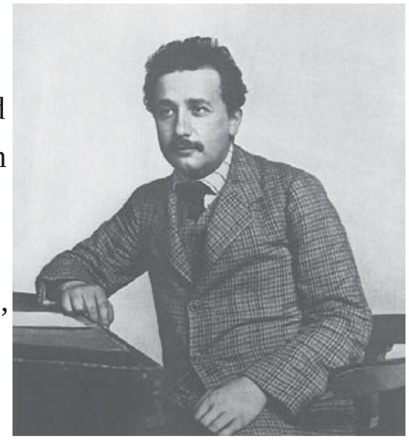
Summary of Lecture 39 – SPECIAL RELATIVITY I

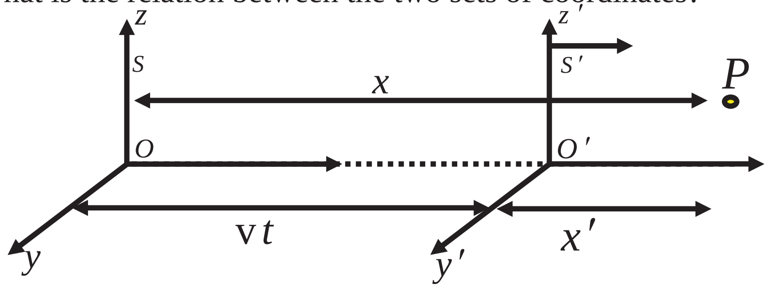


1. The Special Theory of Relativity was created by Albert Einstein, when he was a very young man in 1905. It is one of the most solid pillars of physics and has been tested thousands of times. Special Relativity was a big revolution in understanding the nature of space and time, as well as mass and energy. It is absolutely necessary for a proper understanding of fast-moving particles (electrons, photons, neutrinos..). Einstein's General Theory of Relativity - which we shall not even touch here - goes beyond this and also deals with gravity.

2. Relativity deals with time and space. So let us first get some understanding of how we measure these two fundamental quantities:
   a) *Time* **:** we measure time by looking at some phenomenon that repeats itself. There are endless examples: your heart beat, a pendulum, a vibrating quartz crystal, rotation of the earth around its axis, the revolution of the earth about the sun,... These can all be used as clocks. Of course, an atomic clock is far more accurate than using your heart beat and is accurate to one part in a trillion. Although different systems of measurement have different units it is fortunate that time is always measured in seconds.
   b) *Distance* **:** intuitively we know the difference between short and long. But to do a measurement, we first have to agree on what should be the unit of length. If you use a metre as the unit, then you can use a metre rod and measure any length you want. Of course, sometimes we may use more sophisticated means (such as finding how high a satellite is) but the basic idea is the same: the distance between point A and point B is the number of metre rods (or fractions thereof) that can be made to fit in between the two points.

3. Newton had believed that there was one single time for the entire universe. In other words, time was absolute and could be measured by one clock held somewhere in the centre of the universe. Similarly, he believed that space was absolute and that the true laws of physics could be seen in that particular frame which was fixed to the centre of the universe. (As we shall see, Einstein shocked the world by showing that time and space are not absolute quantities, but depend on the speed of your reference frame. Even more shocking was his proof that time and space are not entirely separate quantities!)

4. An *event* is something that happens at some point in space at some time. With respect to the frame S below, the event P happened at (x,y,z,t). Now imagine a girl running to the right at fixed speed v in frame S′. According to her, the same event P happened at (x′,y′,z′,t′). What is the relation between the two sets of coordinates?



If you were Newton, then you would look at the above figure and say that obviously it is the following: $x' = x - \mathrm{v}t,\ y' = y,\ z' = z,\ t' = t$. These are called *Galilean coordinate transformations*. Note that it is assumed here that the time is the same in both frames because of the Newtonian belief that there is only one true time in the world.

5. There are certain obvious consequences of using the Galilean transformations. So, for example a rod is at rest in S-frame. The length in S-frame $= x_B - x_A$, while the length in the S′-frame $= x'_B - x'_A \ = \ x_B - x_A - \mathrm{v}(t_B - t_A)$. Since $t_B = t_A$, the length is the same in both frames: $x'_B - x'_A \ = \ x_B - x_A$. (As we shall see later this will not be true in Einstein's Special Relativity).

