9.4: More on 4-vectors and 4-tensors

This is a good moment to introduce a formalism that will allow us, in particular, to solve the same proton collision problem in one more (and arguably, the most elegant) way. Much more importantly, this formalism will be virtually necessary for the description of the Lorentz transform of the electromagnetic field, and its interaction with relativistic particles – otherwise the formulas would be too cumbersome.

Let us call the 4-vectors we have used before,

$$\text{Contravariant 4-vectors} \qquad A^\alpha \equiv \{A_0, \mathbf{A}\}, \tag{9.84}$$

contravariant, and denote them with top indices, and introduce also covariant vectors,

$$\text{Covariant 4-vectors} \qquad A_\alpha \equiv \{A_0, -\mathbf{A}\}, \tag{9.85}$$

marked by lower indices. Now if we form a scalar product of these two vectors using the standard (3D-like) rule, just as a sum of the products of the corresponding components, we immediately get

$$A_\alpha A^\alpha \equiv A^\alpha A_\alpha \equiv A_0^2 - A^2. \tag{9.86}$$

Here and below the sign of the sum of four components of the product has been dropped.[37] The scalar product (86) is just the norm of the 4-vector in our former definition, and as we already know, is Lorentz-invariant. Moreover, the scalar product of two different vectors (also a Lorentz invariant), may be rewritten in any of two similar forms:[38]

$$\text{Scalar product's forms} \qquad A_0 B_0 - \mathbf{A}\cdot\mathbf{B} \equiv A_\alpha B^\alpha = A^\alpha B_\alpha; \tag{9.87}$$

again, the only caveat is to take one vector in the covariant, and the other one in the contravariant form.

Now let us return to our sample problem (Fig. 10). Since all components ($\mathscr{E}/c$ and $\mathbf{p}$) of the total 4-momentum of our system are conserved at the collision, its norm is conserved as well:

$$(p_a + p_b)_\alpha (p_a + p_b)^\alpha = (4p)_\alpha (4p)^\alpha. \tag{9.88}$$

Since now the vector product is the usual math construct, we know that the parentheses on the left-hand side of this equation may be multiplied as usual. We may also swap the operands and move constant factors through products as convenient. As a result, we get

$$(p_a)_\alpha (p_a)^\alpha + (p_b)_\alpha (p_b)^\alpha + 2(p_a)_\alpha (p_b)^\alpha = 16 p_\alpha p^\alpha. \tag{9.89}$$

Thanks to the Lorentz invariance of each of the terms, we may calculate it in the reference frame we like. For the first two terms on the left-hand side, as well as for the right hand side term, it is beneficial to use the frames in which that particular proton is at rest; as a result, according to Eq. (77b), each of the two left-hand-side terms equals $(mc)^2$, while the right-hand side equals $16(mc)^2$. On the contrary, the last term on the left-hand side is more easily evaluated in the lab frame, because in it, the three spatial components of the 4-momentum p_b vanish, and the scalar product is just the product of the scalars $\mathscr{E}/c$ for protons a and b. For the latter proton, being at rest, this ratio is just mc so that we get a simple equation,

$$(mc)^2 + (mc)^2 + 2\frac{\mathscr{E}_{\min}}{c} mc = 16(mc)^2, \tag{9.90}$$

immediately giving the final result $\mathscr{E}_{\min} = 7mc^2$, already obtained earlier in two more complex ways.

Let me hope that this example was a convincing demonstration of the convenience of representing 4-vectors in the contravariant (84) and covariant (85) forms,[39] with Lorentz invariant norms (86). To be useful for more complex tasks, this formalism should be developed a little bit further. In particular, it is crucial to know how the 4-vectors change under the Lorentz transform. For contravariant vectors, we already know the answer (54); let us rewrite it in our new notation:

$$A^\alpha = L^\alpha_\beta A'^\beta. \qquad \text{Lorentz transform: contravariant vectors} \tag{9.91}$$

where L^α_β is the matrix (51), generally called the mixed Lorentz tensor:[40]

$$L_{\beta}^{\alpha}=\begin{pmatrix}\gamma & \beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \qquad \text{Mixed Lorentz tensor} \tag{9.92}$$

Note that though the position of the indices α and β in the Lorentz tensor notation is not crucial, because this tensor is symmetric, it is convenient to place them using the general index balance rule: the difference of the numbers of the upper and lower indices should be the same in both parts of any 4-vector/tensor equality. (Please check yourself that all the formulas above do satisfy this rule.)

