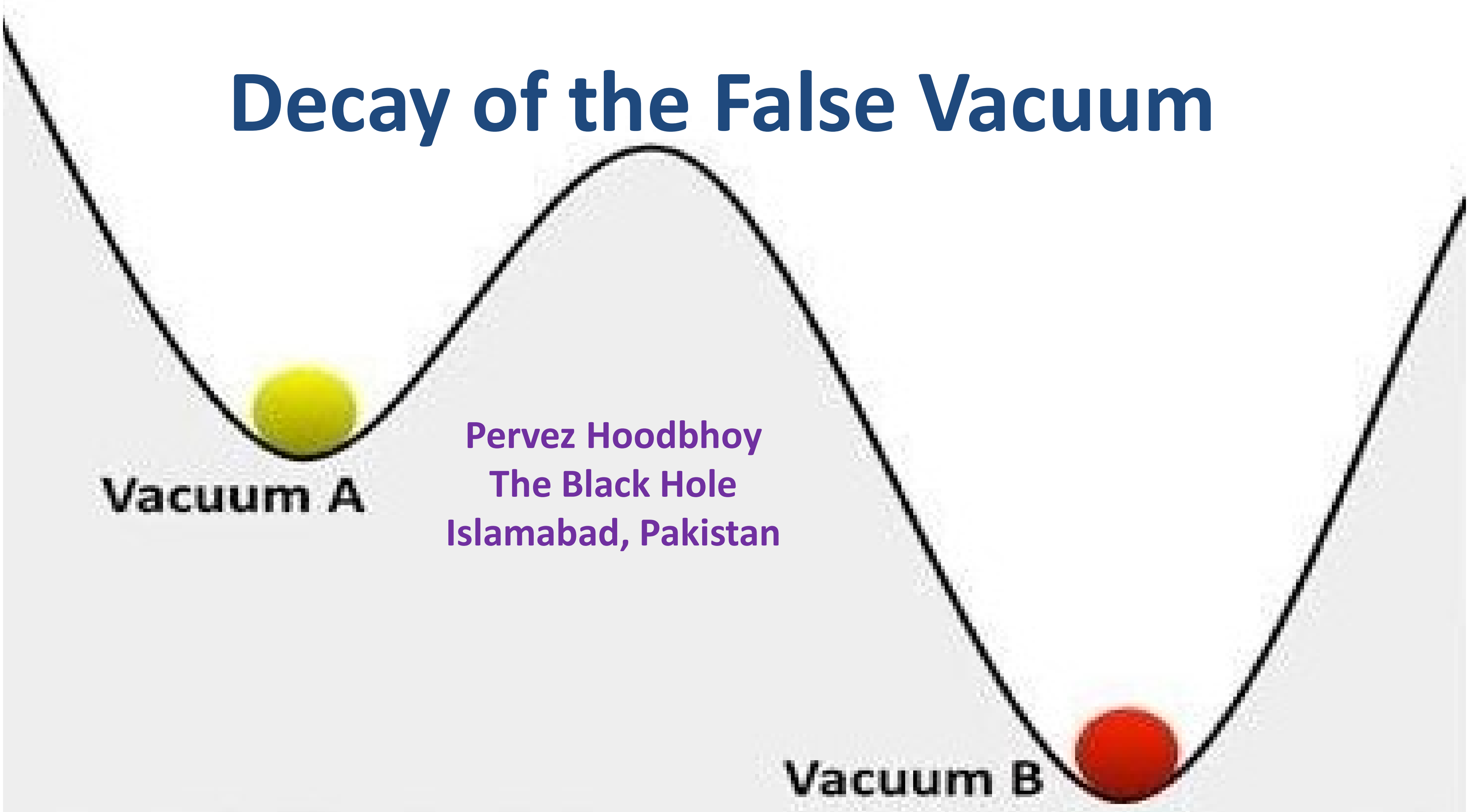
Decay of the False Vacuum



Pervez Hoodbhoy
The Black Hole
Islamabad, Pakistan

Course Outline

Instantons in particle QM

- Intro to path integral ✓
- Imaginary time ✓
- Instantons in a symmetric double well ✓
- Decay of metastable states ✓
- The functional determinant ✓

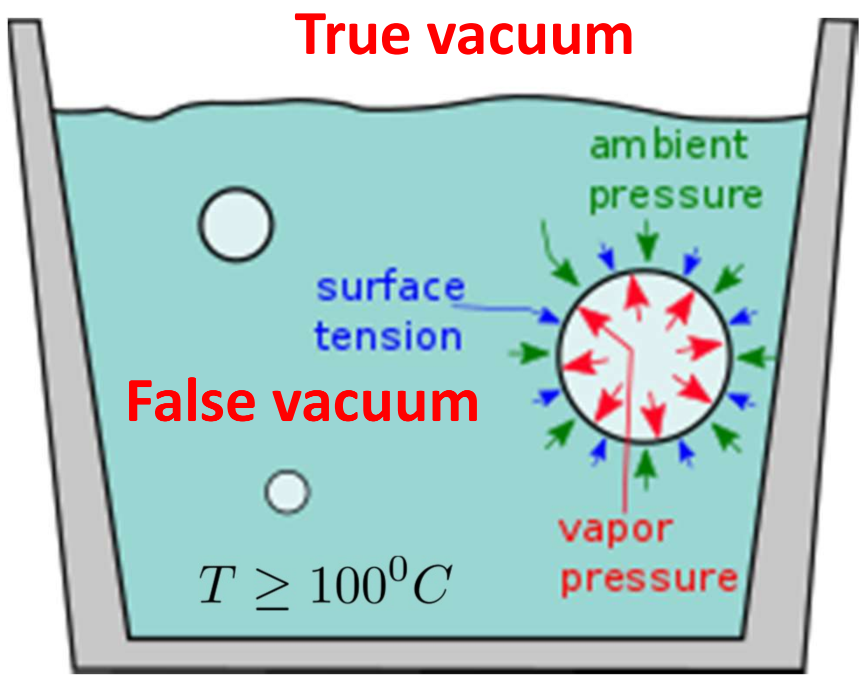
Tunneling of quantum fields

- Basic QFT for a scalar field ✗
- Tunneling of field configurations ✗
- The O(4) instanton ✗
- How/when will the universe end? ✗
- Gauge fields and tunneling
- Effective action