6. Let us now see what the Galilean transformation of coordinates implies for transformations of velocities. Start with $x' = x - \mathrm{v}t$ and differentiate both sides. Then $\frac{dx'}{dt} = \frac{dx}{dt} - \mathrm{v}$. Since $t = t'$, it follows that $\frac{dx'}{dt} = \frac{dx'}{dt'}$ and so $\frac{dx'}{dt'} = \frac{dx}{dt} - \mathrm{v}$. Similarly, $\frac{dy'}{dt'} = \frac{dy}{dt}$ and $\frac{dz'}{dt'} = \frac{dz}{dt}$. Now, $\frac{dx'}{dt'} = u'_x$ is the x-component of the velocity measured in S′-frame, and similarly $\frac{dx}{dt} = u_x$ is the x-component of the velocity measured in S-frame. So we have found that:

$$u'_x = u_x - \mathrm{v},\quad u'_y = u_y,\quad u'_z = u_z \text{ (or, in vector form, } \vec{u}' = \vec{u} - \vec{\mathrm{v}})$$

Taking one further derivative, $\frac{du'_x}{dt'} = \frac{d}{dt}(u_x - \mathrm{v}) = \frac{du_x}{dt}$ (remember that v = constant), we find that the components of acceleration are the same in S and S′:

$$\frac{du'_x}{dt'} = \frac{du_x}{dt},\quad \frac{du'_y}{dt'} = \frac{du_y}{dt},\quad \frac{du'_z}{dt'} = \frac{du_z}{dt}.$$

7. We had learned in a previous chapter that light is electromagnetic wave that travels at a speed measured to be $c = 2.907925 \times 10^8 m/\sec$. Einstein, when he was 16 years old, asked himself the question: in which frame does this light travel at such a speed? If I run holding a torch, will the light coming from the torch also move faster? Yes, if we use the formula for addition of velocities derived in the previous section! So if light actually travels at $2.907925 \times 10^8 m/\sec$ then it is with respect to the frame in which the "aether" (a massless fluid which we cannot feel) is at rest. But there is no evidence for the aether!

8. Einstein made the two following postulates (or assumptions).

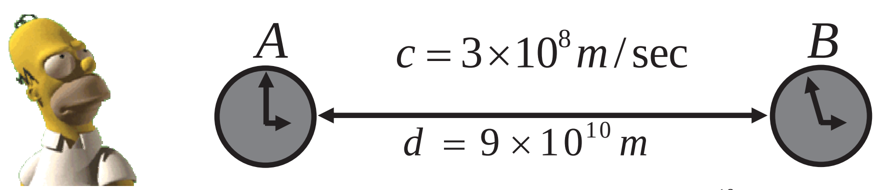
1) The laws of physics have the same form in all inertial frames. (In other words, there is no constant-velocity frame which is preferred, or better, than any other).

2) The speed of light in vacuum has the same value in all inertial systems, independent of the relative motion of source and observer.

[Note: as stressed in the lecture, no postulate of physics can ever be mathematically proved. You have to work out the consequences that follow from the postulates to know whether the postulates are good ones or not.]

9. Let's first get one thing clear: in any one inertial frame, we can imagine that there are rulers and clocks to measure distances and times. We can synchonize all the clocks to read one time, which will be called the time in that frame S. But how do we do this? If are two clocks, then we can set them to read the same time (i.e. synchronize them) by taking account of the time light takes to travel between them.

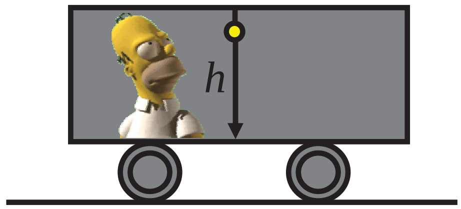
Example: The observer with clock A sees the time on clock B as 2:55pm. But he knows that light took 5 minutes to travel from B to A, and therefore A and B are actually reading exactly the same time.



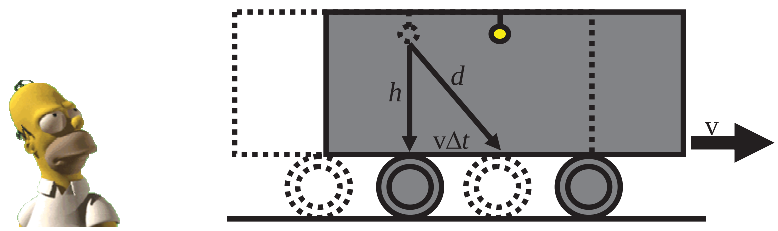
(Time taken by light in going from B to A is equal to $\frac{d}{c} = \frac{9 \times 10^{10} m}{3 \times 10^8 m/\sec} = 300 \sec = 5 \min$.)