In order to rewrite Eq. (91) in a more general form that would not depend on the particular orientation of the coordinate axes (Fig. 1), let us use the contravariant and covariant forms of the 4-vector of the time-space interval (57),

$$dx^{\alpha}=\{cdt, d\mathbf{r}\}, \quad dx_{\alpha}=\{cdt, -d\mathbf{r}\}; \tag{9.93}$$

then its norm (58) may be represented as[41]

$$(ds)^2 \equiv (cdt)^2-(dr)^2=dx^{\alpha}dx_{\alpha}=dx_{\alpha}dx^{\alpha}. \tag{9.94}$$

Applying Eq. (91) to the first, contravariant form of the 4-vector (93), we get

$$dx^{\alpha}=L_{\beta}^{\alpha}dx'^{\beta}. \tag{9.95}$$

But with our new shorthand notation, we can also write the usual rule of differentiation of each component x^{α}, considering it as a function (in our case, linear) of four arguments x'^{β}, as follows:[42]

$$dx^{\alpha}=\frac{\partial x^{\alpha}}{\partial x'^{\beta}}dx'^{\beta}. \tag{9.96}$$

Comparing Eqs. (95) and (96), we can rewrite the general Lorentz transform rule (92) in a new form,

$$\text{Lorentz transform: general form} \qquad A^{\alpha}=\frac{\partial x^{\alpha}}{\partial x'^{\beta}}A'^{\beta}. \tag{9.97a}$$

which does not depend on the coordinate axes' orientation.

It is straightforward to verify that the reciprocal transform may be represented as

$$\text{Reciprocal Lorentz transform} \qquad A'^{\alpha}=\frac{\partial x'^{\alpha}}{\partial x^{\beta}}A^{\beta}. \tag{9.97b}$$

However, the reciprocal transform has to differ from the direct one only by the sign of the relative velocity of the frames, so that for the coordinate choice shown in Fig. 1, its matrix is

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}}=\begin{pmatrix}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{9.98}$$

Since according to Eqs. (84)-(85), covariant 4-vectors differ from the contravariant ones by the sign of their spatial components, their direct transform is given by the matrix (98). Hence their direct and reciprocal transforms may be represented, respectively, as

$$A_{\alpha}=\frac{\partial x'^{\beta}}{\partial x^{\alpha}}A'_{\beta}, \quad A'_{\alpha}=\frac{\partial x^{\beta}}{\partial x'^{\alpha}}A_{\beta}, \text{Lorentz transform: covariant vectors} \tag{9.99}$$

evidently satisfying the index balance rule. (Note that primed quantities are now multiplied, rather than divided as in the contravariant case.) As a sanity check, let us apply this formalism to the scalar product $A_{\alpha}A^{\alpha}$. As Eq. (96) shows, the implicit-sum notation allows us to multiply and divide any equality by the same partial differential of a coordinate, so that we can write:

$$A_{\alpha}A^{\alpha}=\frac{\partial x'^{\beta}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x'^{\gamma}}A'_{\beta}A'^{\gamma}=\frac{\partial x'^{\beta}}{\partial x'^{\gamma}}A'_{\beta}A'^{\gamma}=\delta_{\beta\gamma}A'_{\beta}A'^{\gamma}=A'_{\gamma}A'^{\gamma}, \tag{9.100}$$

i.e. the scalar product $A_{\alpha}A^{\alpha}$ (as well as $A^{\alpha}A_{\alpha}$) is Lorentz-invariant, as it should be.

Now, let us consider the 4-vectors of derivatives. Here we should be very careful. Consider, for example, the following 4-vector operator

$$\frac{\partial}{\partial x^{\alpha}} \equiv \left\{\frac{\partial}{\partial(ct)}, \nabla\right\}, \tag{9.101}$$

As was discussed above, the operator is not changed by its multiplication and division by another differential, e.g., $\partial x'^{\beta}$ (with the corresponding implied summation over all four values of β), so that

$$\frac{\partial}{\partial x^{\alpha}} = \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \frac{\partial}{\partial x'^{\beta}}. \tag{9.102}$$