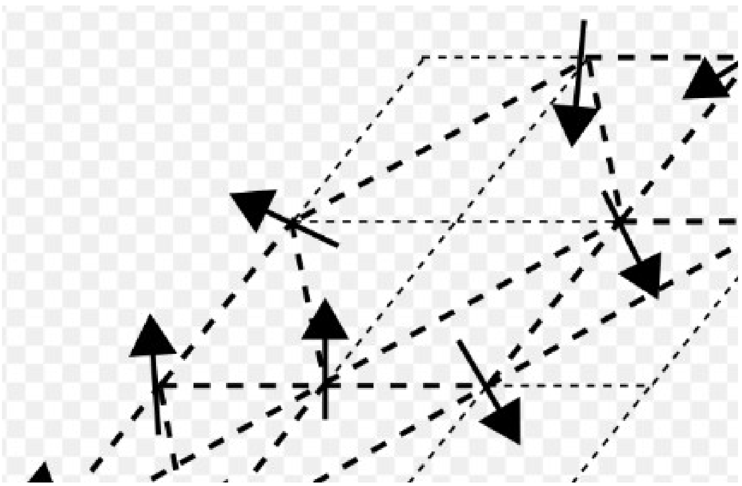
Tunneling in QFT

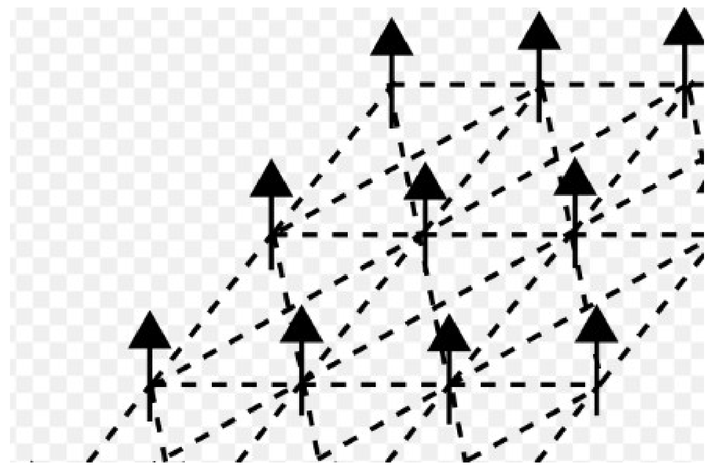
Much richer now that we have an infinite number of degrees of freedom.

- Spin glasses
- Stability of electroweak vacuum
- Multiple vacuaa in QCD
- Superfluid ^{3}He
- Chemical reactions
- $\cdots$


True vacuum
ambient
pressure
surface
tension
False vacuum
vapor
pressure
$T \geq 100^0 C$

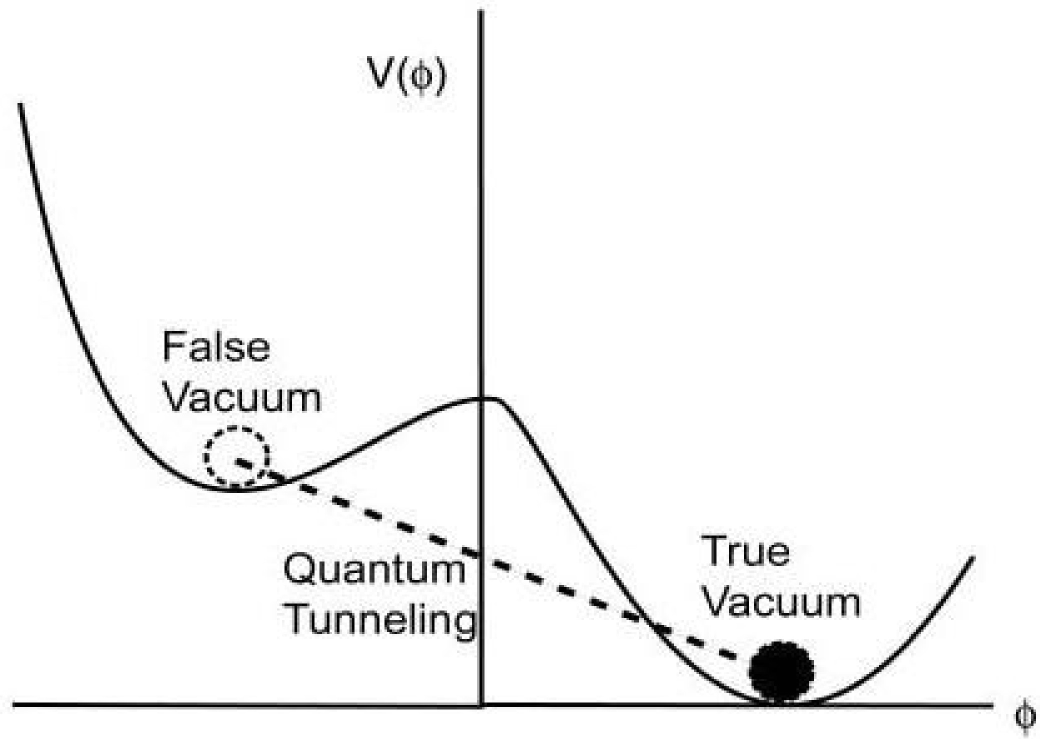
Spin glasses: instanton mediated transition





