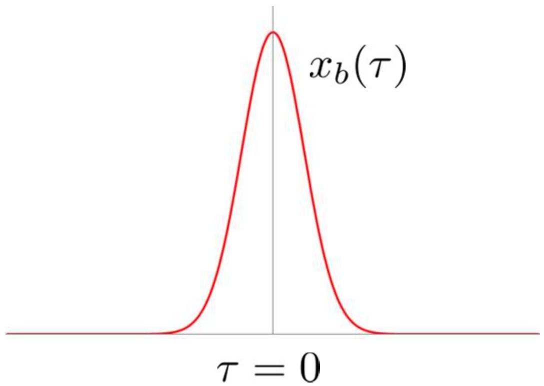
The bounce $x_b(\tau)$ has, $x_b(-\frac{T}{2}) = x_b(\frac{T}{2}) = 0, x_b(0) = a$

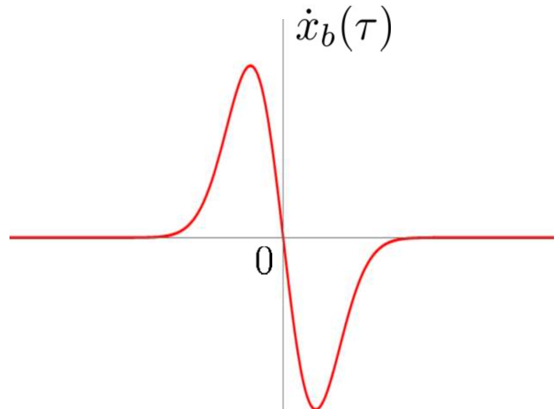
- Expand $-\frac{1}{2}\dot{x_b}^2 + V(x_b) = 0$ around $x = a$ $\rightarrow \dot{x}_b \approx \sqrt{2V(x)} = \omega(x_b - a)$
- The bounce velocity $\dot{x}_b(\tau)$, by virtue of being a solution of the EOM must be an eigenfunction of S'' with zero eigenvalue,

Proof: $\left(-\frac{d^2}{d\tau^2} + V''(x_b)\right)\frac{dx_b}{d\tau} = \frac{d}{d\tau}\left(-\frac{d^2x_b}{d\tau^2} + V'(x_b)\right) = 0$ using the EOM.

Note: $\dot{x}_c(\tau)$ obeys the correct BC's, i.e. $\dot{x}_b(-T/2) = \dot{x}_b(0) = \dot{x}_b(+T/2) = 0.$

But now note a crucial difference





$\dot{x}_b(\tau)$ has zero eigenvalue but it has a node. Hence there must exist some still lower eigenfunction that's without a node and with some negative eigenvalue.

This is easily seen from here: $S_b^{(0)} = \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left(\frac{1}{2} \dot{x_b}^2 + V(x_b) \right)$

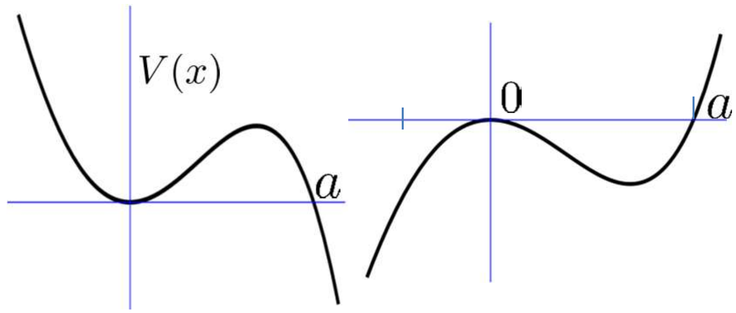
(more nodes in $x_b(\tau)$ means more wiggles, hence more KE and so a bigger λ)

