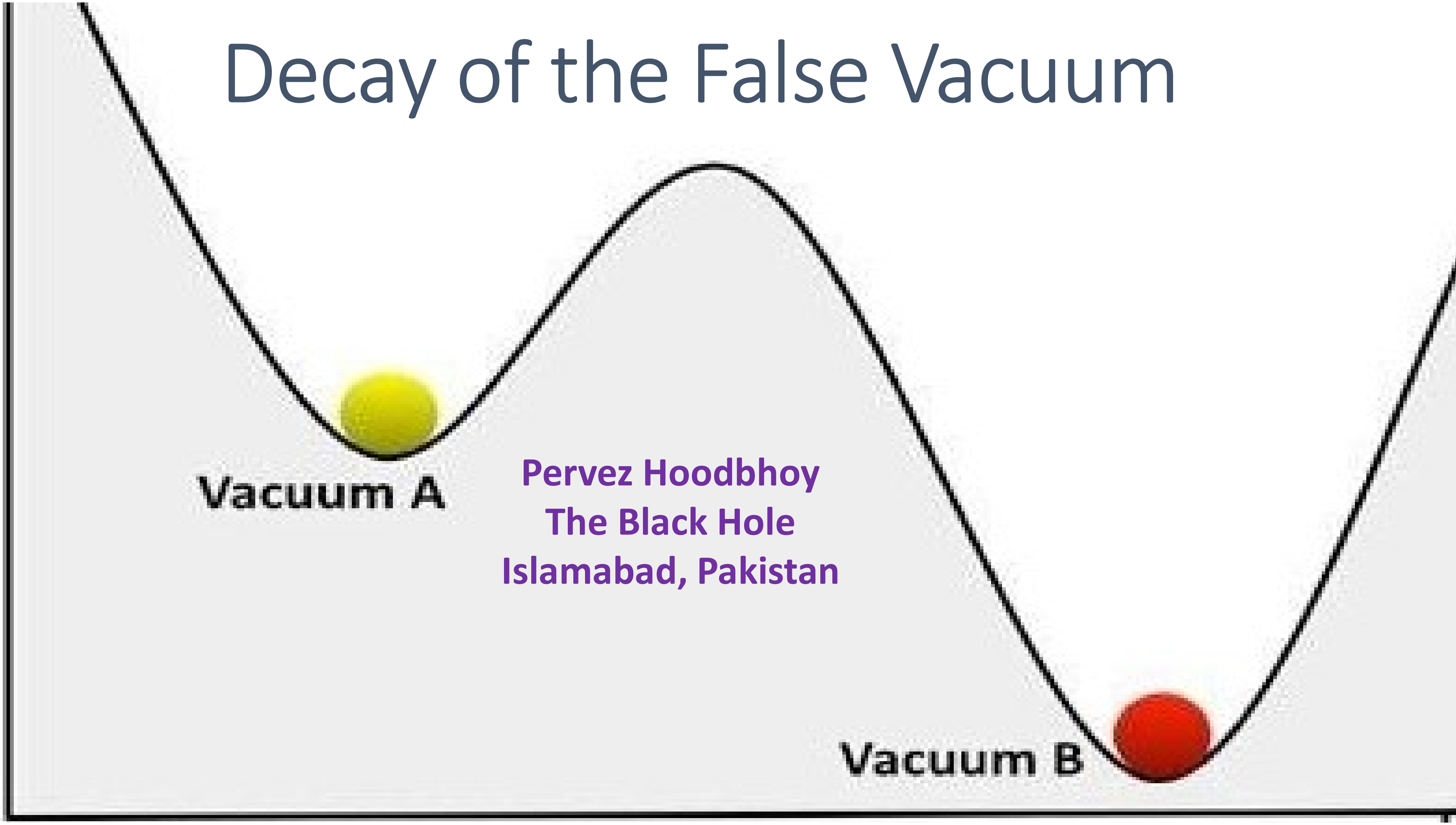
Decay of the False Vacuum



Pervez Hoodbhoy
The Black Hole
Islamabad, Pakistan

Course Outline

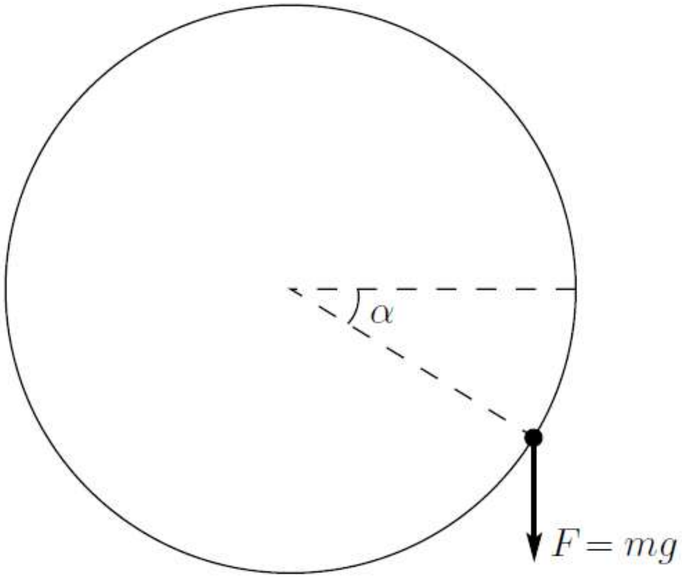
Instantons in particle QM

- Intro to path integral ✓
- Imaginary time ✓
- Instantons in a symmetric double well ✓
- Decay of metastable states ✓
- The functional determinant ✓

Tunneling of quantum fields

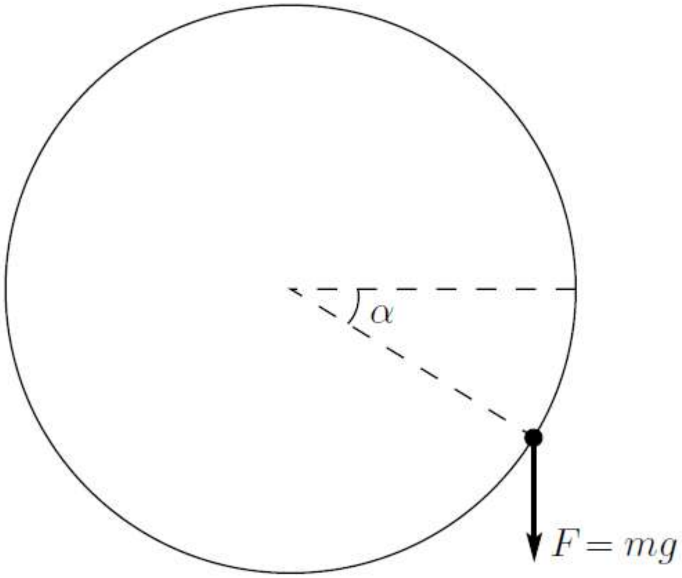
- Basic QFT for a scalar field ✓
- Tunneling of field configurations ✓
- The O(4) instanton ✓
- How/when will the universe end? ✓
- Gauge fields and multiple vacuaa ✗
- Effective action

Instantons in gauge theories

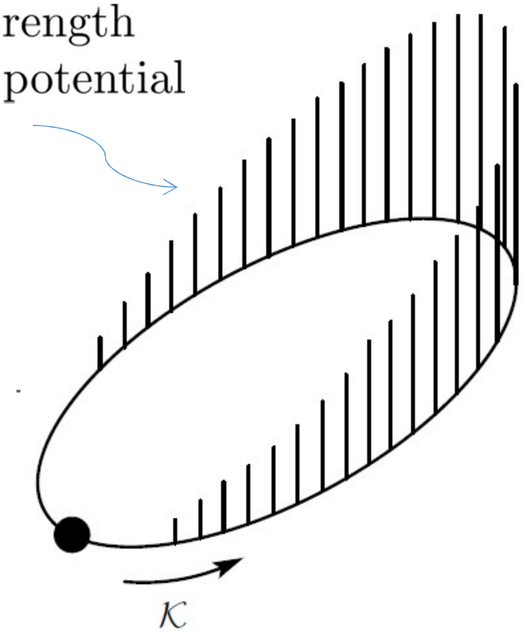


Particle on a 1-dim topologically non-trivial manifold, the circle.