Higgs field and the early universe

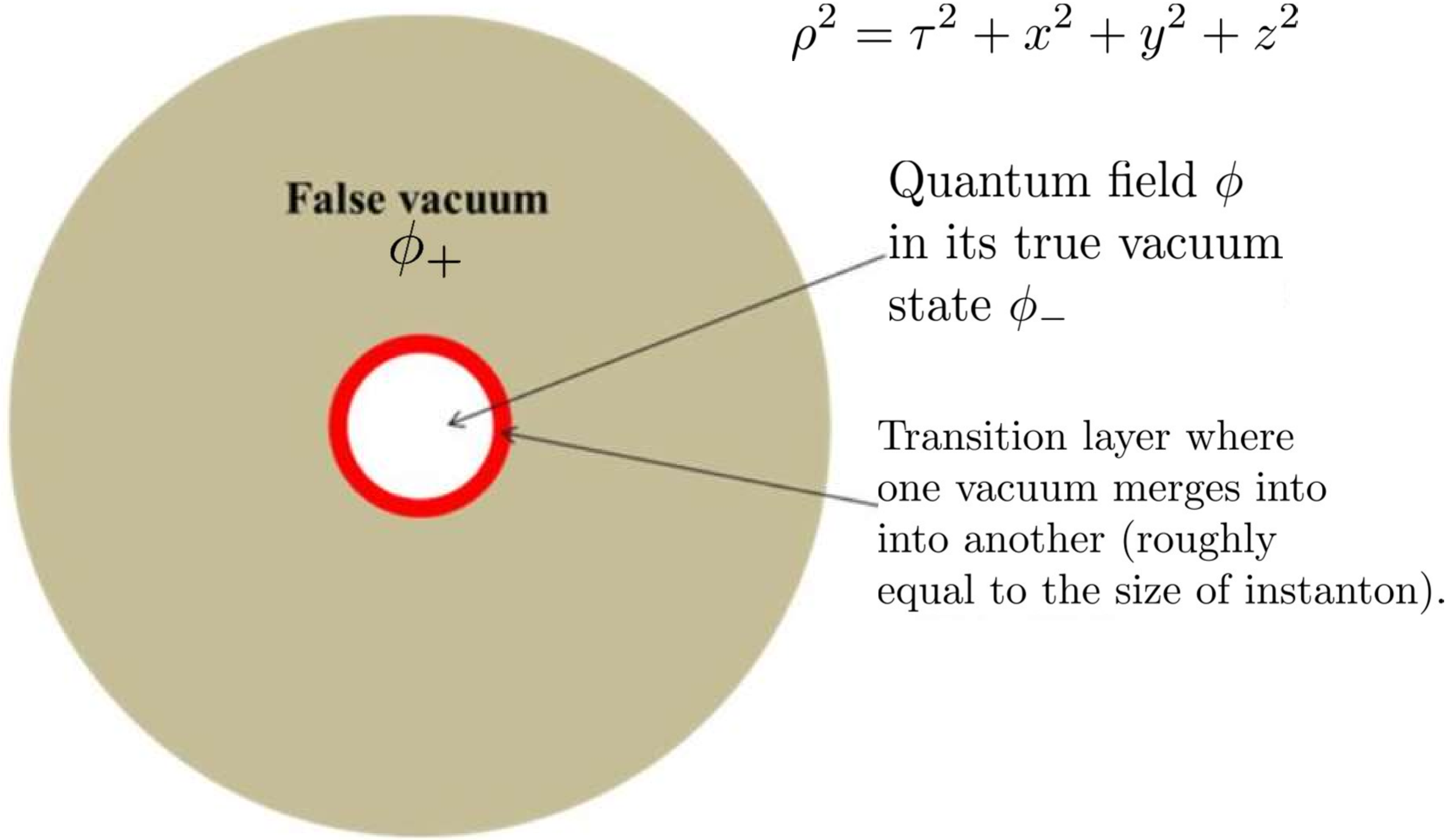
- The Higgs field $\phi(x)$ pervades all space
- The Higgs field $\phi(x)$ has charge under the weak force
- Since $< \phi > \neq 0$ the vacuum also has weak charge
- The Higgs field $\phi(x)$ has a potential $V(\phi)$



Anticipating the O(4) Instanton

$O(4)$ symmetry is invariance under rotations in 4 dimensions, i.e. $\phi = \phi(\rho)$