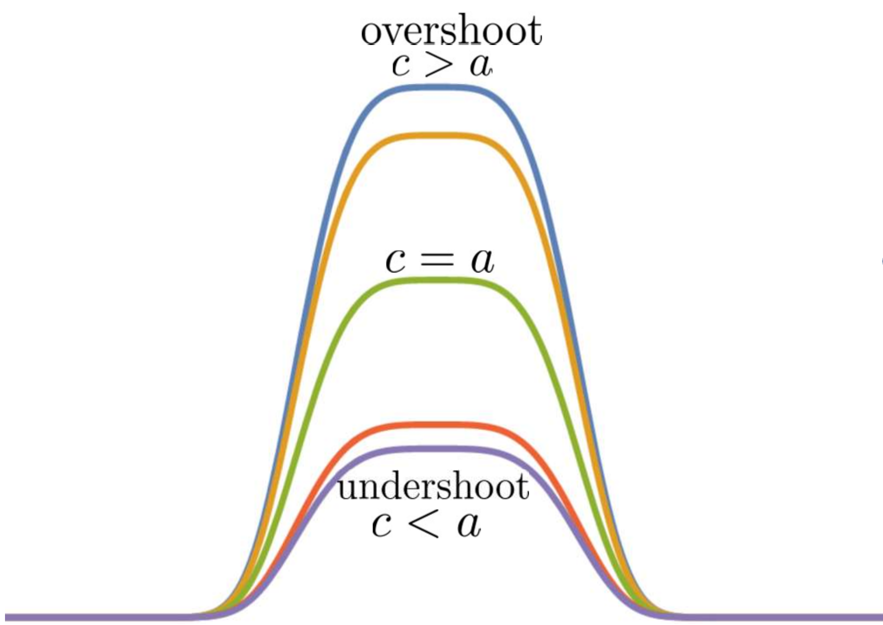
Our previous analysis fails badly:

$$\prod_n \int_{-\infty}^{\infty} \frac{dc_n}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar}\lambda_n c_n^2} = \prod_n \frac{1}{\sqrt{\lambda_n}} = \frac{1}{\sqrt{\det S''}}$$

but now $\lambda_0 = 0$ *and* $\lambda_{-1} < 0$

Is $\int_{-\infty}^{\infty} dc\, e^{|\lambda| c^2} =$ nonsense? Or can it somehow be made meaningful?





- Look at the space of all paths satisfying $x(-T/2) = x(T/2) = 0$ on which $x(\tau)$ takes its maximum at $\tau = 0$. Let $c \equiv x(\tau = 0)$. We will parameterize paths by their value of c.

- A definite value of action is associated with each path, i.e. each value of c. The path which just reaches a extremizes $S(c)$.

$$\left.\frac{dS(c)}{dc}\right|_{c=a} = 0$$

- To give meaning to $\int dc$ we shall have to continue a real integral into the complex plane, i.e. use analytic continuation.

- Loss of probability is associated with $c > a$. We should therefore expect an imaginary part in the probability amplitude.

Review of complex variables and integration - 1

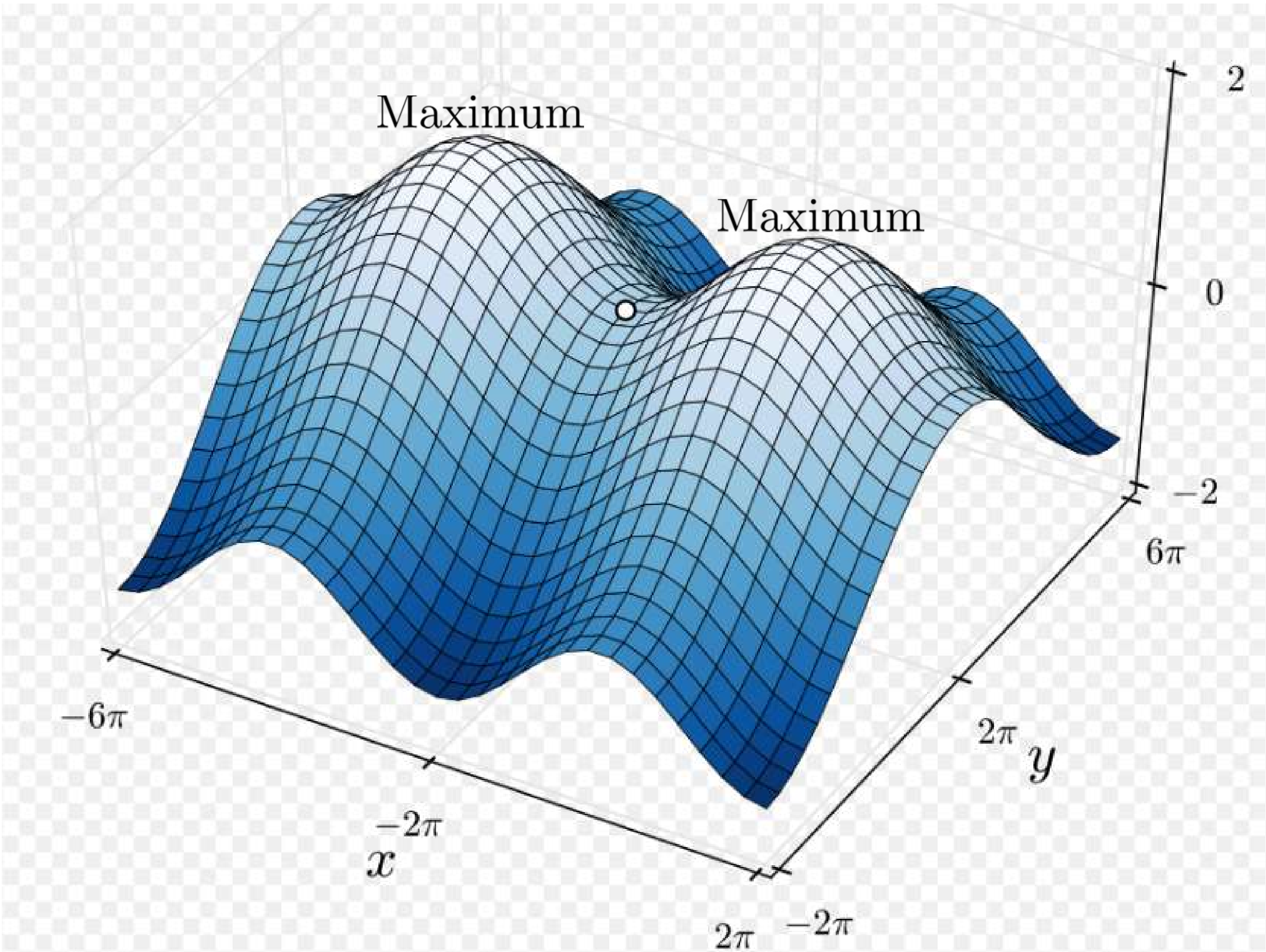
- Suppose $f(z) = u(z) + iv(z)$ where $f(z) = f(x,y), u(z) = u(x,y), v(z) = v(x,y)$
  Then $f'(z)$ exists if $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ (Cauchy Riemann condition)