Instantons in gauge theories

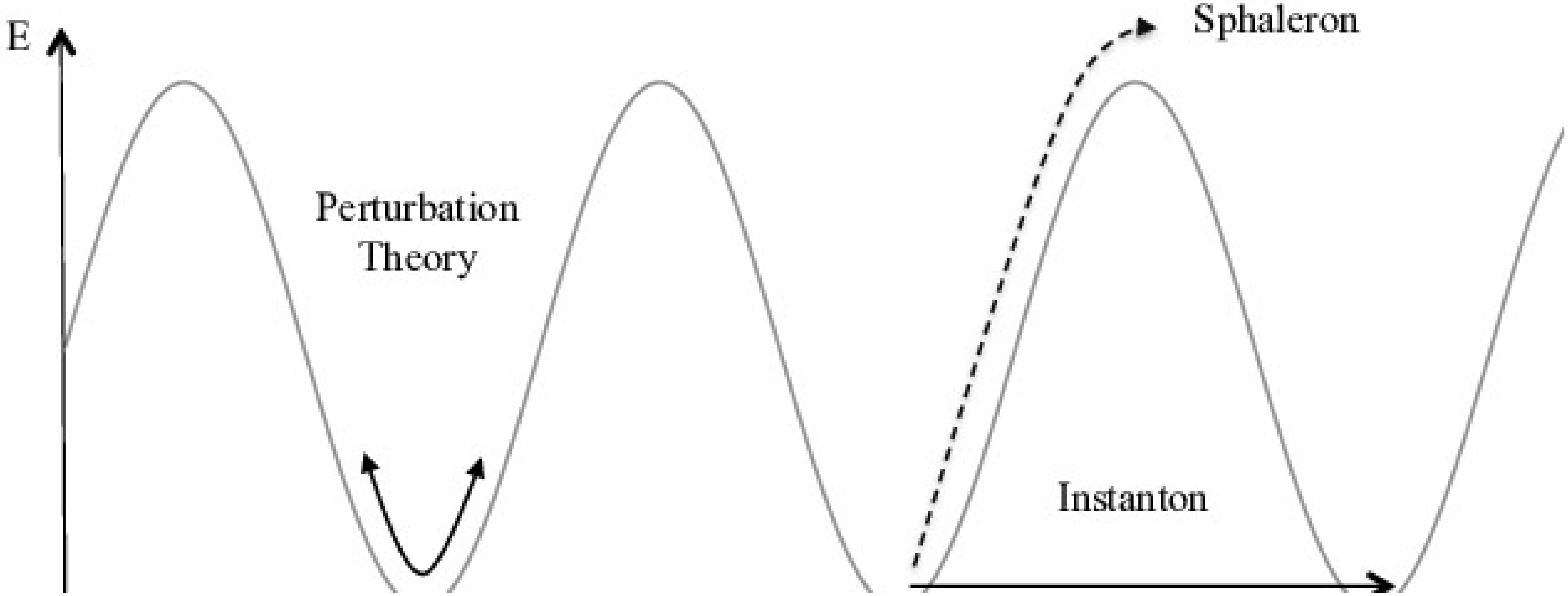


Particle on a 1-dim topologically non-trivial manifold, the circle.

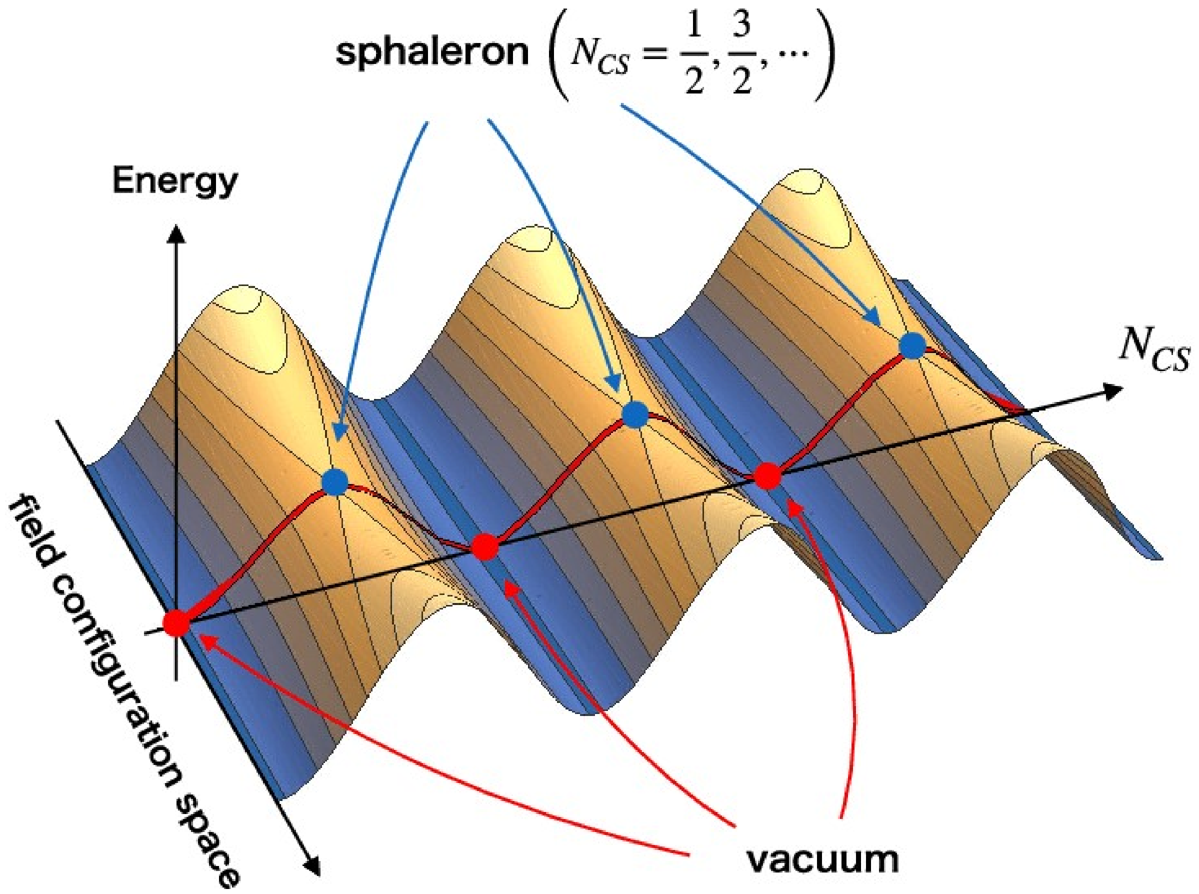


indicates strength
of "gauge" potential

Nontrivial topology in the space of gauge fields in the κ (Chern-Simons current) direction.