$$\rho^2 = \tau^2 + x^2 + y^2 + z^2$$

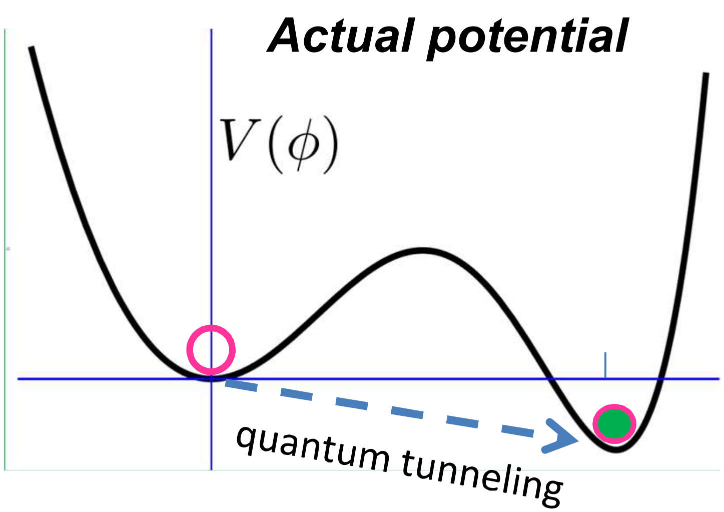


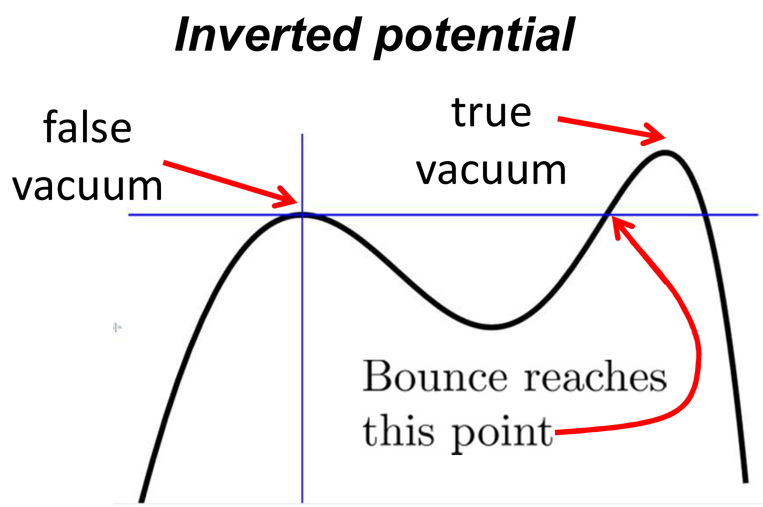
Tunnelling via bouncing instantons – peek at the final result

$$\Gamma = A\, e^{-B}$$ is the probability of tunnelling out of the unstable vacuum per unit time per unit volume of space.

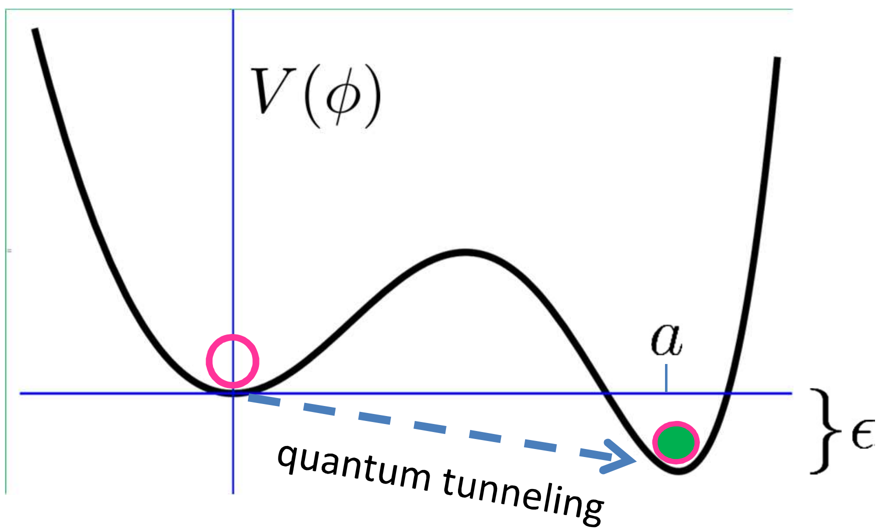
$$B = \frac{S[\phi_B]}{\hbar} = \frac{1}{\hbar}\int d^4x\, V(\phi_B)$$ is the bounce action

From this we see that the false vacuum cannot tunnel into a spatially homogeneous true vacuum. The only possibility is through nucleation, i.e. bubble formation.





Thin wall approximation



An explicit solution for Γ is possible provided ϵ, i.e. the difference in energy per unit volume of space between the true and false vacuum, is small.

$$\Gamma = A\, e^{-B}$$

$$B = \frac{27\pi^2 S_1^4}{2\epsilon^3} \qquad S_1 = \int_0^a d\phi \sqrt{2\,V(\phi)}$$

Basics of relativistic QFT in imaginary time

$x^\mu = (t, \vec{x})$
$\phi = \phi(t, \vec{x})$
is the field

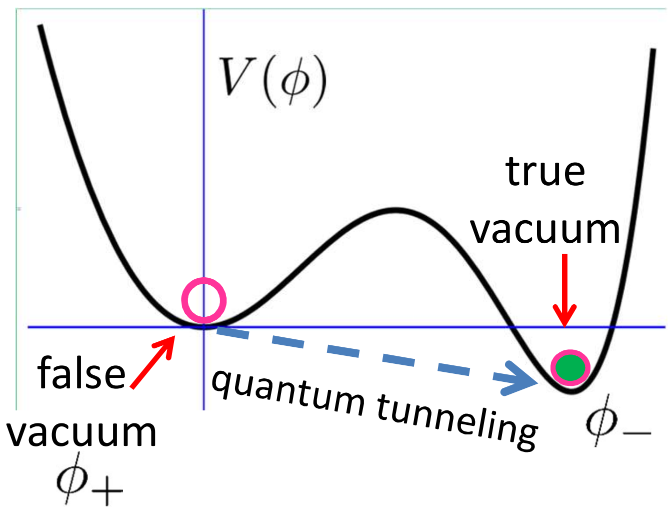
$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$$

is the Lagrangian density

$$S = \int d^4x \mathscr{L}$$

is the action

Imaginary time: let, $t \to -i\tau$. $\mathcal{L} \to \mathcal{L}_E = -\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$



$$d^4x_E = d\tau d^3x \qquad S_E[\phi] = \int d^4x_E \mathscr{L}_E$$

$$\mathscr{A}_{i\to f} = \int_{\phi_+}^{\phi_-} [D\phi(x)] \exp\left(-\frac{1}{\hbar}S_E[\phi(x)]\right)$$

This integration is over all possible paths connecting ϕ_+ to ϕ_- at each spacetime point x. Henceforth suppress “Euclidean”.