10. Now I shall derive the famous formula which shows that a moving clock runs slow. This will be something completely different from the older Newtonian conception of time. Einstein derived this formula using a "gedanken" experiment, meaning an experiment which can imagine but not necessarily do. So imagine the following: a rail carriage has a

a bulb that is fixed to the ceiling. The bulb suddenly flashes, and the light reaches the floor. Time taken according to the observer inside the train is $\Delta t' = h/c$.



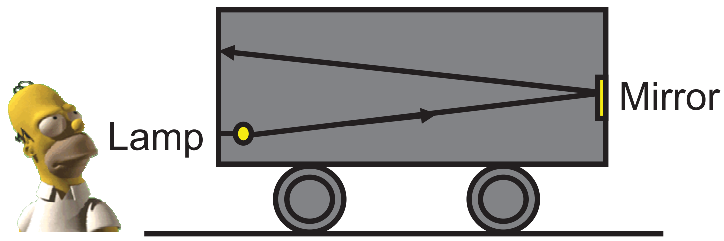
Now suppose that the same flash is observed by an observer S standing on the ground. According to S, the train is moving with speed v. Let's consider the same light ray. Clearly, the train has moved forward between the time when the light left the ceiling and when it reached the floor. According to S, the time taken is $\Delta t = \frac{d}{c} = \frac{\sqrt{h^2 + (\mathrm{v}\Delta t)^2}}{c}$. Now



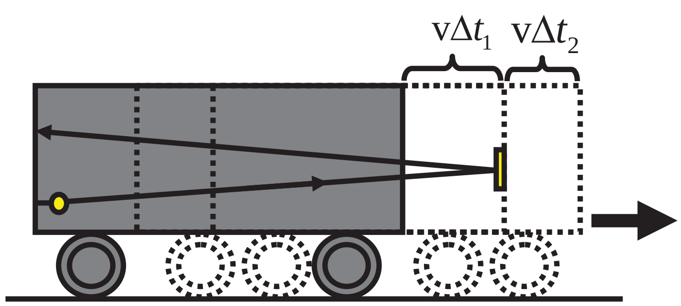
square both sides: $(c\Delta t)^2 = h^2 + (\mathrm{v}\Delta t)^2$, which gives $\Delta t = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\frac{h}{c} = \gamma\Delta t'$. Here γ is the relativistic factor, $\gamma = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}$ and is a number that is always bigger than one. As v gets closer and closer to c, the value of γ gets larger and larger. For v=4c/5, γ=5/3. So, if 1 sec elapses between the ticks of a clock in S′ (i.e. $\Delta t' = 1$), the observer in S will see 5/3 seconds between the ticks. In other words, he will think that the moving clock is slow!

11. The muon is an unstable particle. If at rest, it decays in just 10^{-18} seconds. But if traveling at 3/5 the speed of the light, it will last 25% longer because $\gamma = \frac{1}{\sqrt{1-(3/5)^2}} = \frac{5}{4}$. If it is traveling at v=0.999999c, it will last 707 times longer. We can observe these shifts due to time dilation quite easily, and they are an important confirmation of Relativity.

12. Another amazing prediction of Relativity is that objects are shortened (or contracted) along the direction of their motion. Einstein reached this astonishing conclusion on the basis of yet another gedanken experiment. Again, consider a moving railway carriage with a bulb



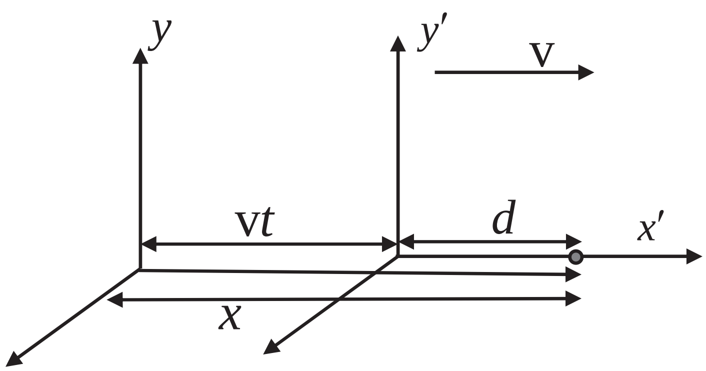
at one end that suddenly flashes. Let Δx be the length of the carriage according to ground observer S, and $\Delta x'$ be the length according to the observer S′ inside the carriage. So, $\Delta t' = \frac{2\Delta x'}{c}$ is the time taken for the light to go from one end to the other, and then return after being reflected by a mirror. Now let's look at this from the point of view of S, who is fixed to the ground. Let Δt_1 be the time for the signal to reach the front end. Then, because the mirror is moving forward, $\Delta t_1 = \frac{\Delta x + v\Delta t_1}{c}$. Call Δt_2 the return time. Then,