E
Perturbation Theory
Sphaleron
Instanton

Standard Model also has sphalerons



Zero energy vacuaa connected by tunneling transitions that violate baryon and lepton number conservation

Possible model for baryogenesis in the early universe

Basics of Gauge Theory - I

Suppose we have two independent real scalar fields ϕ_1, ϕ_2 with,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - V(\phi_1,\phi_2) \text{ with } V = V(\phi_1^2+\phi_2^2)$$

Let the fields take complex values and assemble them into column and row vectors:

$$\phi = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \quad \text{and, } \phi^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1^\dagger & \phi_2^\dagger\end{bmatrix} \qquad \mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - V(\phi^\dagger\phi).$$

Suppose $U(N)$ is a **constant** $N \times N$ matrix ($N = 2$ here) with $U^\dagger U = 1$.

$$\phi(x) \to U\phi(x),\ \ \phi^\dagger(x) \to \phi^\dagger(x)U^\dagger \ \text{ and so } \phi^\dagger\phi \to \phi^\dagger\phi,\ \partial_\mu\phi^\dagger\partial^\mu\phi \to \partial_\mu\phi^\dagger\partial^\mu\phi$$

The Lagrangian is therefore invariant under global transformations, $\mathcal{L} \to \mathcal{L}$

The Yang-Mills idea (1953): make $\mathcal{L}$ invariant under **local** transformations. The original motivation came from electrodynamics but now extends far beyond. All of modern particle physics depends on gauge theories.

Basics of Gauge Theory - 2

Now suppose $U = U(x)$ with $U^\dagger(x)U(x) = 1$. Then under, $\phi(x) \to U\phi(x)$, $\partial_\mu \phi \to \partial_\mu(U\phi) = (\partial_\mu U)\phi + U\partial_\mu \phi$, and $\partial_\mu \phi^\dagger \to (\partial_\mu \phi^\dagger)U^\dagger + \phi^\dagger \partial_\mu U^\dagger$

- This means that $\partial_\mu \phi^\dagger \partial^\mu \phi \not\longrightarrow \partial_\mu \phi^\dagger \partial^\mu \phi$. So should one give up hope?
- YM said no, let's invent a new Lagrangian, $\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - V(\phi^\dagger \phi)$ where, $D_\mu \phi(x) = \partial_\mu \phi(x) - iA_\mu(x)\phi(x)$ (called covariant derivative)
- Now under the combined transformation, $\phi \to U\phi$ **and** $A_\mu \to U^\dagger A_\mu U + iU^\dagger \partial_\mu U$ we do have the desired invariance, i.e, $\mathcal{L} \to \mathcal{L}$. (Plug in and verify!)
- Up to this point the gauge field A_μ is a "dead field", i.e. it has no dynamics. How to make it come alive? Quick answer: Include this term in the Lagrangian,

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \text{ where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$$

Basics of Gauge Theory - 3

Up to this point the gauge field A_μ is a "dead field", i.e. it has no dynamics. How to make it come alive? Quick answer: Include this term in the Lagrangian,

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \text{ where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$$

So the invariant Lagrangian of a scalar field coupled to a gauge field is,

$$\mathcal{L} = D_\mu\phi^\dagger D^\mu\phi - V(\phi^\dagger\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$D_\mu\phi(x) = \partial_\mu\phi(x) - iA_\mu(x)\phi(x)$$

But why this particular structure of $F_{\mu\nu}$? Yang and Mills did it by trial and error. Is it possible to do better?

NEW MATH FOR QFT

Physics is becoming so unbelievably complex that it is taking longer and longer to train a physicist. It is taking so long, in fact, to train a physicist to the place where he understands the nature of physical problems that he is already too old to solve them.

Eugene P. Wigner

“OLD” Math

Calculus
Linear Algebra
Differential Equations
Complex Variables
Optional: Probability Theory
Optional: Group Theory

“NEW” Math

Differential Geometry = Calculus + Linear Algebra
Differential Topology = Differential Geometry + Topology
Fibre Bundles = Group Theory + Topology
Knot Theory, Braid Theory, Quantum Groups, Harmonic Analysis, Cohomology……plus more stuff that is still to find names

Does this “heavy stuff” really help ?

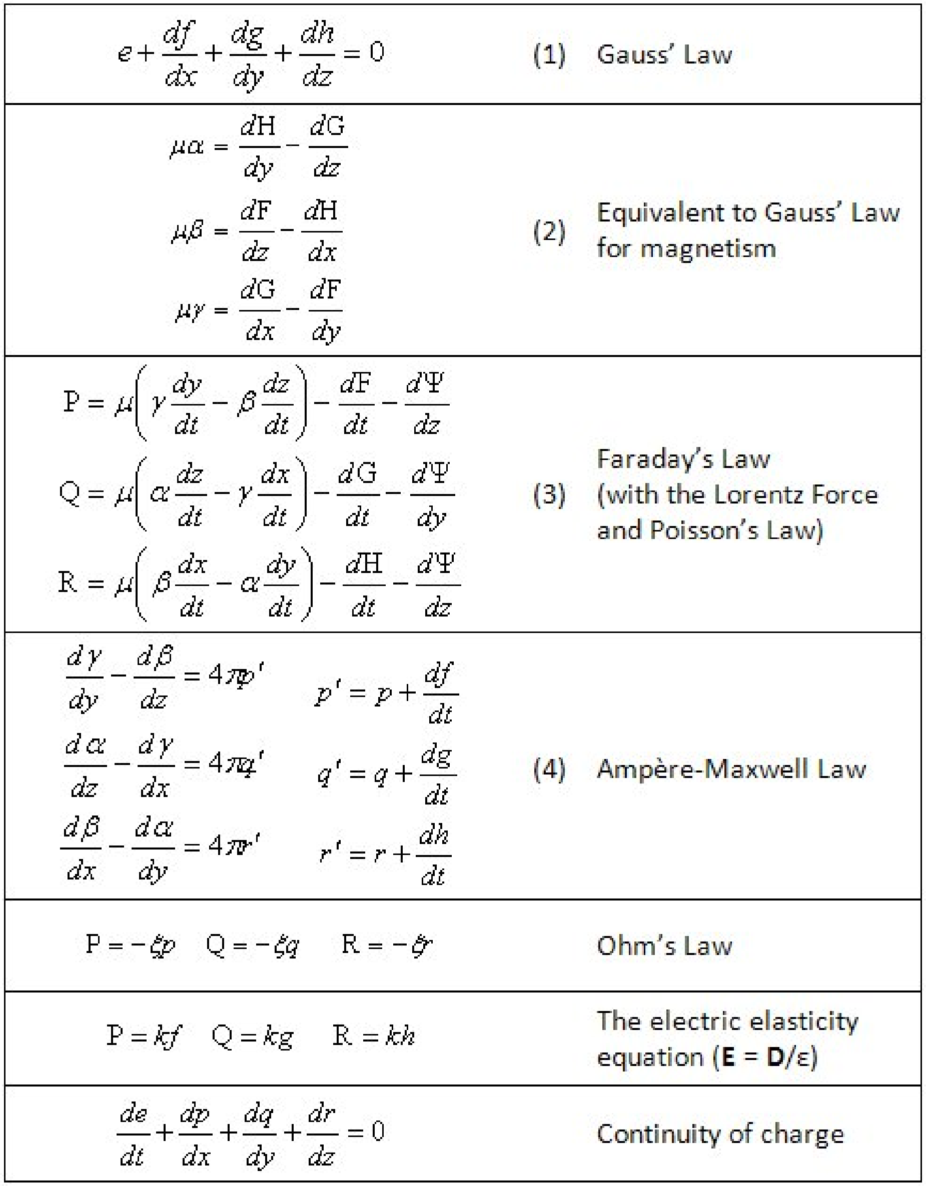
PHILOSOPHICAL
TRANSACTIONS:

A Dynamical Theory of the Electromagnetic Field

J. Clerk Maxwell

Phil. Trans. R. Soc. Lond. 1865 **155**, 459-512, published 1 January 1865

The Original (Twenty) Maxwell's Equations (1865)