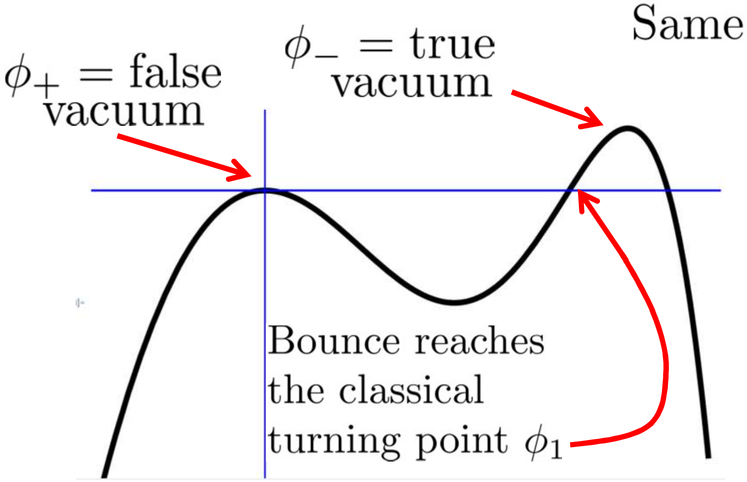
Expand the action $S[\phi]$ to second order in a Taylor series:

$$S[\phi+\delta\phi] = S[\phi] + \int d\phi(x')\, \frac{\delta S[\phi(x)]}{\delta\phi(x')}\delta\phi(x') + \frac{1}{2}\int d\phi(x')d\phi(x'')\frac{\delta^2 S[\phi(x)]}{\delta\phi(x')\delta\phi(x'')}\delta\phi(x')\delta\phi(x'') + \cdots$$

$$\frac{\delta S[\phi(x)]}{\delta\phi(x')} = 0 \quad \Rightarrow \quad \left(\frac{\partial^2}{\partial\tau^2} + \nabla^2\right)\phi_b = V'(\phi_b)$$

$\phi_b(\tau, \vec{x})$ is the bounce (classical) path.

Same as: $\partial_\mu\partial_\mu\phi_b = V'(\phi_b)$



Boundary conditions for bounce solutions:

$$\phi_b\left(\vec{x}, -\frac{T}{2}\right) = \phi_b\left(\vec{x}, \frac{T}{2}\right) = \phi_+ \text{ as } T \to \infty$$

$$\left.\frac{d\phi_b(\vec{x},\tau)}{d\tau}\right|_{\tau=0} = 0$$ bounce velocity is zero at the classical turning point

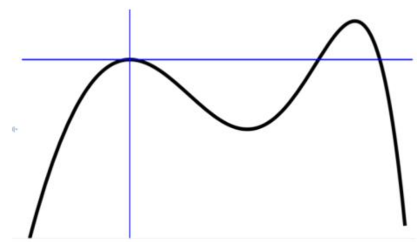
$$\phi_b(|\vec{x}|, \tau) = \phi_+ \quad \text{as } |\vec{x}| \to \infty$$

The O(4) Instanton - Introduction

Bounce solutions needed for EOM, $\left(\frac{\partial^2}{\partial\tau^2} + \nabla^2\right)\phi_b = V'(\phi_b)$ or, $\partial_\mu\partial_\mu\phi_b = V'(\phi_b)$

BC's are: $\lim_{\tau\to\pm\infty} \phi_b(\tau,\vec{x}) = \phi_F$, $\left.\frac{\partial\phi_b}{\partial\tau}\right|_{\tau=0} = 0$, $\lim_{|\vec{x}|\to\pm\infty} \phi_b(\tau,\vec{x}) = 0$

Laplacian in 4-D: $\frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial}{\partial\rho} + \cdots$ where, $\rho^2 = \tau^2 + \vec{x}^{\,2}$



EOM simplifies to this with above BC's $\frac{d^2\phi_b}{d\rho^2} + \frac{3}{\rho}\frac{d\phi_b}{d\rho} = V'(\phi_b)$ gives $\phi_b(\rho) = \phi_b(\sqrt{\tau^2 + \vec{x}^{\,2}})$

$$S[\phi] = \int d^4x\, \mathscr{L} = 2\pi^2 \int_0^\infty \rho^3 d\rho\, \mathscr{L} = 2\pi^2 \int_0^\infty \rho^3 d\rho \left[\frac{1}{2}\left(\frac{\partial\phi_b}{\partial\tau}\right)^2 + \frac{1}{2}(\nabla\phi_b)^2 + V(\phi_b)\right]$$

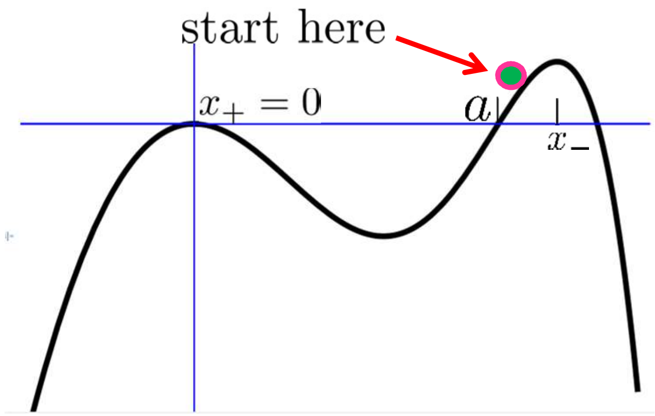
$$= 2\pi^2 \int_0^\infty \rho^3 d\rho \left[\frac{1}{2}\left(\frac{\partial\phi_b}{\partial\rho}\right)^2 + V(\phi_b)\right]$$

Must impose BC: $\left.\frac{d\phi_b}{d\rho}\right|_{\rho=0} = 0$ to avoid EOM disaster.