But, according to the first of Eqs. (99), this is exactly how the covariant vectors are Lorentz-transformed! Hence, we have to consider the derivative over a contravariant space-time interval as a covariant 4-vector, and vice versa.[43] (This result might be also expected from the index balance rule.) In particular, this means that the scalar product

$$\frac{\partial}{\partial x^{\alpha}} A^{\alpha} \equiv \frac{\partial A_0}{\partial(ct)} + \nabla \cdot \mathbf{A} \tag{9.103}$$

should be Lorentz-invariant for any legitimate 4-vector. A convenient shorthand for the covariant derivative, which complies with the index balance rule, is

$$\frac{\partial}{\partial x^{\alpha}} \equiv \partial_{\alpha}, \tag{9.104}$$

so that the invariant scalar product may be written just as $\partial_{\alpha} A^{\alpha}$. A similar definition of the contravariant derivative,

$$\partial^{\alpha} \equiv \frac{\partial}{\partial x_{\alpha}} = \left\{\frac{\partial}{\partial(ct)}, -\nabla\right\}, \tag{9.105}$$

allows us to write the Lorentz-invariant scalar product (103) in any of the following two forms:

$$\frac{\partial A_0}{\partial(ct)} + \nabla \cdot \mathbf{A} = \partial^{\alpha} A_{\alpha} = \partial_{\alpha} A^{\alpha}. \tag{9.106}$$

Finally, let us see how does the general Lorentz transform change 4-tensors. A second-rank 4×4 matrix is a legitimate 4-tensor if the 4-vectors it relates obey the Lorentz transform. For example, if two legitimate 4-vectors are related as

$$A^{\alpha} = T^{\alpha\beta} B_{\beta}, \tag{9.107}$$

we should require that

$$A'^{\alpha} = T'^{\alpha\beta} B'_{\beta}, \tag{9.108}$$

where A^{α} and A'^{α} are related by Eqs. (97), while B_{β} and B'_{β}, by Eqs. (99). This requirement immediately yields

Lorentz transform of 4-tensors

$$T^{\alpha\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\gamma}} \frac{\partial x^{\beta}}{\partial x'^{\delta}} T'^{\gamma\delta}, \quad T'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} T^{\gamma\delta}, \tag{9.109}$$

with the implied summation over two indices, γ and δ. The rules for the covariant and mixed tensors are similar.[44]

Reference

[37] This compact notation may take some time to be accustomed to, but is very convenient (compact) and can hardly lead to any confusion, due to the following rule: the summation is implied when (and only when) an index is repeated twice, once on the top and another at the bottom. In this course, this shorthand notation will be used only for 4-vectors, but not for the usual (spatial) vectors.

[38] Note also that, by definition, for any two 4-vectors, $A_{\alpha} B^{\alpha} = B^{\alpha} A_{\alpha}$.

[39] These forms are 4-vector extensions of the notions of contravariance and covariance, introduced in the 1850s by J. Sylvester (who also introduced the term "matrix") for the description of the change of the usual geometric (3-component) vectors at the

transfer between different reference frames – e.g., resulting from the frame rotation. In this case, the contravariance or covariance of a vector is uniquely determined by its nature: if the Cartesian coordinates of a vector (such as the non-relativistic velocity $\mathbf{v} = d\mathbf{r}/dt$) are transformed similarly to the radius-vector $\mathbf{r}$, it is called contravariant, while the vectors (such as ∇f) that require the reciprocal transform, are called covariant. In the Minkowski space, both forms may be used for any 4-vector.

[40] Just as the 4-vectors, 4-tensors with two top indices are called contravariant, and those with two bottom indices, covariant. The tensors with one top and one bottom index are called mixed.

[41] Another way to write this relation is $(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta = g^{\alpha\beta}dx_\alpha dx_\beta$, where double summation over indices α and β is implied, and g is the so-called metric tensor,

$$g^{\alpha\beta} \equiv g_{\alpha\beta} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which may be used, in particular, to transfer a covariant vector into the corresponding contravariant one and back: $A^\alpha = g^{\alpha\beta}A_\beta$, $A_\alpha = g_{\alpha\beta}A^\beta$. The metric tensor plays a key role in general relativity, in which it is affected by gravity – "curved" by particles' masses.

[42] Note that in the index balance rule, the top index in the denominator of a fraction is counted as a bottom index in the numerator, and vice versa.

[43] As was mentioned above, this is also a property of the reference-frame transform of the "usual" 3D vectors.