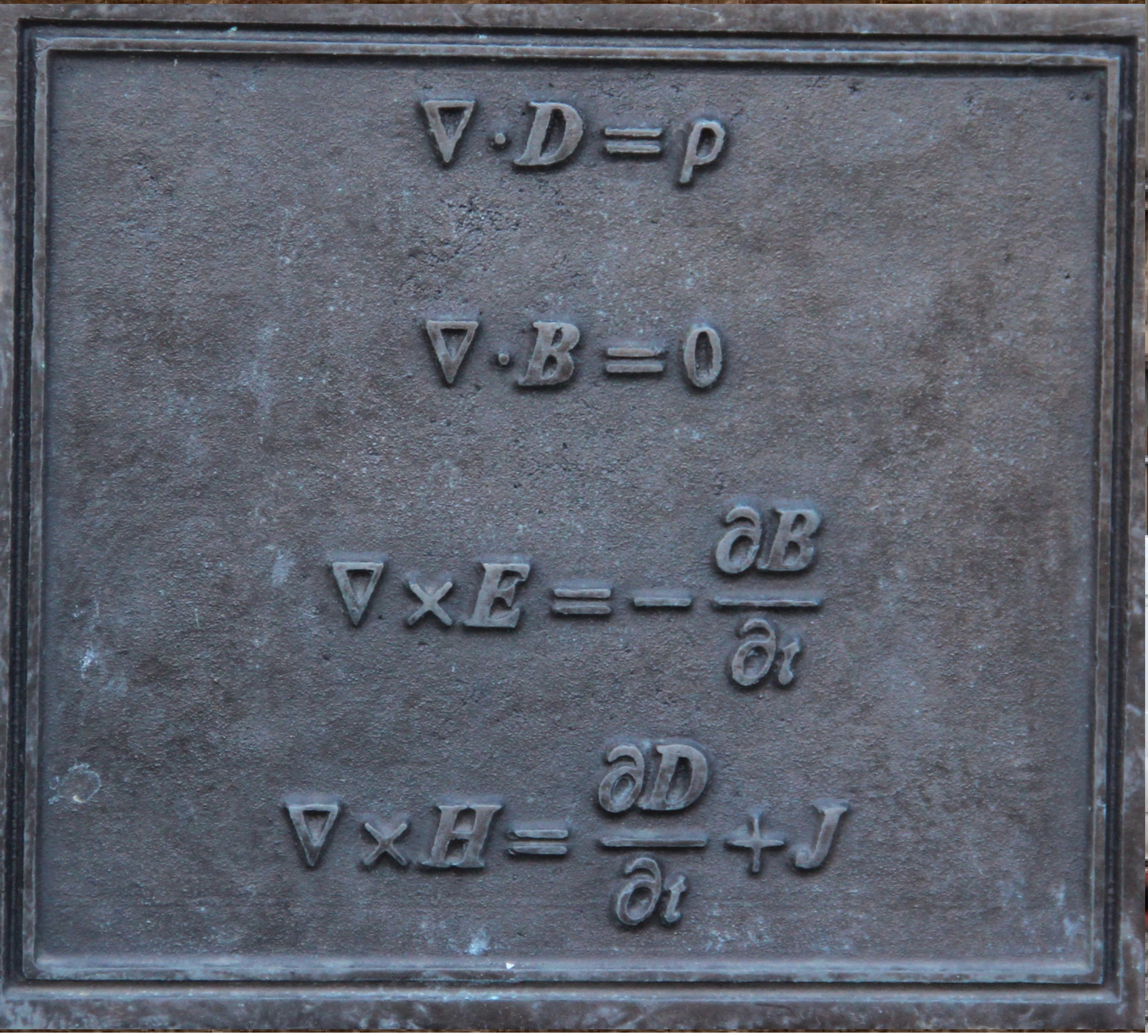
$e+\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0$	(1)	Gauss' Law
$\mu\alpha=\frac{d\mathrm{H}}{dy}-\frac{d\mathrm{G}}{dz}$ $\mu\beta=\frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{dx}$ $\mu\gamma=\frac{d\mathrm{G}}{dx}-\frac{d\mathrm{F}}{dy}$	(2)	Equivalent to Gauss' Law for magnetism
$\mathrm{P}=\mu\left(\gamma\frac{dy}{dt}-\beta\frac{dz}{dt}\right)-\frac{d\mathrm{F}}{dt}-\frac{d\Psi}{dz}$ $\mathrm{Q}=\mu\left(\alpha\frac{dz}{dt}-\gamma\frac{dx}{dt}\right)-\frac{d\mathrm{G}}{dt}-\frac{d\Psi}{dy}$ $\mathrm{R}=\mu\left(\beta\frac{dx}{dt}-\alpha\frac{dy}{dt}\right)-\frac{d\mathrm{H}}{dt}-\frac{d\Psi}{dz}$	(3)	Faraday's Law (with the Lorentz Force and Poisson's Law)
$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=4\pi p'$ $\quad p'=p+\frac{df}{dt}$ $\frac{d\alpha}{dz}-\frac{d\gamma}{dx}=4\pi q'$ $\quad q'=q+\frac{dg}{dt}$ $\frac{d\beta}{dx}-\frac{d\alpha}{dy}=4\pi r'$ $\quad r'=r+\frac{dh}{dt}$	(4)	Ampère-Maxwell Law
$\mathrm{P}=-\xi p \quad \mathrm{Q}=-\xi q \quad \mathrm{R}=-\xi r$		Ohm's Law
$\mathrm{P}=kf \quad \mathrm{Q}=kg \quad \mathrm{R}=kh$		The electric elasticity equation (**E** = **D**/ε)
$\frac{de}{dt}+\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}=0$		Continuity of charge