The O(4) Instanton – informal proof of existence

Does $\dfrac{d^2\phi_b}{d\rho^2} + \dfrac{3}{\rho}\dfrac{d\phi_b}{d\rho} = V'(\phi_b)$ with $\lim_{\rho\to\infty} \phi_b(\rho) = \phi_+$, $\left.\dfrac{\partial\phi_b}{\partial\rho}\right|_{\rho=0} = 0$ have a solution?

To see why it does, rewrite this as: $\ddot{x} + \dfrac{3}{t}\dot{x} = V'(x(t))$ with BC's: $\dot{x}(0) = 0$, $x(\infty) = 0$.



- If start is too close to x_- you overshoot
- If start too far from x_- you undershoot
- In between lies the right starting value!

Note: $\dfrac{d}{dt}\left[\dfrac{1}{2}\dot{x}^2 - V(x)\right] = -\dfrac{3}{t}\dot{x}^2$ $\therefore$ frictional energy loss

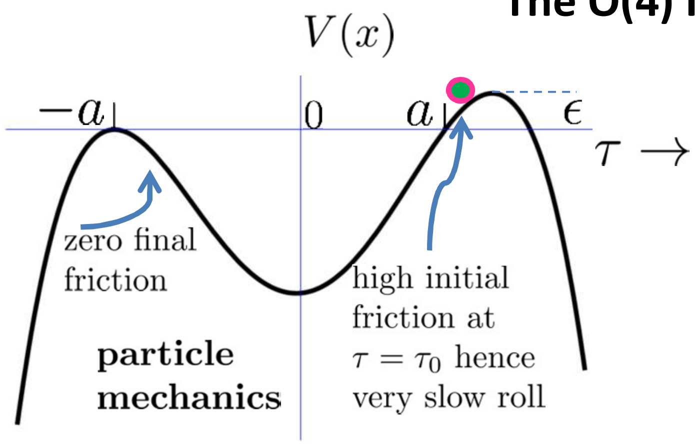
Suppose the ball is initially ($t = 0$) at $x(0) > a$ and passes a at some finite time t_0.

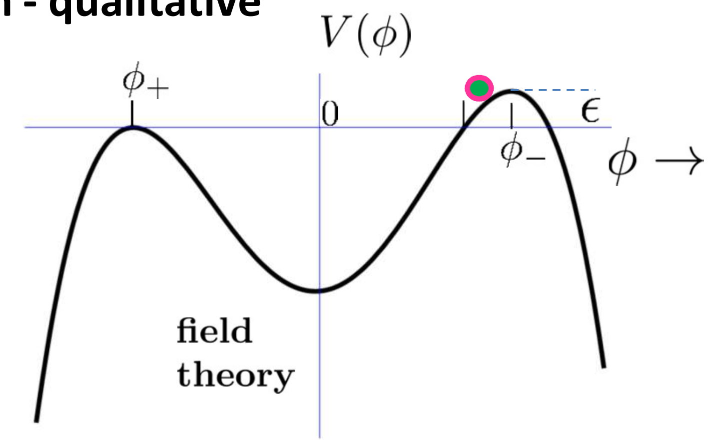
$$\text{Energy loss} = \int_{t_0}^{t_f} dt\, \frac{3}{t}\left|\frac{dx}{dt}\right| < \frac{3}{t_0}\int_{t_0}^{t_f} dt\, \left|\frac{dx}{dt}\right|$$

$$\approx \frac{3}{t_0}\int_{t_0}^{t_f} dt\, \left|\frac{dx}{dt}\right| \approx \frac{3a}{t_0}$$

So friction can be neglected for large t_0.

The O(4) Instanton - qualitative





$$V(x) = V_S(x) + \frac{\epsilon}{2a}(x-a)$$

$$V_S(x) = V_S(-x), V_S'(\pm a) = 0, V_S''(\pm a) = \omega^2$$

$$\text{Example of } V_S: \; V_S(x) = \frac{\omega^2}{8a^2}(x^2-a^2)^2$$

$$V(\phi) = V_S(\phi) + \frac{\epsilon}{2a}(\phi-a)$$

$$V_S(\phi) = V_S(-\phi), V_S'(\pm a) = 0, V_S''(\pm a) = \omega^2$$

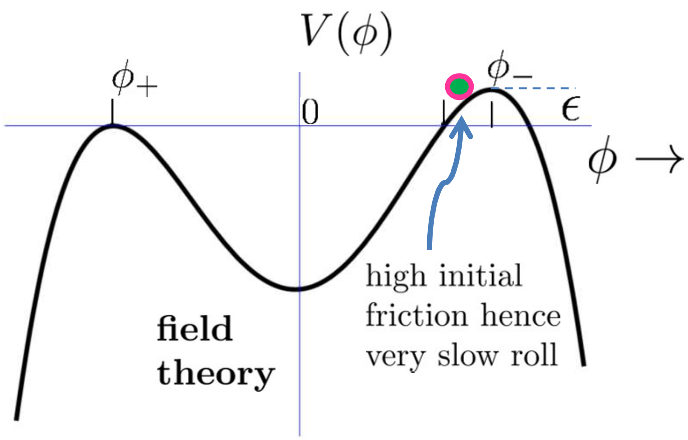
$$\text{Example of } V_S: \; V_S(\phi) = \frac{\omega^2}{8a^2}(\phi^2-a^2)^2$$

Very similar! What exactly separates classical particle mechanics from classical field theory?

a) $\phi = \phi(t, \vec{x}) \to \phi(\tau, \vec{x}) \to \phi(\rho)$. Now we have an infinite number of degrees of freedom.

b) The gradient term $|\nabla\phi|^2$ gives us an extra kinetic energy density. Hence an infinite amount of energy is needed to change ϕ everywhere because we must integrate over all spatial $\vec{x}$.