$\Delta t_2 = \frac{\Delta x - v\Delta t_2}{c}$. Solving for Δt_1 and Δt_2: $\Delta t_1 = \frac{\Delta x}{c - v}$ and $\Delta t_2 = \frac{\Delta x}{c + v}$. The total time is therefore $\Delta t = \Delta t_1 + \Delta t_2 = \frac{2\Delta x / c}{\left(1 - v^2/c^2\right)}$. Now, from the time dilation result derived earlier, $\Delta t' = \sqrt{1 - v^2/c^2}\,\Delta t$, and so $\Delta t' = \sqrt{1 - v^2/c^2}\left(\frac{2\Delta x / c}{\left(1 - v^2/c^2\right)}\right) = \frac{2\Delta x / c}{\sqrt{1 - v^2/c^2}}$. This gives, $\Delta x' = \frac{\Delta x}{\sqrt{1 - v^2/c^2}}$, or $\Delta x = \Delta x'/\gamma$. This an astonishing result! Suppose that there is a metre rod. Then the observer riding with it has $\Delta x' = 1$. But according to someone who sees the metre rod moving towards/away from him, the length is $1/\gamma$. This is less than 1 metre!

13. Although an object shrinks in the direction of motion (both while approaching and receding), the dimensions perpendicular to the velocity are not contracted. It is easy to conceive of a gedanken experiment that will demonstrate this. One of the exercises will guide you in this direction.

14. We shall now derive the "Lorentz Transformation", which is the relativistic version of the Galilean transformation discussed earlier. Consider an event that occurs at position x (as measured in S) at a distance d (again, as seen in S) from the origin of S′. Then, $x = d + \text{v}t$. From the Lorentz contraction formula derived earlier, $d = \frac{x'}{\gamma}$ where x' is the distance measured in S′. This gives $x' = \gamma(x - \text{v}t)$. Now, by the same logic, $x' = d' - \text{v}t'$.



Here $d' = \frac{x}{\gamma}$. This gives $x = \gamma(x' + \text{v}t') = \gamma(\gamma(x - \text{v}t) + \text{v}t')$. We find that the time in S′ is related to the time in S by $t' = \gamma\left(t - \frac{\text{v}}{\text{c}^2}x\right)$. Note that if we make c very large, then $t' = t$.

To summarize:

LORENTZ TRANSFORMATION

$$x' = \gamma(x - \text{v}t)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{\text{v}}{c^2}x\right).$$

(Note: in various books you will find slightly different derivations of the above Lorentz transformation. You should look at one of your choice and understand that as well.)

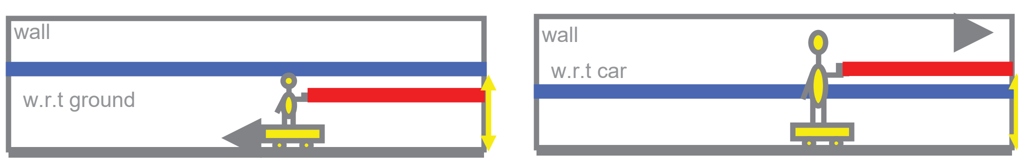
QUESTIONS AND EXERCISES – 39

1. Instead of using a ruler to measure distances, can we use a laser together with an accurate clock instead? Discuss.

2. What must be the value of v/c if a) $\gamma = 1.01$, b) $\gamma = 10$, c) $\gamma = 1000$.

3. Two identical spaceships carrying identical clocks pass by each other at a speed of 0.99c.
   a) Each sees the other's clock as running slow. By how much?
   b) Each sees the other's spaceship shortened. By how much?
   c) How is it possible for the observers on both spaceships to be correct in saying that the other's clock is slow, and the other's spaceship is shortened?

4. Devise a logical argument that distances perpendicular to the direction of motion do not shrink. As a help, consider the following gedanken experiment: a train passes through a tunnel where, at a certain height above the ground a blue stripe has been painted. A man in the train has a paintbrush with red paint on it which leaves a red stripe as the train moves at high speed. Make your case using the equivalence of rest frames.



5. a) Two events happen at the same point in the same inertial frame S. To be specific, a man stays still as blows his nose (event 1) and then wipes it (event 2). In any other frame S′ that is moving in the x direction, show that the sequence of these two events will remain unchanged.
   b) Will this still be true if the man, instead of standing still, is walking very fast at half the speed of light?