Oliver Heaviside: 1850-1925

Profession: Electrician

Education: High School

Under Maxwell's Statue

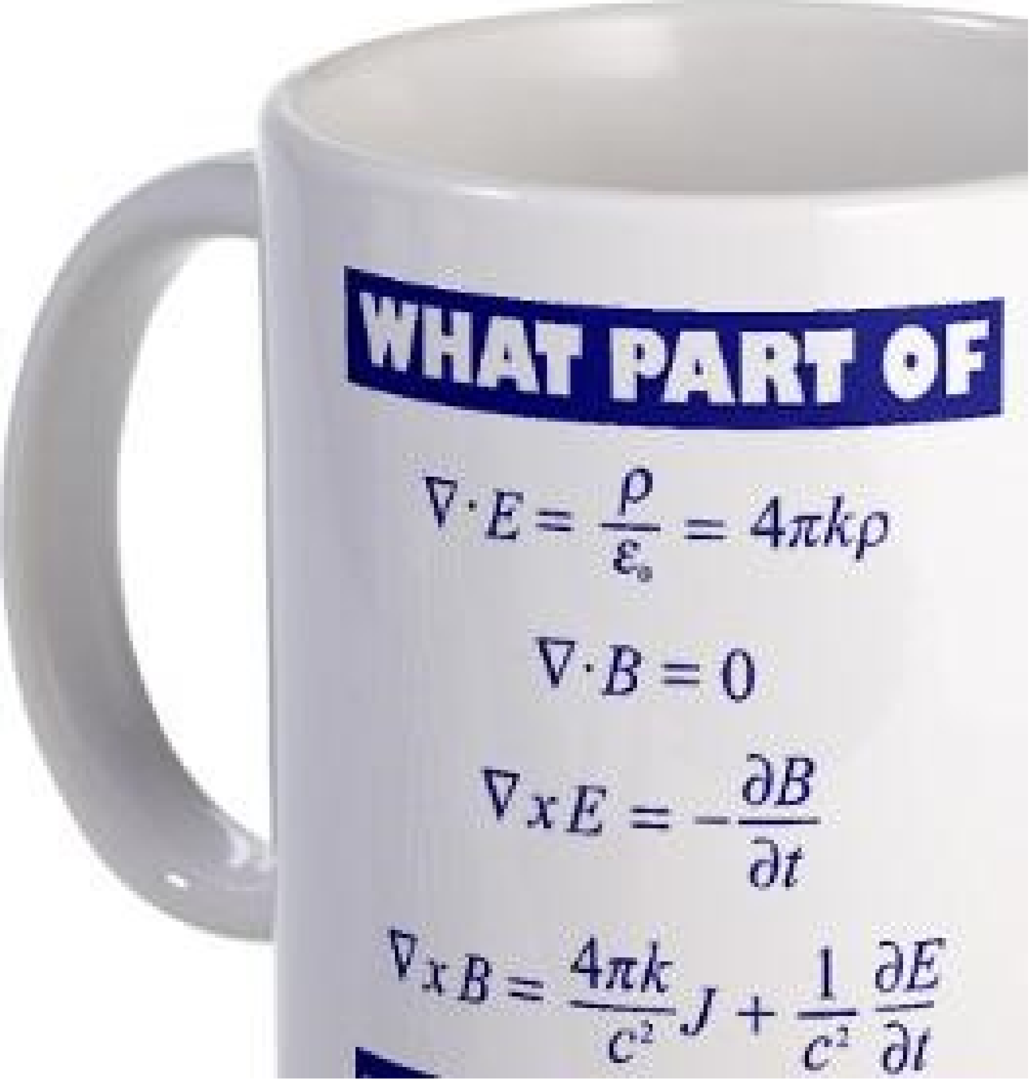


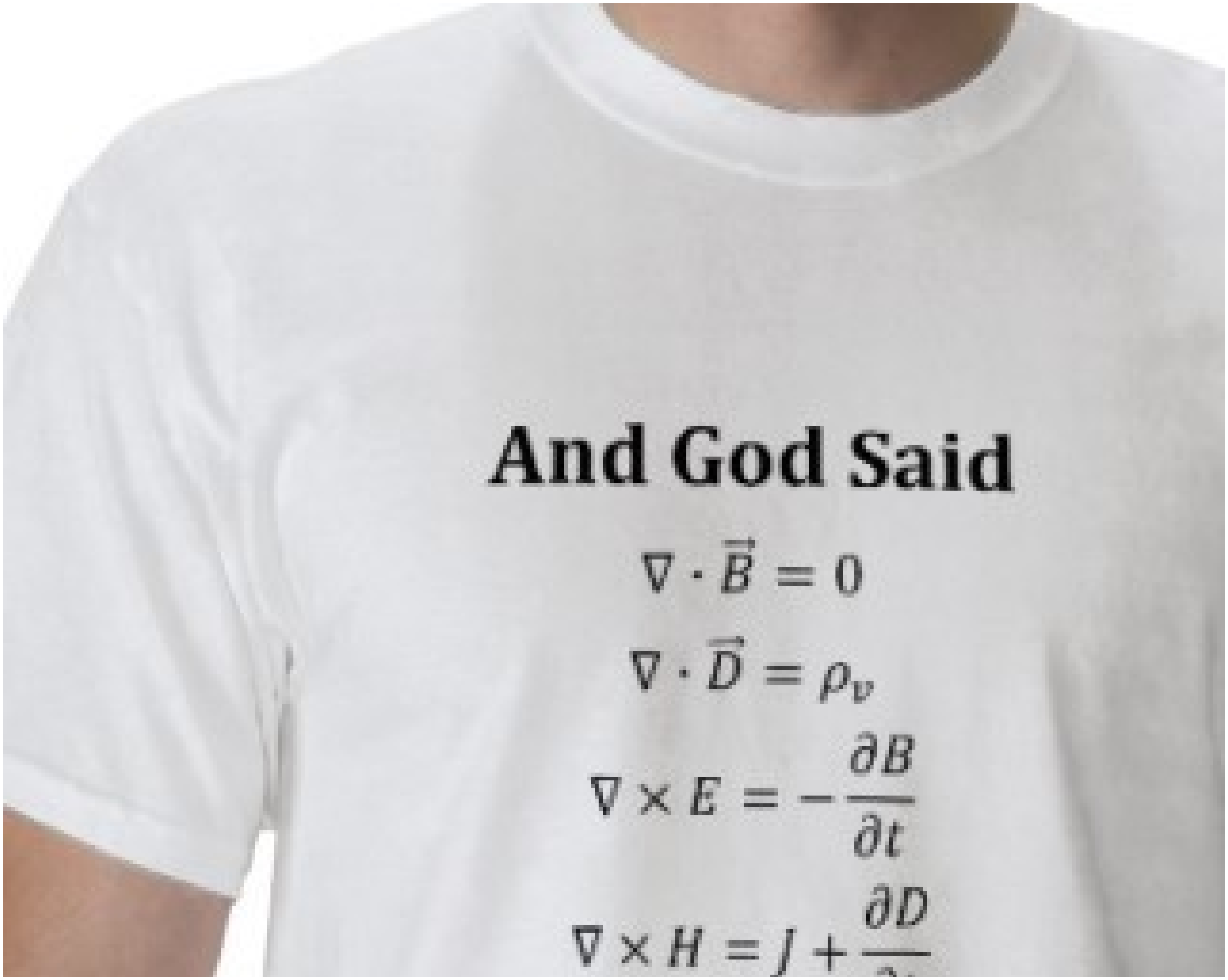
$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times H = \frac{\partial D}{\partial t} + J$$


WHAT PART OF
$\nabla \cdot E = \frac{\rho}{\varepsilon_0} = 4\pi k \rho$
$\nabla \cdot B = 0$
$\nabla x E = -\frac{\partial B}{\partial t}$
$\nabla x B = \frac{4\pi k}{c^2} J + \frac{1}{c^2} \frac{\partial E}{\partial t}$


And God Said
$\nabla \cdot \vec{B} = 0$
$\nabla \cdot \vec{D} = \rho_v$
$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \times H = J + \frac{\partial D}{}$


KEEP
CALM
AND
LISTEN TO
MAXWELL

If you have bought one of those T-shirts with Maxwell's equations on the front, you may have to worry about its going out of style, but not about its becoming false. We will go on teaching Maxwellian electrodynamics as long as there are scientists.

(Steven Weinberg)

This is how we teach Maxwell's equations these days

You start with the quantity A_μ (called the gauge potential).

Then, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ (called the field tensor).

Define: $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ (called the dual field tensor).

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad \partial_\mu F^{\mu\nu} = j^\nu}$$

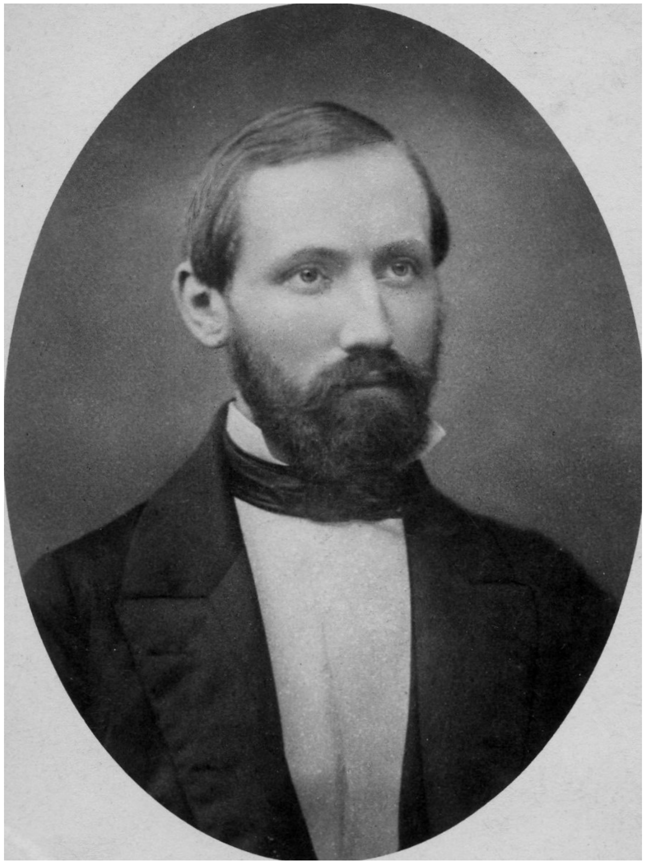
Much better! But is it good enough?