The O(4) Instanton - quantitative



To zero'th order in ϵ, $\dfrac{d^2\phi}{d\rho^2} = V_S'(\phi)$ is the EOM.

$$\therefore \frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 - V_S(\phi) = 0 \quad \rightarrow \quad d\rho = \frac{d\phi}{\sqrt{2|V_S(\phi)|}}$$

$$\rho = \int \frac{d\phi}{\sqrt{2|V_S(\phi)|}}$$

Expand $V_S(\phi)$ about $\phi = 0$,

$$V_S(\phi) = 0 + 0 + \frac{1}{2}\omega^2\phi^2$$

$$\therefore \rho = \pm\frac{1}{\omega}\int\frac{d\phi}{\phi} = \pm\frac{1}{\omega}\log\,\phi(\tau) \qquad \therefore \phi \sim e^{-\omega\rho},\ e^{\omega\rho}$$

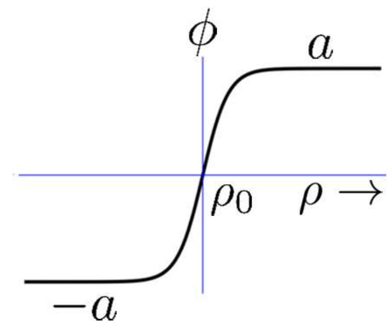
Now remember time translation invariance of EOM says we can send $\rho \to \rho - \rho_0$

For $V_S(x) = \dfrac{\omega^2}{8a^2}(x^2 - a^2)^2$

Solution of $\dfrac{d^2\phi}{d\rho^2} = V_S'(\phi)$ is,

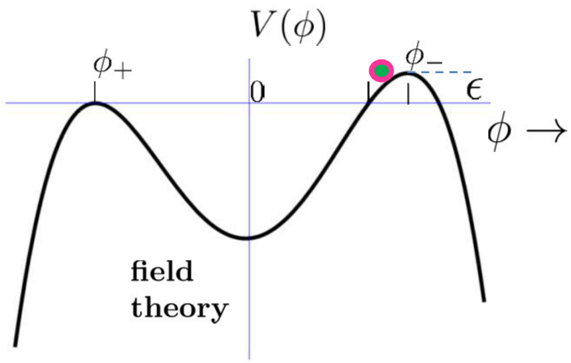
$$\phi(\rho) = a\tanh\left(\frac{\omega\rho}{2}\right)$$

$$\phi_b(\rho) = \begin{cases} a(1 - 2e^{-\omega(\rho-\rho_0)}), & \rho \gg \rho_0 \\ a\tanh\left(\dfrac{\omega(\rho-\rho_0)}{2}\right), & \rho \approx \rho_0 \\ -a(1 - 2e^{-\omega(\rho_0-\rho)}), & \rho_0 \gg \rho \end{cases}$$



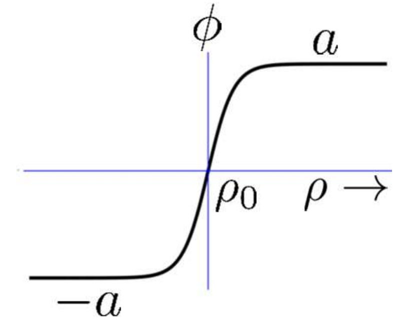
The location of the instanton ρ_0 appears arbitrary. Is it?

The O(4) Instanton – deciding the location



$$V(\phi) = V_S(\phi) + \frac{\epsilon}{2a}(\phi - a)$$

$$\phi_b(\rho) = \begin{cases} a, \quad \rho \gg \rho_0 \\ a \tanh\left(\frac{\omega(\rho - \rho_0)}{2}\right), \; \rho \approx \rho_0 \\ -a \quad \rho \ll \rho_0 \end{cases}$$



$$S[\phi] = 2\pi^2 \int_0^\infty \rho^3 d\rho \left[\frac{1}{2}\left(\frac{\partial \phi_b}{\partial \rho}\right)^2 + V_S(\phi_b) + \frac{\epsilon}{2a}(\phi_b - a)\right] = \int_0^{\rho_o - \delta} d\rho + \int_{\rho_o - \delta}^{\rho_o + \delta} d\rho + \int_{\rho_o + \delta}^{\infty} d\rho$$

$$= 2\pi^2 \int_0^{\rho_0 - \delta} \rho^3 d\rho \, (-\epsilon) + 2\pi^2 \rho_0^3 \int_{\rho_o - \delta}^{\rho_o + \delta} d\rho \, \mathcal{L} \quad + 2\pi^2 \int_{\rho_0 + \delta}^{\infty} \rho^3 d\rho \, (0)$$

$$= -\frac{1}{2}\pi^2 \rho_0^4 \, \epsilon + 2\pi^2 \rho_0^3 S_1 \;\; \text{where, } S_1 \approx \int_{-\infty}^{+\infty} d\rho \left[\frac{1}{2}\left(\frac{\partial \phi_b}{\partial \rho}\right)^2 + V_S(\phi_b)\right] = \int_{-a}^{+a} d\phi \sqrt{2V_S(\phi)}$$

$$\text{Minimize wrt } \rho_0 : \; 0 = \frac{dS}{d\rho_o} = -2\pi^2 \rho_0^3 \, \epsilon + 6\pi^2 \rho_0^2 S_1 \longrightarrow \boxed{\rho_0 = 3\frac{S_1}{\epsilon} \quad \text{and } S = \frac{27\pi^2 S_1^4}{2\epsilon^3}}$$