6. The Lorentz transformation is $x' = \gamma(x - \mathrm{v}t),\ y' = y,\ z' = z,\ t' = \gamma\left(t - \frac{\mathrm{v}}{c^2}x\right)$. Solve for the inverse transformation, i.e express x, y, z, t in terms of x', y', z', t'.

Summary of Lecture 40 – SPECIAL RELATIVITY II

1. Recall the Lorentz Transformation: $x' = \gamma(x - \mathrm{v}t)$ and $t' = \gamma\left(t - \frac{\mathrm{v}}{c^2}x\right)$. Suppose we take the space interval between two events $\Delta x = x_1 - x_2$, and the time interval $\Delta t = t_1 - t_2$. Then, these intervals will be seen in S′ as $\Delta t' = \gamma\left(\Delta t - \frac{\mathrm{v}}{c^2}\Delta x\right)$ and $\Delta x' = \gamma(\Delta x - \mathrm{v}\Delta t)$.

   Now consider two particular cases:

   a) Suppose the two events occur at the same place (so $\Delta x = 0$) but at different times (so $\Delta t \neq 0$). Note that in S′ they do not occur at the same point: $\Delta x' = \gamma(0 - \mathrm{v}\Delta t)$!

   b) Suppose the two events occur at the same time (so $\Delta t = 0$) but at different places (so $\Delta x \neq 0$). Note that in S′ they are not simultaneous: $\Delta t' = \gamma\left(0 - \frac{\mathrm{v}}{c^2}\Delta x\right)$.

2. As seen in the frame S, suppose a particle moves a distance dx in time dt. Its velocity u is then $u = \frac{dx}{dt}$ (in S-frame). As seen in the S′-frame, meanwhile, it has moved a distance dx' where $dx' = \gamma(dx - \mathrm{v}dt)$ and the time that has elapsed is $dt' = \gamma\left(dt - \frac{\mathrm{v}}{c^2}dx\right)$. The velocity in S′-frame is therefore $u' = \frac{dx'}{dt'} = \frac{\gamma(dx - \mathrm{v}dt)}{\gamma\left(dt - \frac{\mathrm{v}}{c^2}dx\right)} = \frac{dx/dt - \mathrm{v}}{1 - \frac{\mathrm{v}}{c^2}dx/dt} = \frac{u - \mathrm{v}}{1 - \frac{u\mathrm{v}}{c^2}}$. This is the Einstein velocity addition rule. It is an easy exercise to solve this for u in terms of u',

$$u = \frac{u' + \mathrm{v}}{1 + \frac{u'\mathrm{v}}{c^2}}.$$

3. Note one very interesting result of the above: suppose that a car is moving at speed v and it turns on its headlight. What will the speed of the light be according to the observer on the ground? If we use the Galilean transformation result, the answer is v+c (wrong!). But using the relativistic result we have $u' = c$ and $u = \frac{c + \mathrm{v}}{1 + \frac{c\mathrm{v}}{c^2}} = \frac{c + \mathrm{v}}{1 + \frac{\mathrm{v}}{c}} = c\frac{c + \mathrm{v}}{c + \mathrm{v}} = c$. In other words, the speed of the source makes no difference to the speed of light in your frame. Note that if either u or v is much less than c, then u' reduces to the familiar result: $u' = \frac{u - \mathrm{v}}{1 - \frac{u\mathrm{v}}{c^2}} \rightarrow u - \mathrm{v}$, which is the Galilean velocity addition rule.

4. The Lorentz transformations have an interesting property that we shall now explore. Take the time and space intervals between two events as observed in frame S, and the corresponding quantities as observed in S′. We will now prove that the quantities defined respectively as $I = (c\Delta t)^2 - (\Delta x)^2$ and $I' = (c\Delta t')^2 - (\Delta x')^2$ are equal. Let's start with I':

$$I' = (c\Delta t')^2 - (\Delta x')^2 = \gamma^2\left((c\Delta t)^2 + (\Delta x)^2 \mathrm{v}^2/c^2 - \cancel{2\mathrm{v}\Delta t\Delta x} - (\Delta x)^2 - (\mathrm{v}\Delta t)^2 + \cancel{2\mathrm{v}\Delta t\Delta x}\right)$$

