MATHEMATICAL PHYSICS

SEMESTER 2

2016–2017

MP352
Special Relativity

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Time allowed: $1\frac{1}{2}$ hours

Answer **ALL** questions

1. Let Σ and Σ' be inertial frames. Frame Σ' moves at velocity v with respect to Σ, in the common (positive) x direction. Measurements of events in the two frames, denoted respectively by (x, y, z, t) and (x', y', z', t'), are related by the Lorentz transformation

$$x' = \gamma_v(x - vt) \ ; \quad y' = y \ ; \quad z' = z \ ; \quad t' = \gamma_v(t - vx/c^2)$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) A particle has velocity $\vec{u}' = (0, 0, c)$ relative to Σ'.
Find the velocity of the particle relative to Σ.
Explain how your result is consistent with the constancy of the speed of light.

[18 marks]

(b) A body of mass m is at rest in the frame Σ'.
Write down its four-momentum in the frame Σ'.
Write down its four-momentum in the frame Σ.
Show that the norm of the four-momentum is the same in the two frames.

[20 marks]

(c) In the Σ frame, two events occur at times $t_1 = \dfrac{L}{c}$ and $t_2 = \dfrac{L}{2c}$, and have spatial coordinates

$$(x_1 = 4L, y_1 = 0, z_1 = 0) \quad \text{and} \quad (x_2 = L, y_2 = d, z_2 = 0)$$

respectively. What is the speed v if these two events are found to be simultaneous in frame Σ'?

[12 marks]

2. (a) Consider the set of 4×4 matrices Λ which satisfy the relation

$$\Lambda^T g \Lambda = g \,,$$

where g is the metric tensor. Show that this set forms a group under matrix multiplication. Verify all four group properties.

[20 marks]

(b) A particle of rest mass M, while at rest, decays into a particle of mass m and speed v, and a photon of frequency f, moving in opposite directions. Relativistic momentum and energy are conserved in this process.

Write down the equations for momentum conservation and for energy conservation.

Use these equations to show that $m = M\sqrt{(c-v)/(c+v)}$.

[20 marks]

(c) The kinetic energy of motion of a particle is the relativistic total energy minus the rest energy. Find the speed of a particle whose kinetic energy is twice as large as its rest energy.

[10 marks]

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PARTIAL SOLUTIONS AND HINTS

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1. Question 1.

   (a) Question 1(a)

   A particle has velocity $\vec{u}' = (0, 0, c)$ relative to Σ'.

   Find the velocity of the particle relative to Σ.

   Explain how your result is consistent with the constancy of the speed of light.

   [18 marks]

   [**Sample Answer:**]

   The velocity components relative to Σ are $\left(v, 0, \dfrac{c}{\gamma_v}\right)$.

   (Found using the velocity addition formulae.)

   The particle moves with the speed of light relative to Σ', hence is a photon. For consistency with the postulates of relativity, its speed should be c relative to Σ as well. Using the velcity components, the speed relative to Σ is seen to be

$$\sqrt{v^2 + 0^2 + \left(\frac{c}{\gamma_v}\right)^2} \;=\; \sqrt{v^2 + c^2\left(1 - \frac{v^2}{c^2}\right)^2} \;=\; \sqrt{c^2} \;=\; c$$

   Hence consistent with the postulate of constancy of the speed of light.

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(b) Question 1(b)

A body of mass m is at rest in the frame Σ'.

Write down its four-momentum in the frame Σ'.

Write down its four-momentum in the frame Σ.

Show that the norm of the four-momentum is the same in the two frames.

[20 marks]

[Sample Answer:]

In the frame Σ':

The body has momentum 0 and energy $\gamma_0 mc^2 = mc^2$. Hence the four-momentum is

$$(\frac{mc^2}{c}, 0, 0, 0) \;=\; (mc, 0, 0, 0)$$

The norm is

$$(mc)^2 - 0^2 - 0^2 - 0^2 = m^2c^2$$

In the frame Σ:

The body has velocity v in the x-direction, hence three-momentum components

$$(\gamma_v mv, 0, 0)$$

The energy of the body is $\gamma_v mc^2$.

The four-momentum is thus

$$(\frac{\gamma_v mc^2}{c}, \gamma_v mv, 0, 0) \;=\; (\gamma_v mc, \gamma_v mv, 0, 0)$$

The norm is

$$(\gamma_v mc)^2 - (\gamma_v mv)^2 - 0^2 - 0^2 \;=\; (\gamma_v mc)^2(1 - v^2/c^2) \;=\; \gamma_v^2 m^2 c^2 \gamma_v^{-2} \;=\; m^2c^2$$

The norm is thus the same in the two frames.

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(c) Question 1(c)

In the Σ frame, two events occur at times $t_1 = \dfrac{L}{c}$ and $t_2 = \dfrac{L}{2c}$, and have spatial coordinates

$$(x_1 = 4L, y_1 = 0, z_1 = 0) \quad \text{and} \quad (x_2 = L, y_2 = d, z_2 = 0)$$

respectively. What is the speed v if these two events are found to be simultaneous in frame Σ'?