Instanton energetics

Consider an observer who is moving with the wall. At time of formation there no KE and bubble radius is $r = \rho_0$. The energy per unit area of the bubble is,

$$\mathcal{E} = \frac{1}{4\pi\rho_0^2}\int d^3x\left(\frac{1}{2}|\nabla\phi_b|^2 + V_S(\phi_b)\right) \approx \frac{1}{4\pi\rho_0^2}\int_{r\approx\rho_0} 4\pi r^2 dr\left(\frac{1}{2}|\nabla\phi_b|^2 + V_S(\phi_b)\right)$$

$$\approx \int_{-\infty}^{+\infty} dr\left(\frac{1}{2}|\nabla\phi_b|^2 + V_S(\phi_b)\right) = S_1$$

So the single instanton action and the instanton energy are equal at $t = \tau = 0$.

How fast does the wall move?
Recall: constant $= \rho_0^2 = -c^2t^2 + r^2$

Energy transforms to a moving frame as,

$$\mathcal{E} \to \gamma\,\mathcal{E},\ \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Now need v:

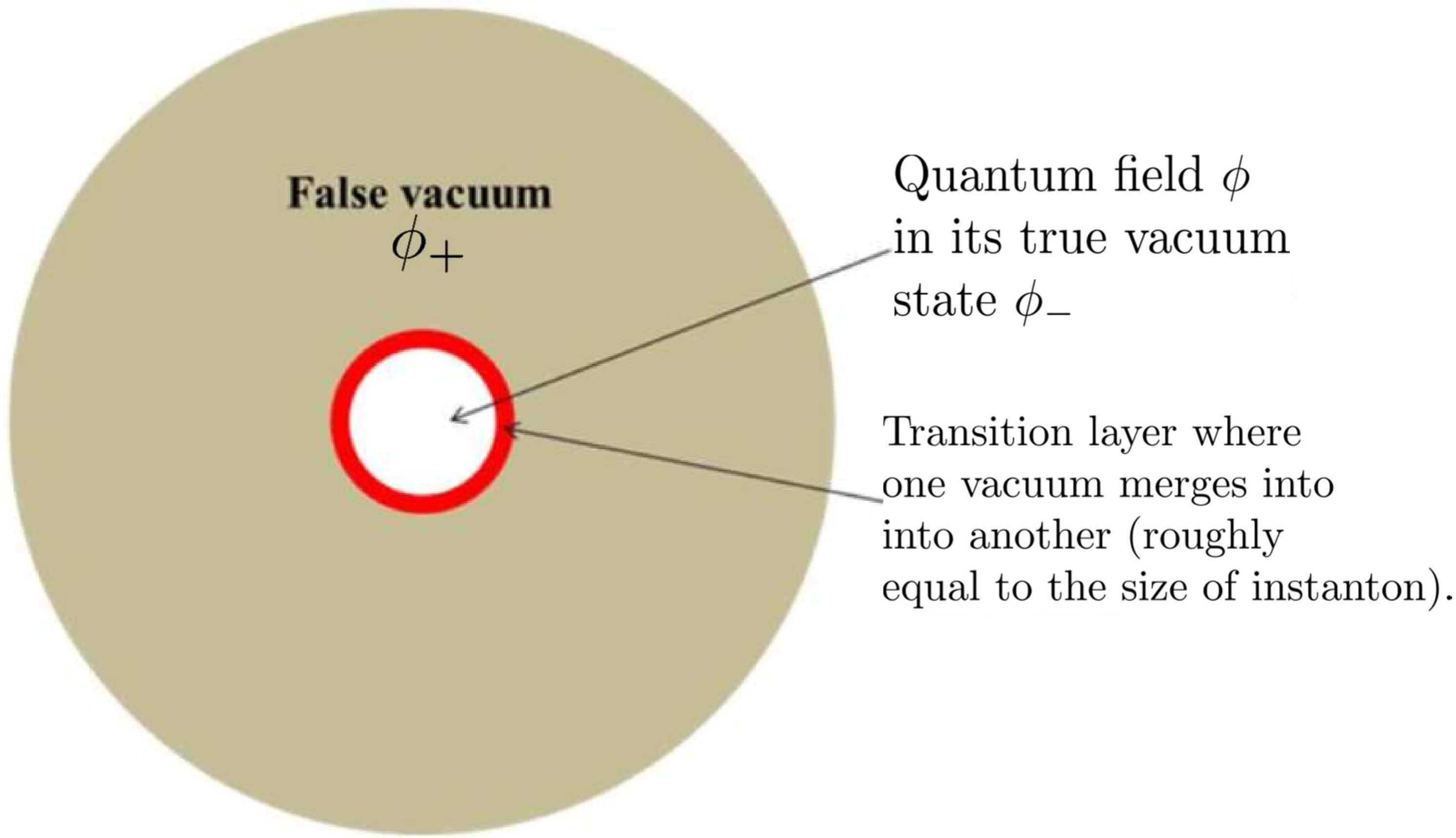
$$v = \frac{dr}{dt} = c^2\frac{t}{r} = c\frac{ct}{\sqrt{\rho_0^2 + c^2t^2}}$$

From this, $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \dfrac{r}{\rho_0}$

$$\therefore \mathcal{E} = \frac{r}{\rho_0}\,4\pi r^2\ S_1 = \frac{4}{3}\pi r^3\,\frac{3S_1}{\rho_0} = \frac{4}{3}\pi r^3\epsilon$$

- The bubble is expanding at speed $\approx c$
- There's only true vacuum inside
- All energy from false vacuum goes into the wall

Once the false to true transition happens everything will be over, so bye-bye!



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