- If CR holds then $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$. (Check by differentiating.)

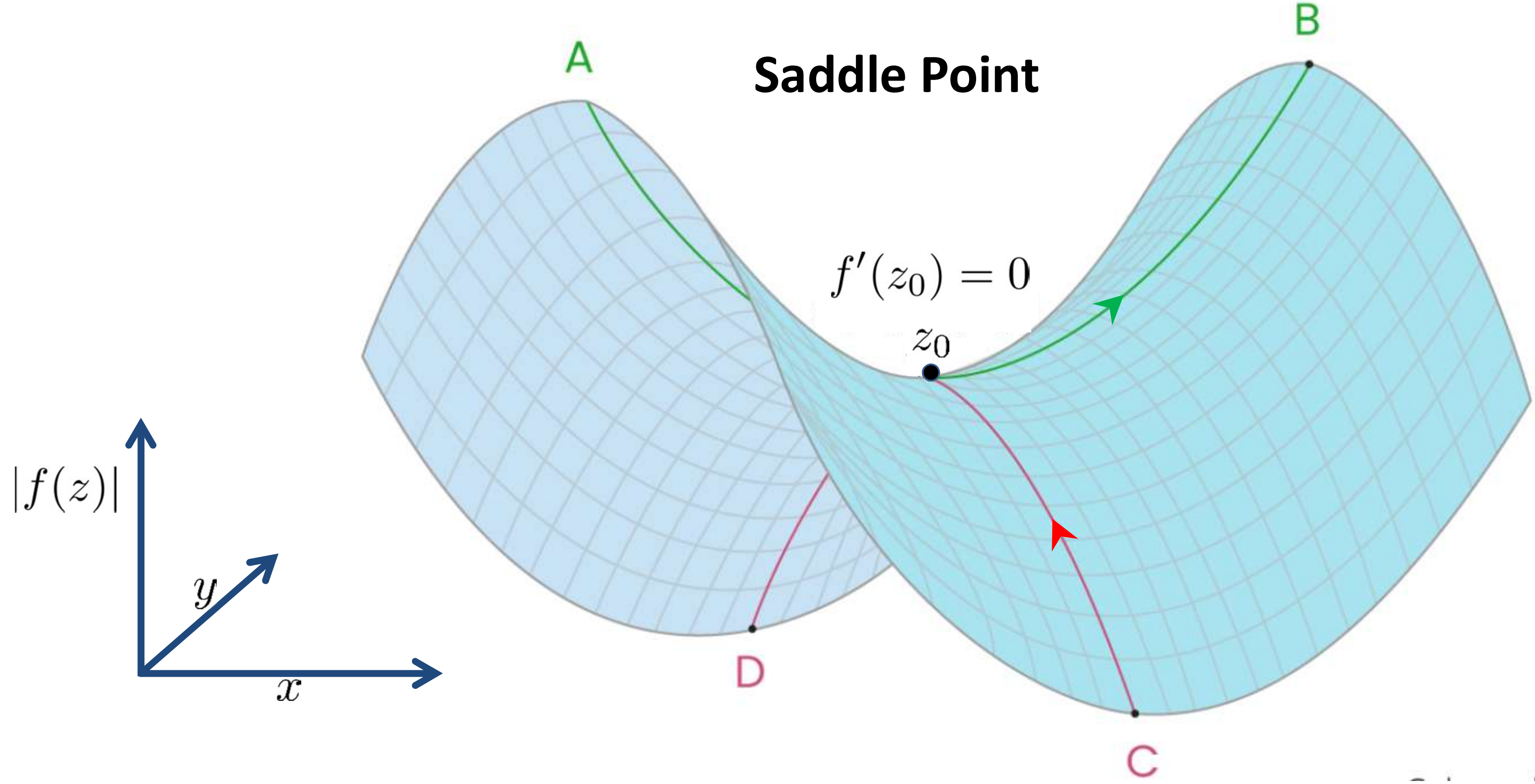
- Suppose $f'(z_0) = 0$, i.e. if $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$. Then this is an extreme point.