I hate indices and coordinates!

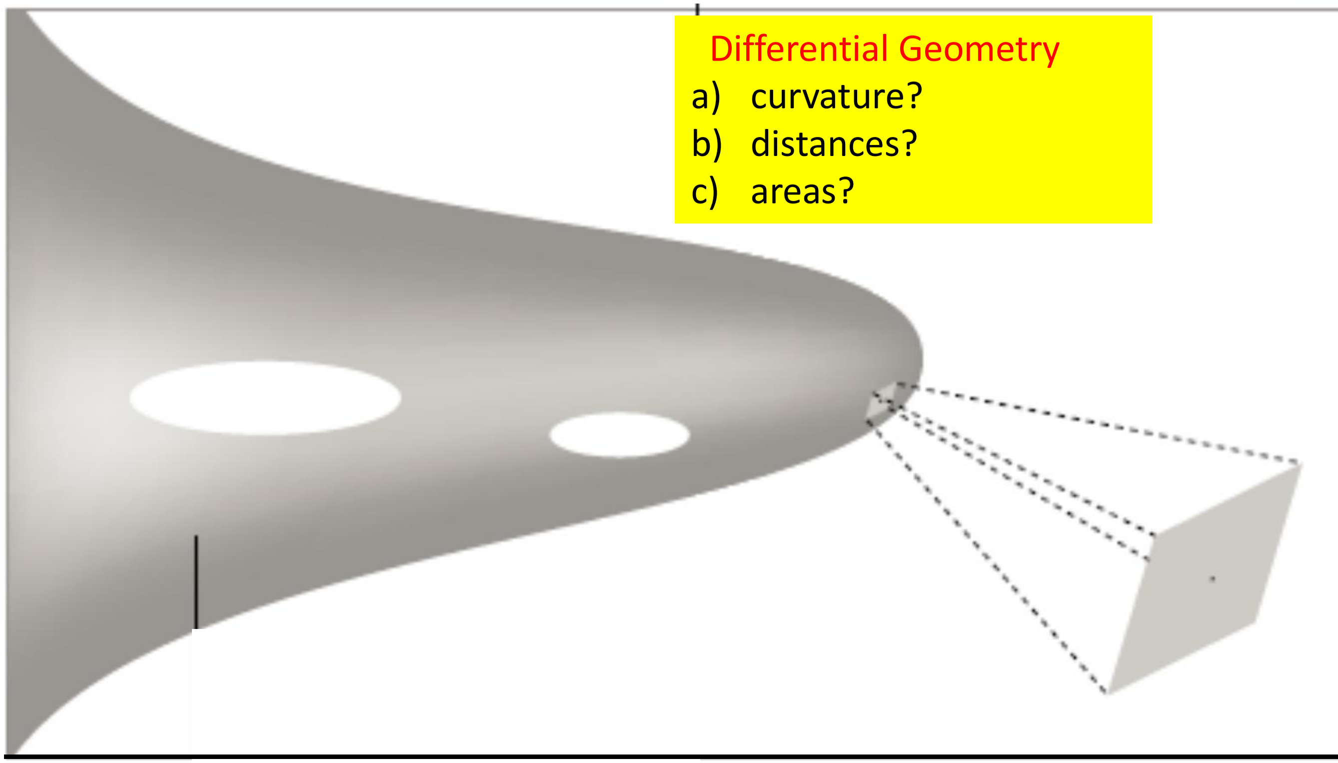
$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

Differential Geometry and Differential Forms



Bernhard Riemann 1826-1866



Differential Geometry

a) curvature?
b) distances?
c) areas?

Differential Forms

$A = A_\mu dx^\mu$ is called a one-form.

Under $x \to x'$, A doesn't change:

$$A = A_\mu dx^\mu = A_\mu \frac{dx^\mu}{dx'^\nu} dx'^\nu = A'_\mu dx'^\mu$$

$A = \frac{1}{2!} A_{\mu\nu} dx^\mu \wedge dx^\nu$ is called a two-form.

Antisymmetry under multiplication: $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$

Under $x \to x'$, A also doesn't change:

$$A = \frac{1}{2!} A_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2!} A'_{\mu\nu} dx'^\mu \wedge dx'^\nu$$

Differential Forms

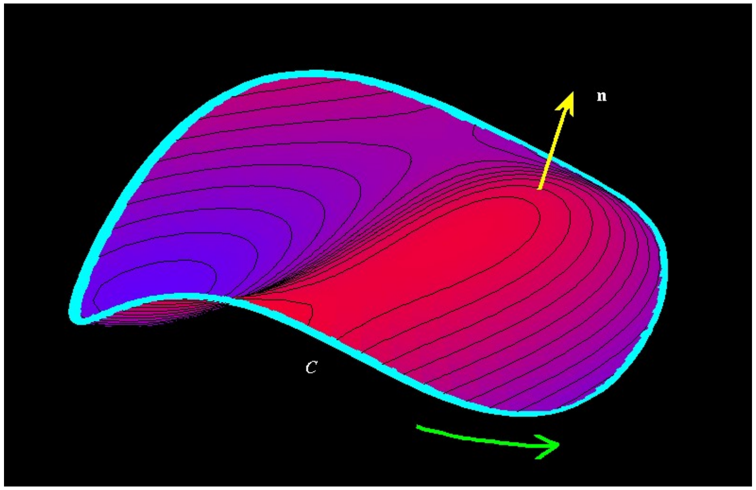
Define d (called the exterior derivative).

$dA = \frac{1}{2!}\partial_\rho A_{\mu\nu}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$ makes a 3-form if A is a 2-form

$ddA = 0 \quad \Rightarrow \quad \boxed{d^2 = 0}$

Poincare Lemma: if $dH = 0$ then $H = dK$ locally.

Integration: $\int_M dA = \int_{\partial M} A$ A is a $(d-1)$ form



Calculus

$$\int_{R} d\omega = \int_{\partial R} \omega$$

Topology

Rewriting Maxwell's equations using differential forms

1) You start with the one-form A, then $A = A_\alpha dx^\alpha$. Define $F \equiv dA$.

$d^2 = 0 \;\Rightarrow\; dF = d^2 A = 0$ 50% of Maxwell's equations hence derived!

2) Notice $A \to A + d\alpha$ doesn't change F.

3) Define Hodge $*$ operation on a zero-form A, a one-form A, a two-form A, etc:

$$* A \equiv \sqrt{-g}\, \varepsilon_{\mu_1\mu_2\mu_3\mu_4} A\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} \qquad \text{(4-form)}$$

$$* A \equiv \sqrt{-g}\, \varepsilon_{\mu_1\mu_2\mu_3\mu_4} A^{\mu_4} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \qquad \text{(3-form)}$$

$$* A \equiv \sqrt{-g}\, \varepsilon_{\mu_1\mu_2\mu_3\mu_4} A^{\mu_3\mu_4} dx^{\mu_1} \wedge dx^{\mu_2} \qquad \text{(2-form)}$$

$$* A \equiv \sqrt{-g}\, \varepsilon_{\mu_1\mu_2\mu_3\mu_4} A^{\mu_2\mu_3\mu_4} dx^{\mu_1} \qquad \text{(1-form)}$$

$$* A \equiv \sqrt{-g}\, \varepsilon_{\mu_1\mu_2\mu_3\mu_4} A^{\mu_1\mu_2\mu_3\mu_4} \qquad \text{(0-form)}$$

Rewriting Maxwell’s equations using differential forms

4) Create an action S invariant under $A \to A + d\alpha$ and couple A to a current one-form j :

$$S = \int_M \left(-\tfrac{1}{2} F \wedge *F + A \wedge *j \right) \qquad \text{(coordinate free!!).}$$

5) Then minimize S by variation: $A \to A + \delta A.$

$$dF = 0 \qquad d * F = * j$$

An immediate payback is generalized electrodynamics

1) Now start with the p-form A.