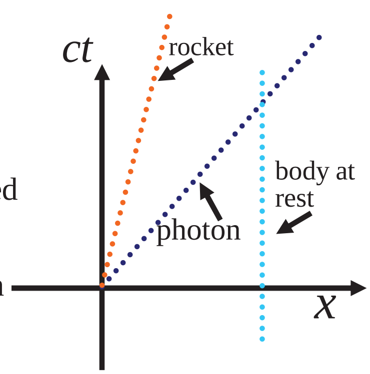
$$= \frac{1}{(1-\mathrm{v}^2/c^2)}\left((c^2-\mathrm{v}^2)(\Delta t)^2 - (\Delta x)^2(1-\mathrm{v}^2/c^2)\right)$$

$$= (c\Delta t)^2 - (\Delta x)^2 = I$$

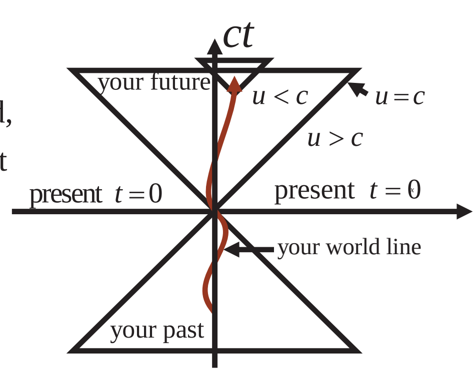
This guarantees that all inertial observers measure the same speed of light !!

5. If the time separation is large, then $I > 0$ and we call the interval *timelike*.
If the space separation is large, then $I < 0$ and we call the interval *spacelike*.
If $I = 0$ and we call the interval *lightlike*.
Note : a) If an interval is timelike in one frame, it is timelike in all other frames as well.
b) If interval between two events is timelike, their time ordering is absolute.
c) If the interval is spacelike the ordering of two events depends on the frame from which they are observed.

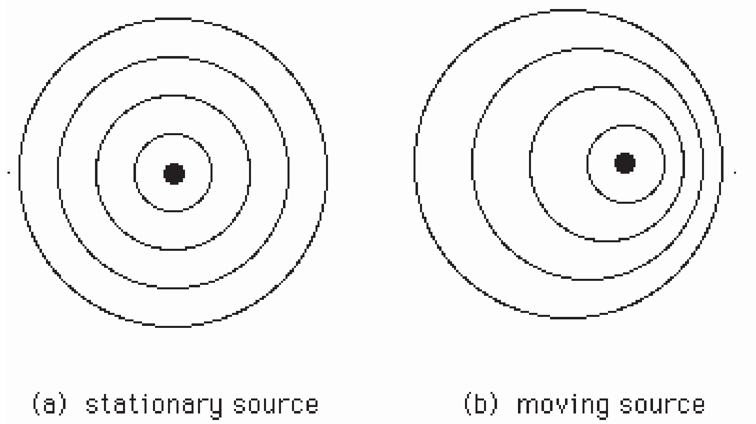
6. It is sometimes nice to look at things graphically. Here is a graph of position versus time for objects that move with different speeds along a fixed direction. First look at an object at rest. Its position is fixed even though time (plotted on the vertical axis) keeps increasing. Then look at the rocket moving at constant speed (which has to be less than c), and finally a photon (which can only move at c).



7. The trajectory of a body as it moves through space-time is called its world-line. Let's take a rocket that is at $x = 0$ at $t = 0$. It moves with non-constant speed, and that is why its world-line is wavy. A photon that moves to the right will have a world line with slope equal to +1, and that to the right with slope -1. The upper triangle (with t positive) is called the future light cone (don't forget we also have y and x). The lower light cone consists of past events.



8. Earlier on we had discussed the Doppler effect in the context of sound. Now let us do so for light. Why are they different? Because light always moves with a fixed speed while the speed of sound is different according to a moving and a fixed observer. Consider the case of a source of light at rest, and one that is moving to the right as shown below:



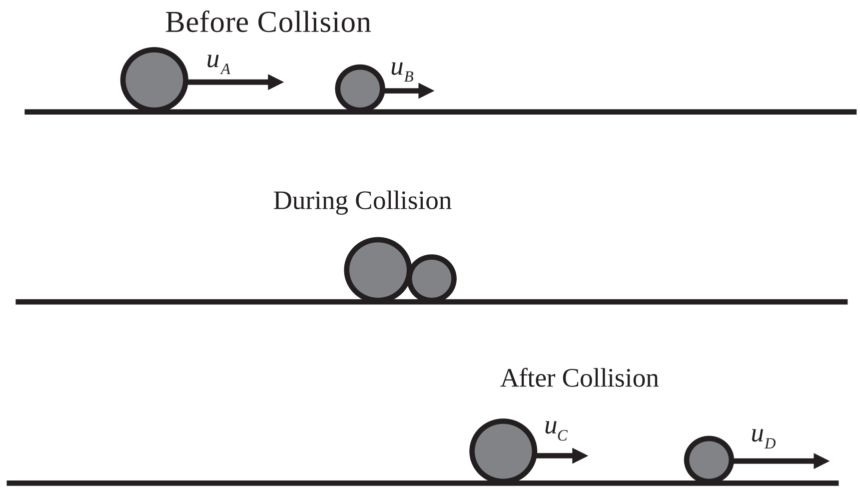
Let $\nu_0 = \frac{1}{T_0}$ be the frequency measured in the source's rest frame S, where T_0 is the time for one complete cycle in S. We want to calculate ν, the frequency as seen by the observer in S′ moving to the right at speed v. Call λ the distance between two successive wave crests (i.e. the wavelength according to the observer). In time T the crests ahead of the source move a distance cT, even as the source moves a shorter distance $\mathrm{v}T$ in the same direction. Hence $\lambda = (c - \mathrm{v})T$ and so $\nu = \frac{c}{\lambda} = \frac{c}{(c-\mathrm{v})T}$. Now, as discussed earlier, the time measured by observer will not be T_0 because of time dilation. Instead,

$$T = \frac{T_0}{\sqrt{1-\mathrm{v}^2/c^2}} = \frac{cT_0}{\sqrt{c^2-\mathrm{v}^2}}. \text{ Hence, } \nu = \frac{c}{(c-\mathrm{v})T} = \left(\frac{c}{c-\mathrm{v}}\right)\frac{\sqrt{c^2-\mathrm{v}^2}}{c}\nu_0 = \sqrt{\frac{c+\mathrm{v}}{c-\mathrm{v}}}\nu_0.$$

(If source moves away from the observer just change the sign of v: $\nu = \sqrt{\frac{c-\mathrm{v}}{c+\mathrm{v}}}\nu_0$.)

This is the famous Doppler effect formula. In the lecture I discussed some applications such as finding the speed at which stars move away from the earth, or finding the speed of cars or aircraft.

9. We now must decide how to generalize the concept of momentum in Relativity theory. The Newtonian definition of momentum is $\vec{p} = m\vec{u}$. The problem with this definition is that we are used to having momentum conserved when particles collide with each other, and this old definition will simply not work when particles move very fast. Consider the collision of two particles as in the diagrams below:



In the frame fixed to the lab (S-frame) conservation of momentum implies: $m_A u_A + m_B u_B = m_C u_C + m_D u_D$. Mass is also conserved: $m_A + m_B = m_C + m_D$. Now suppose we wish to observed the collision from a frame S′ moving at speed v. Then, from the relativistic addition of velocities formula, $u' = \dfrac{u - \text{v}}{1 - \dfrac{u\text{v}}{c^2}}$ and, from it, $u = \dfrac{u' + \text{v}}{1 + \dfrac{u'\text{v}}{c^2}}$. Insert this into the equation of conservation of momentum (using $p = m\text{v}$ as the definition):

$$m_A\left(\frac{u'_A + \text{v}}{1 + u'_A \text{v}/c^2}\right) + m_B\left(\frac{u'_B + \text{v}}{1 + u'_B \text{v}/c^2}\right) = m_C\left(\frac{u'_C + \text{v}}{1 + u'_C \text{v}/c^2}\right) + m_D\left(\frac{u'_D + \text{v}}{1 + u'_D \text{v}/c^2}\right)$$

This is clearly not the equation $m_A u'_A + m_B u'_B = m_C u'_C + m_D u'_D$ So momentum will not be conserved relativistically if we insist on using the old definition!!

10. Can we save the situation and make the conservation of momentum hold by finding some suitable new definition of momentum? The new definition must have two properties:
    1) At low speeds it must reduce to the old one.
    2) At all speeds momentum must be conserved.

    Let's see if the definition $\vec{p} = \dfrac{m\vec{u}}{\sqrt{1 - u^2/c^2}} = \gamma m\vec{u}$ will do the job. Obviously if u<<c we get $\vec{p} = m\vec{u}$, so requirement 1 is clearly satisfied. . Let's now see if the conservation of momentum equation will hold if the new definition of momentum is used:

    $$m_A \gamma_A u_A + m_B \gamma_B u_B = m_C \gamma_C u_C + m_D \gamma_D u_D.$$

    After doing some algebra you find,

    $$m_A\left(\frac{1}{\gamma}\gamma'_A u'_A + \gamma_A \text{v}\right) + m_B\left(\frac{1}{\gamma}\gamma'_B u'_B + \gamma_B \text{v}\right) = m_C\left(\frac{1}{\gamma}\gamma'_C u'_C + \gamma_C \text{v}\right) + m_D\left(\frac{1}{\gamma}\gamma'_D u'_D + \gamma_D \text{v}\right)$$

    This gives $m_A \gamma'_A u'_A + m_B \gamma'_B u'_B = m_C \gamma'_C u'_C + m_D \gamma'_D u'_D$, which is just what we want.