  But is it a maximum or minimum? Answer: it can be both! That's because if $\dfrac{\partial^2 u}{\partial x^2} < 0$ (i.e. u has a max along x) then $\dfrac{\partial^2 u}{\partial y^2} > 0$ (i.e. u has a min along y.) This is called a saddle point.

- Note from CR, $\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y} = \nabla u \cdot \nabla v = 0$ and so contours of constant phase are also contours of steepest descent.



Maximum
Maximum
2
0
−2
6π
2π
y
−6π
−2π
x
2π
−2π



On $A - z_0 - B$, note that z_0 is the *minimum* point
On $C - z_0 - D$, note that z_0 is the *maximum* point

Review of complex variables and integration - 2

- If $f(z)$ is differentiable in a region it is *analytic* there. Cauchy's integral theorem then holds: $\oint_\Gamma dz f(z) = 0$ provided $f(z)$ is analytic inside the region Γ. This allows us to deform the integration path if $f(z)$ is analytic inside Γ.

- We are interested in calculating integrals like $\int_{-\infty}^{\infty} \frac{dc}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar}\lambda c^2}$ with $\lambda < 0$.

- To make sense we go to the complex plane and deform the integration path as we choose - provided that we avoid all singularities such as poles or branch cuts. So with a, b complex numbers that we can change at will, consider a general integral of this kind,

$$I = \int_a^b dz\, e^{-\frac{1}{\epsilon} f(z)} \quad \text{with } \epsilon \text{ small.}$$

Review of complex variables and integration - 3

Expand $f(z)$ about z_0: $f(z) = f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2$ with $f'(z_0) = 0$.

Consider: $I = \int_a^b dz\, e^{-\frac{1}{\epsilon} f(z)} = \exp\left\{-\frac{1}{\epsilon} f(z_0)\right\} \int_a^b dz\, \exp\left\{-\frac{1}{2\epsilon} f''(z_0)(z - z_0)^2\right\}$

Now change coordinates: $z - z_0 = re^{i\theta}$ $\quad = \exp\left\{-\frac{1}{\epsilon} f(z_0)\right\} \int dr e^{i\theta} \exp\left\{-\frac{1}{2\epsilon} f''(z_0) r^2 e^{2i\theta}\right\}$

If $f''(z_0)$ is negative rewrite as, $f''(z_0) = -|f''(z_0)|$

$$\therefore I = \exp\left\{-\frac{1}{\epsilon} f(z_0)\right\} \int dr e^{i\theta} \exp\left\{\frac{1}{2\epsilon} |f''(z_0)| r^2 e^{i2\theta}\right\}$$

Choose $\theta = \pi/2$ for path of steepest descent.

$$= i \exp\left\{-\frac{1}{\epsilon} f(z_0)\right\} \int_0^\infty dr\, \exp\left\{-\frac{1}{2\epsilon} |f''(z_0)| r^2\right\}$$

$$= i\, \frac{1}{2} \exp\left\{-\frac{1}{\epsilon} f(z_0)\right\} \sqrt{\frac{2\pi\epsilon}{|f''(z_0)|}} \longrightarrow \boxed{i\, \frac{1}{2} \exp\left\{-\frac{1}{\hbar} S_b\right\} \sqrt{\frac{1}{|\lambda_{-1}|}}}$$

note

Now put it all together and move towards the final result

Including fluctuations, and after integrating over all bounce times between $-T/2$ and $+T/2$, the correct single bounce amplitude is:

$$\mathcal{A}^{(1)}_{(-\frac{T}{2},0)\to(\frac{T}{2},0)} = \int [dx] e^{-\frac{1}{\hbar}(S^{(0)}+S^{(2)})} = \mathcal{N} e^{-\frac{1}{\hbar}S_b} \sqrt{\frac{S_b}{2\pi\hbar}}\, T\, \frac{i}{2} \prod_n{}' \frac{1}{\sqrt{\lambda_n}}$$