Define $F \equiv dA$.

The conserved current j is now a p-form.

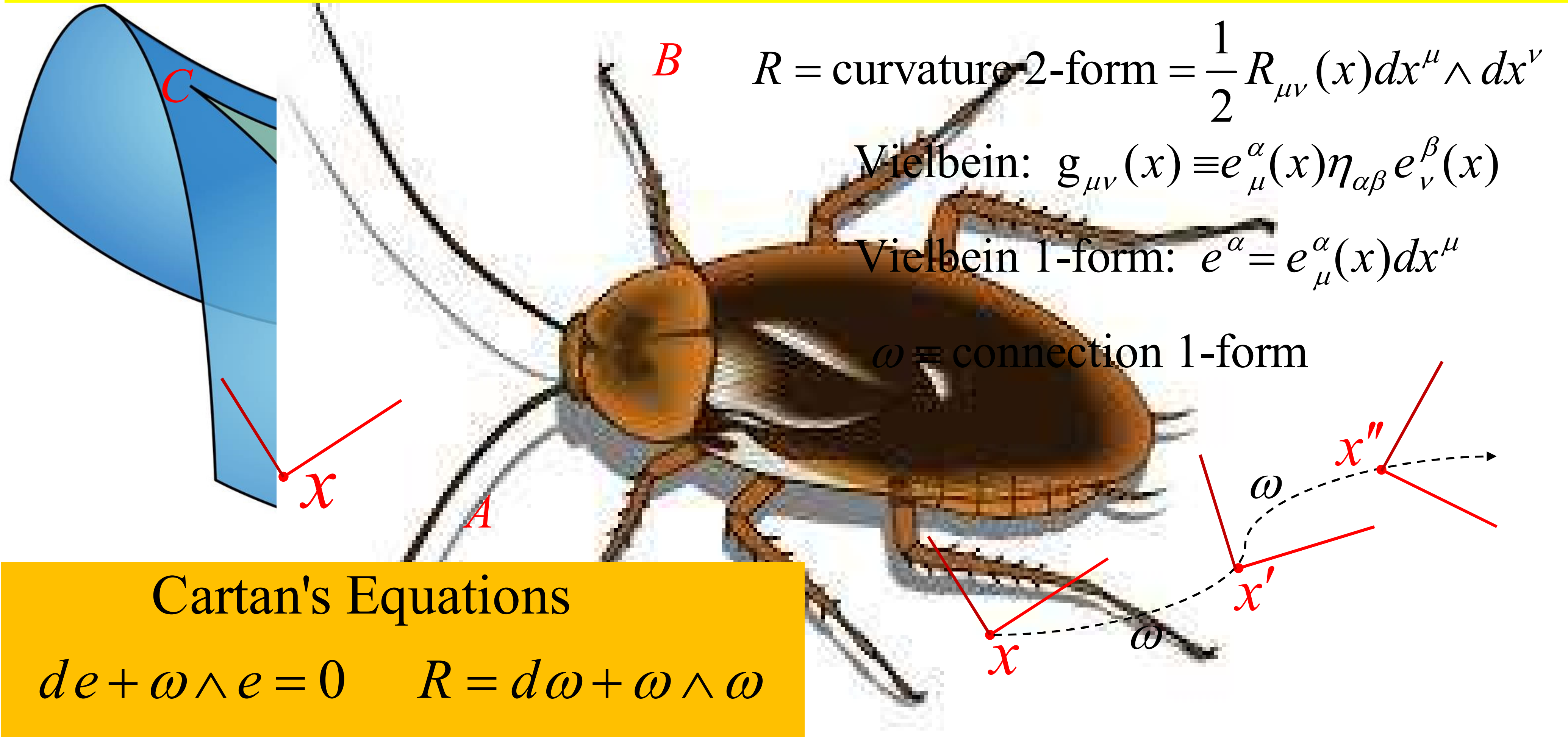
2) Gauge invariant action :

$$S = \int_M \left(\tfrac{1}{2} F \wedge *F + (-1)^p A \wedge *j \right)$$

Still not convinced?

یارب! وہ نہ سمجھے ہیں نہ سمجھیں گے مری بات،
دے اور دل ان کو جو نہ دے مجھ کو زباں اور۔

Vielbeins and Cartan’s Equations



$R = \text{curvature 2-form} = \frac{1}{2} R_{\mu\nu}(x) dx^\mu \wedge dx^\nu$

Vielbein: $g_{\mu\nu}(x) \equiv e^{\alpha}_{\mu}(x) \eta_{\alpha\beta} e^{\beta}_{\nu}(x)$

Vielbein 1-form: $e^\alpha = e^{\alpha}_{\mu}(x) dx^\mu$

$\omega \equiv$ connection 1-form

Cartan's Equations

$$de + \omega \wedge e = 0 \qquad R = d\omega + \omega \wedge \omega$$

What did this buy for you?

Example: Easy derivation of various identities involving the Riemann curvature tensor, $R^{\sigma}{}_{\lambda\mu\nu}$

Start with: $de+\omega\wedge e=0 \quad \Rightarrow \quad d(\,de+\omega\wedge e\,)=0 \quad \Rightarrow \quad \cancel{dd}\,e+d\omega\wedge e-\omega\wedge de=0$

$\Rightarrow \quad d\omega\wedge e+\omega\wedge\omega\wedge e=0 \quad \Rightarrow \quad (d\omega+\omega\wedge\omega)\wedge e=0 \quad \Rightarrow \quad R\wedge e=0$

Restore Lorentz indices: $R^{\alpha}{}_{\beta\mu\nu}\,e^{\beta}{}_{\lambda}\,dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda}=0 \quad \Rightarrow \quad R^{\alpha}{}_{\lambda\mu\nu}\,dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda}=0$

or $\quad e^{\alpha}{}_{\sigma}R^{\sigma}{}_{\lambda\mu\nu}\,dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda}=0$

$$R^{\sigma}{}_{\lambda\mu\nu}+R^{\sigma}{}_{\nu\lambda\mu}+R^{\sigma}{}_{\mu\nu\lambda}=0$$

Now you can rewrite the usual Einstein-Hilbert action without using coordinates or space-time indices:

$$S = \int_M R_{\alpha\beta} * (e^\alpha \wedge e^\beta)$$

Why? Because:

a) You cannot have more than one power of the curvature 2-form R.

b) The only other available form is e^α.

c) $2 + (d - 2) = d$

Principal Fibre Bundles and Non-Abelian Gauge Theories

It’s the natural math to use because:

a) The gauge potential is the connection

b) Global properties of gauge fields can be explored.