11. We shall now consider how energy must be redefined relativistically. The usual expression for the kinetic energy $K = \frac{1}{2}mu^2$ is not consistent with relativistic mechanics (it does not satisfy the law of conservation of energy in relativity). To discover a new definition, let us start from the basics: the work done by a force F when it moves through distance dx adds up to an increase in knetic energy, $K = \int F dx = \int \frac{dp}{dt} dx = \int \frac{dx}{dt} dp = \int u dp$. Now use:

$$dp = md\left(\frac{u}{\sqrt{1-u^2/c^2}}\right) = \frac{mdu}{\sqrt{1-u^2/c^2}} + \frac{m\left(u^2/c^2\right)du}{\left(1-u^2/c^2\right)^{3/2}} = \frac{mudu}{\left(1-u^2/c^2\right)^{3/2}}$$

This give $K = \int_0^u udp = m\int_0^u \frac{udu}{\left(1-u^2/c^2\right)^{3/2}} = \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2$

Note that K is zero if there is no motion, or $K = E - E_0$ where $E = \frac{mc^2}{\sqrt{1-u^2/c^2}}$ and $E_0 = mc^2$. Now expand $\frac{1}{\sqrt{1-u^2/c^2}} = \left(1-u^2/c^2\right)^{-1/2} = 1 + \frac{u^2}{2c^2} + \cdots$

Hence, $K = mc^2\left(1 + \frac{u^2}{2c^2} + \cdots\right) - mc^2 \to \frac{1}{2}mu^2$ as $u/c \to 0$.

12. Now that we have done all the real work, let us derive some alternative expressions using our two main formulae: $\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} = \gamma m\vec{u}$ and $E = \frac{mc^2}{\sqrt{1-u^2/c^2}} = \gamma mc^2$.

a) $\vec{u} = \frac{\vec{p}}{\gamma m} = \frac{\vec{p}c^2}{E}$ or $pc = E\left(\frac{u}{c}\right)$

b) Clearly $(pc)^2 = E^2\left(\frac{u}{c}\right)^2 = \gamma^2 m^2 c^4\left(1 - 1/\gamma^2\right) = \gamma^2 m^2 c^4 - m^2c^4 = E^2 - m^2c^4$. Hence,

$E^2 = p^2c^2 + m^2c^4$.

c) For a massless particle ($m = 0$), $E = pc$.

13. A particle with mass m has energy mc^2 even though it is at rest. This is called its rest energy. Since c is a very large quantity, even a small m corresponds to a very large energy. We interpret this as follows: suppose all the mass could somehow be converted into energy. Then an amount of energy equal to mc^2 would be released.

QUESTIONS AND EXERCISES – 40

1. a) In measuring the length of a metre stick, how does the requirement of simultaneity enter?
   b) What is the equivalent requirement in the meassurement of time?

2. A spaceship moving away from earth at 0.5c fires two missiles at speed 0.6c as measured in its rest frame. One missile is aimed away from earth and the other is towards it. What will be the speed of the missiles as measured on earth.

3. How much work must be done to increase the speed of a particle of mass from 0.2c to 0.3c? From 0.8c to 0.9c? In each case the increase in speed is the same amount, so why is the work done different?

4. A particle of mass m moving with velocity v collides with another particle moving with velocity -v. The two particles stick together. What will be the mass of the single particle that remains?

5. A particle of mass M decays while at rest into two identical particles that move off in opposite directions. What is the mass of those particles?

6. Two identical clocks are set to the same time. One remains on earth while the other is in an aircraft that travels at 1000km/h around the world. Will the the two clocks read the same or a different time? [Actually this experiment has been done with extremely accurate atomic clocks, and the time difference has been measured.]

7. How fast must a car approach a red light so that it is Doppler shifted and appears green?

8. Clocks placed at equal distances in frame S are synchronized to read the same value of time by sending a light pulse from the origin of S. The clocks in frame S′ are similarly synchronized. To an observer in S, will the clocks in S′ be synchronized? What about the observer in S?