$K \sim$ ratio of the SHO determinant to that of the actual determinant (minus the zero and negative mode)

$$K \equiv \frac{1}{2} \sqrt{\frac{S_b}{2\pi\hbar}} \frac{1}{\sqrt{|\lambda_{-1}}} \frac{\det\left[-\frac{d^2}{d\tau^2}+\omega^2\right]^{\frac{1}{2}}}{\det'\left[-\frac{d^2}{d\tau^2}+V''(x_c)\right]^{\frac{1}{2}}}$$

Note

$$\therefore \mathcal{A}^{(1)} = \frac{\mathcal{N}}{\det\left[-\frac{d^2}{d\tau^2}+\omega^2\right]^{\frac{1}{2}}} \times i e^{-\frac{1}{\hbar}S_b} K T \xrightarrow[T\to\infty]{} \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega T}{2}} \times i e^{-\frac{1}{\hbar}S_b} K T$$

Now sum over all bounces: $Z(T) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega T}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} i^n T^n K^n e^{-\frac{n S_b}{\hbar}}$

$$\boxed{Z(T) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega T}{2}} \exp\left[iTKe^{-\frac{S_b}{\hbar}}\right]}$$

(dilute bounce approximation)

The final touches

Recall: $Z(T) = \langle x=0|e^{-\frac{HT}{\hbar}}|x=0\rangle$ and insert $\hat{\mathbf{1}} = \sum |n\rangle\langle n$

$$= |\phi_0(0)|^2 e^{-\frac{E_0 T}{\hbar}} + |\phi_1(0)|^2 e^{-\frac{E_1 T}{\hbar}} + \cdots$$

$$\therefore \log(Z(T)) \xrightarrow[T\to\infty]{} -\frac{E_0 T}{\hbar}$$

Now look at our hard won result: $Z(T) = \left(\dfrac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega T}{2}} \exp\left[iTKe^{-\frac{S_b}{\hbar}}\right]$

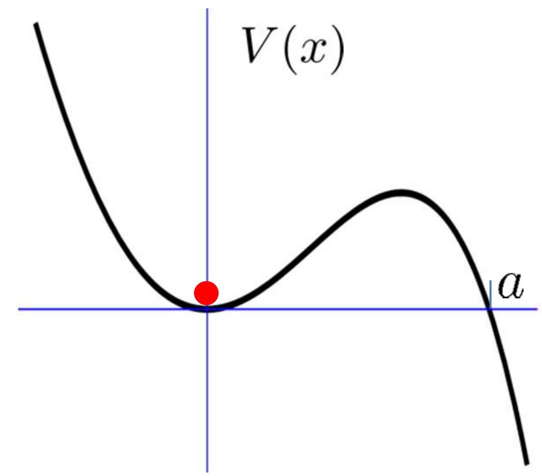
$\log(Z(T)) \xrightarrow[T\to\infty]{} -\dfrac{\frac{1}{2}\hbar\omega}{\hbar}T + i\dfrac{\hbar K}{\hbar}e^{-\frac{S_b}{\hbar}}T$ Compare this against

$-\dfrac{E_0 T}{\hbar} = -\dfrac{(\epsilon_R - i\Gamma/2)T}{\hbar}$ to get $E_0 = \dfrac{1}{2}\hbar\omega - iKe^{-\frac{S_b}{\hbar}}$

Recall: $S_b = \displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left(\frac{1}{2}\dot{x_b}^2 + V(x_b)\right)$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau\, \dot{x}_b^2 \quad = 2\int_0^a dx_b \frac{d\tau}{dx_b}\left(\frac{dx_b}{d\tau}\right)^2 \quad = 2\int_0^a dx \sqrt{2V(x)}$$

The Grand Result



$$\Gamma \equiv Ae^{-B}$$ = probability per unit time for tunneling out of the unstable vacuum.

$$A = \sqrt{\frac{S_b}{2\pi\hbar}}\sqrt{\frac{\det S_0''}{\det' S_b''}}\exp\left(-\frac{S_b}{\hbar}\right)$$

$$B = \frac{S_b}{\hbar} = \frac{2}{\hbar}\int_0^a dx\sqrt{2V(x)}$$

References

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