[12 marks]

[Sample Answer:]
The LT also holds for the difference of events, thus

$$\Delta t' = \gamma_v(\Delta t - v\Delta x/c^2) \ .$$

(Alternatively, can write the LT for the two events and then subtract.)
If the events are simultaneous in frame Σ', we have $\Delta t' = 0$, so that

$$\gamma_v(\Delta t - v\Delta x/c^2) = 0$$
$$\implies \frac{v}{c^2} = \frac{\Delta t}{\Delta x} = \frac{t_1 - t_2}{x_1 - x_2} = \frac{L/2c}{3L} = \frac{1}{6c} \qquad \implies \qquad v = c/6$$

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2. Question 2.

(a) Question 2(a).
Consider the set of 4×4 matrices Λ which satisfy the relation $\Lambda^T g \Lambda = g$, where g is the metric tensor. Show that this set forms a group under matrix multiplication. Verify all four group properties.

[20 marks]

[**Sample Answer:**]
To be a group, the properties of Closure, Associativity, Existence of Identity, and Existence of Inverse must be satisfied.
Closure: If Λ_1 and Λ_2 are members of the set, then $\Lambda_1^T g \Lambda_1 = g$ and $\Lambda_2^T g \Lambda_2 = g$. Then

$$\Big(\Lambda_1\Lambda_2\Big)^T g\Big(\Lambda_1\Lambda_2\Big) = \Big(\Lambda_2^T\Lambda_1^T\Big)g\Big(\Lambda_1\Lambda_2\Big) = \Lambda_2^T\Big(\Lambda_1^T g\Lambda_1\Big)\Lambda_2 = \Lambda_2^T g\Lambda_2 = g$$

which means that $\Lambda_1\Lambda_2$ is also a member of the set.
Associativity: Matrix multiplication is known to be associative.
Identity: The 4×4 identity matrix I is a member of the set, because $IgI = g$. Hence the set contains an identity element.
Inverse: If Λ is an element of the set, $\Lambda^T g \Lambda = g$ by definition. To show that the matrix inverse Λ^{-1} also belongs to the set, multiply both sides by $(\Lambda^{-1})^T$ on the left and by Λ^{-1} on the right:

$$\big(\Lambda^{-1}\big)^T\Big(\Lambda^T g\Lambda\Big)\Lambda^{-1} = \big(\Lambda^{-1}\big)^T g\Lambda^{-1}$$

The left side is

$$\Big(\big(\Lambda^{-1}\big)^T\Lambda^T\Big)g\Big(\Lambda\Lambda^{-1}\Big) = \Big(\Lambda\Lambda^{-1}\Big)^T gI = I^T g = g$$

so that we have obtained

$$g = \big(\Lambda^{-1}\big)^T g\Lambda^{-1}$$

i.e., the inverse of Λ satisfies the defining equation and hence belongs to the set.

We have thus shown that each of the four defining properties of a group is satisfied by the set under matrix multiplication. This implies that the set defined by $\Lambda^T g \Lambda = g$ is a group under matrix multiplication.

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(b) Question 2(b).

A particle of rest mass M, while at rest, decays into a particle of mass m and speed v, and a photon of frequency f, moving in opposite directions. Relativistic momentum and energy are conserved in this process.

Write down the equations for momentum conservation and for energy conservation.

Use these equations to show that $m = M\sqrt{(c-v)/(c+v)}$.

[20 marks]

[Sample Answer:]

$$\text{Momentum conservation:} \qquad 0 = \frac{hf}{c} - \gamma(v)mv$$

$$\text{Energy conservation:} \qquad Mc^2 = hf + \gamma(v)mc^2$$

Eliminating hf yields

$$Mc^2 = \gamma(v)mvc + \gamma(v)mc^2$$

$$\implies \qquad m = \frac{Mc}{\gamma(v)\times(c+v)} = M\frac{c\sqrt{1-(v/c)^2}}{c+v} = M\sqrt{\frac{c-v}{c+v}}$$

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(c) Question 2(c).
The kinetic energy of motion of a particle is the relativistic total energy minus the rest energy. Find the speed of a particle whose kinetic energy is twice as large as its rest energy.

[10 marks]

[Sample Answer:]
The kinetic energy is

$$T = \begin{pmatrix}\text{energy of}\\ \text{moving particle}\end{pmatrix} - \begin{pmatrix}\text{energy of}\\ \text{particle at rest}\end{pmatrix} = \gamma_v mc^2 - mc^2$$

We are given that $T = 2mc^2$; thus

$$\begin{aligned} \gamma_v mc^2 - mc^2 &= 2mc^2 \\ \implies \quad \gamma_v &= 3 \\ \implies \quad \frac{1}{\sqrt{1-(v/c)^2}} &= 3 \\ \implies \quad (v/c)^2 &= 1 - \frac{1}{3^2} = \frac{8}{9} \\ \implies \quad v^2 = \frac{8}{9}c^2 \quad &\implies \quad v = \frac{2\sqrt{2}}{3}c \end{aligned}$$

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