Basic Non-Abelian Gauge Theory

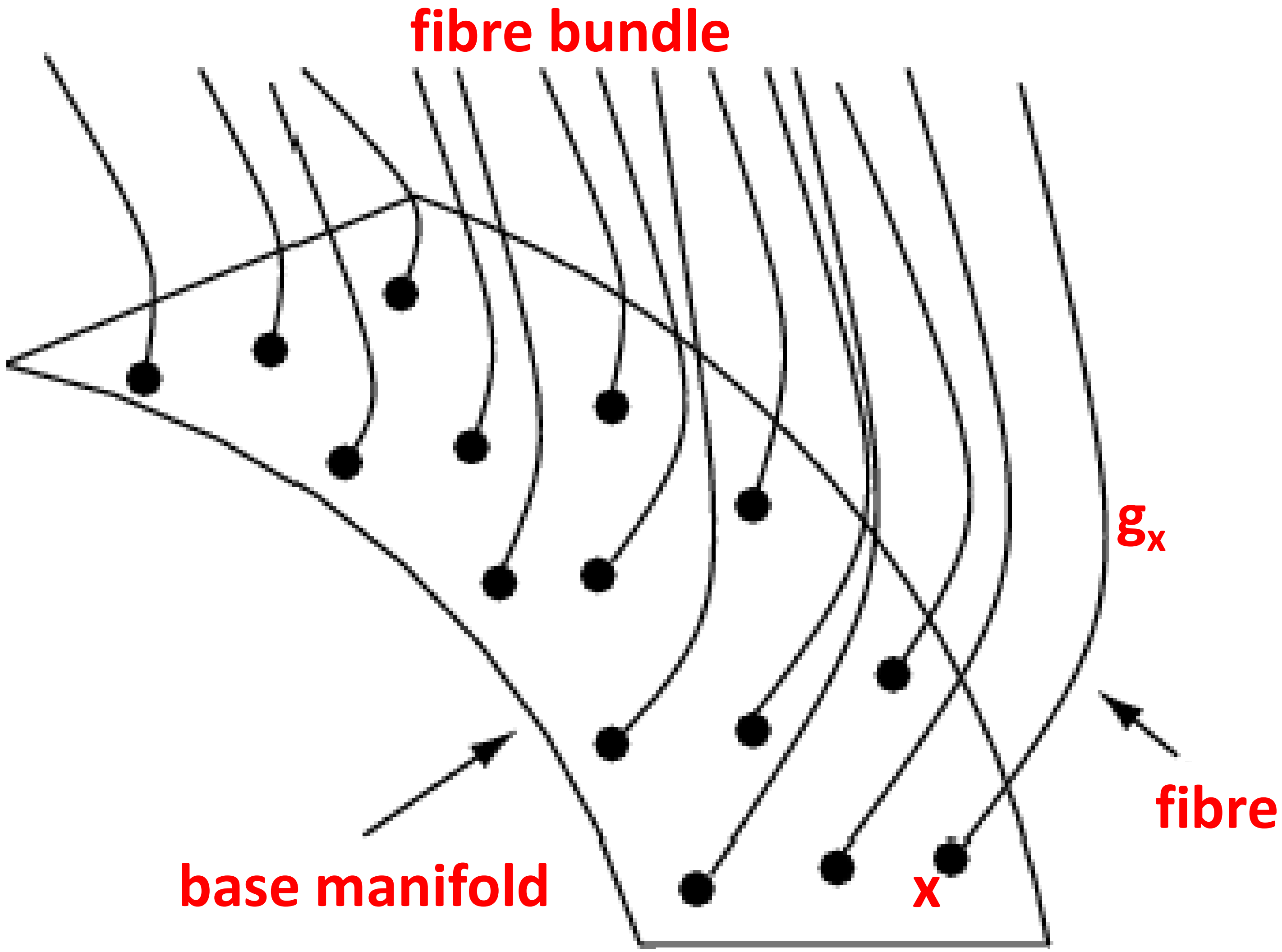
- Define one-form (matrix valued): $A = A_\mu dx^\mu$.

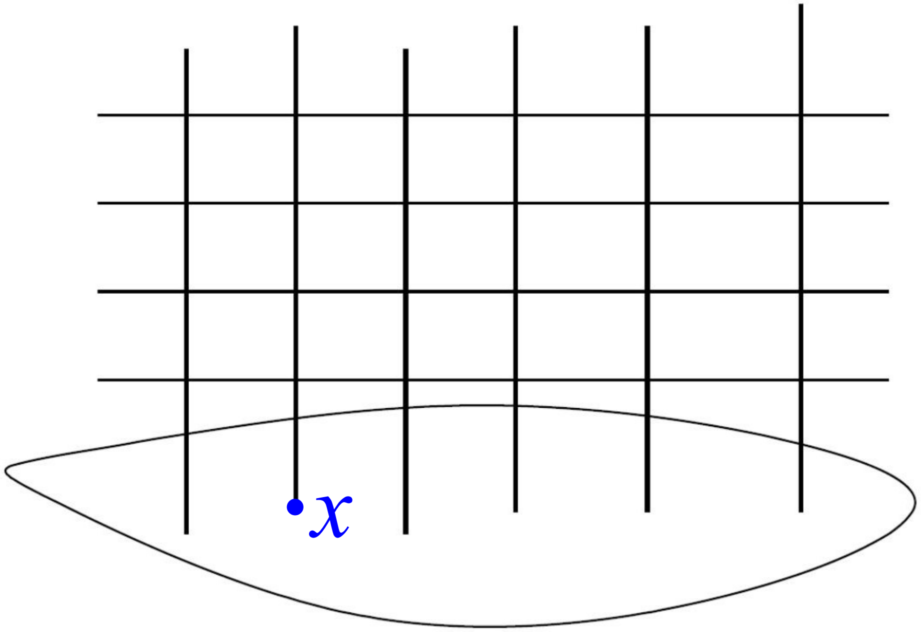
  Note that $A \wedge A = A_\mu A_\nu \, dx^\mu \wedge dx^\nu = \tfrac{1}{2}[A_\mu , A_\nu] dx^\mu \wedge dx^\nu$
- Create $F = d\,A + A \wedge A$ (remember Cartan!)

  If $A \to U\,A\,U^\dagger + U\,d\,U^\dagger$ then $F \to U\,F\,U^\dagger$

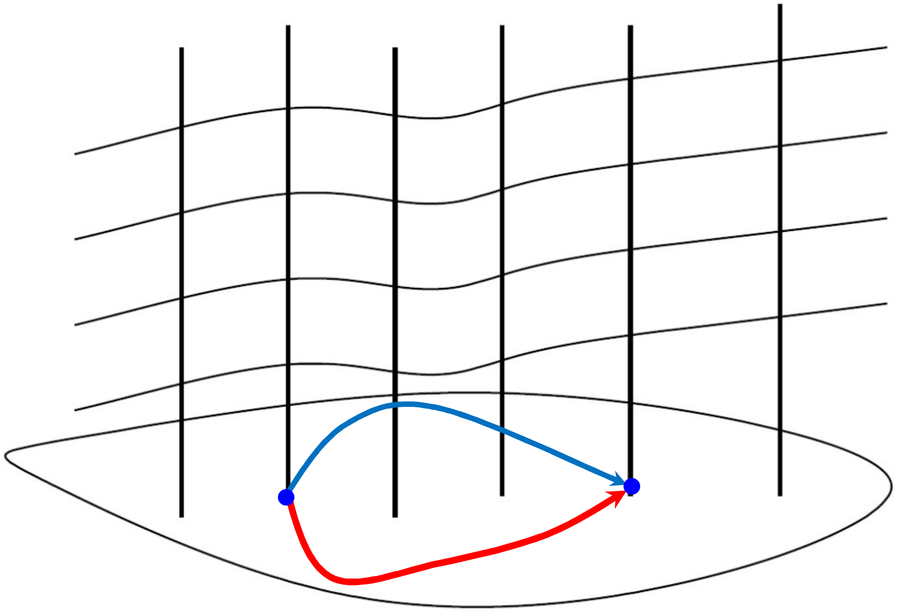
  Equivalently: $F_{\mu\nu}{}^a\, t^a \to U\,F_{\mu\nu}{}^a\, t^a\, U^\dagger$
- Then $L = -\dfrac{1}{2} Tr \displaystyle\int F \wedge *F$ is a gauge-invariant Lagrangian.

But if *A* is to be understood as a connection, then in what space is it a connection ?


fibre bundle
g_x
fibre
base manifold
x



M^4 (base space)



The Great Philosophical Puzzle

“The Unreasonable Effectiveness of Mathematics in the Natural Sciences”

Eugene P. Wigner