[44] It is straightforward to check that transfer between the contravariant and covariant forms of the same tensor may be readily achieved using the metric tensor g: $T_{\alpha\beta} = g_{\alpha\gamma}T^{\gamma\delta}g_{\delta\beta}, T^{\alpha\beta} = g^{\alpha\gamma}T_{\gamma\delta}g^{\delta\beta}$.

9.5: The Maxwell Equations in the 4-form

This 4-vector formalism background is already sufficient to analyze the Lorentz transform of the electromagnetic field. Just to warm up, let us consider the continuity equation (4.5),

$$\frac{\partial \rho}{\partial t}+\nabla \cdot \mathbf{j}=0, \tag{9.110}$$

which expresses the electric charge conservation, and, as we already know, is compatible with the Maxwell equations. If we now define the contravariant and covariant 4-vectors of electric current as

$$\text{4-vector of electric current} \qquad j^{\alpha} \equiv\{\rho c, \mathbf{j}\}, \quad j_{\alpha} \equiv\{\rho c,-\mathbf{j}\}, \tag{9.111}$$

then Eq. (110) may be represented in the form

$$\text{Continuity equation: 4-form} \qquad \partial^{\alpha} j_{\alpha}=\partial_{\alpha} j^{\alpha}=0, \tag{9.112}$$

showing that the continuity equation is form-invariant[45] with respect to the Lorentz transform.

Of course, such a form's invariance of a relation does not mean that all component values of the 4-vectors participating in it are the same in both frames. For example, let us have some static charge density ρ in frame 0; then Eq. (97b), applied to the contravariant form of the 4-vector (111), reads

$$j^{\prime \alpha}=\frac{\partial x^{\prime \alpha}}{\partial x^{\beta}} j^{\beta}, \quad \text{with } j^{\beta}=\{\rho c, 0,0,0\}. \tag{9.113}$$

Using the particular form (98) of the reciprocal Lorentz matrix for the coordinate choice shown in Fig. 1, we see that this relation yields

$$\rho^{\prime}=\gamma \rho, \quad j_{x}^{\prime}=-\gamma \beta \rho c=-\gamma \nu \rho, \quad j_{y}^{\prime}=j_{z}^{\prime}=0. \tag{9.114}$$

Since the charge velocity, as observed from frame 0', is $(-\mathbf{v})$, the non-relativistic results would be $\rho^{\prime}=\rho$, $\mathbf{j}^{\prime}=-\mathbf{v} \rho$. The additional γ factor in the relativistic results is caused by the length contraction: $d x^{\prime}=d x / \gamma$, so that to keep the total charge $d Q=\rho d^{3} r=\rho d x d y d z$ inside the elementary volume $d^{3} r=d x d y d z$ intact, ρ (and hence j_{x}) should increase proportionally.

Next, at the end of Chapter 6 we have seen that Maxwell equations for the electromagnetic potentials ϕ and $\mathbf{A}$ may be represented in similar forms (6.118), under the Lorenz (again, not "Lorentz", please!) gauge condition (6.117). For free space, this condition takes the form

$$\nabla \cdot \mathbf{A}+\frac{1}{c^{2}} \frac{\partial \phi}{\partial t}=0. \tag{9.115}$$

This expression gives us a hint of how to form the 4-vector of the potentials:[46]

$$A^{\alpha} \equiv\left\{\frac{\phi}{c}, \mathbf{A}\right\}, \quad A_{\alpha} \equiv\left\{\frac{\phi}{c},-\mathbf{A}\right\}; \qquad \text{4-vector of potentials} \tag{9.116}$$

indeed, this vector satisfies Eq. (115) in its 4-form:

$$\partial^{\alpha} A_{\alpha}=\partial_{\alpha} . A^{\alpha}=0 \qquad \text{Lorenz gauge: 4-form} \tag{9.117}$$

Since this scalar product is Lorentz-invariant, and the derivatives (104)-(105) are legitimate 4-vectors, this implies that the 4-vector (116) is also legitimate, i.e. obeys the Lorentz transform formulas (97), (99). Even more convincing evidence of this fact may be obtained from the Maxwell equations (6.118) for the potentials. In free space, they may be rewritten as

$$\left[\frac{\partial^{2}}{\partial(c t)^{2}}-\nabla^{2}\right] \frac{\phi}{c}=\frac{\rho c}{\varepsilon_{0} c^{2}} \equiv \mu_{0}(\rho c), \quad\left[\frac{\partial^{2}}{\partial(c t)^{2}}-\nabla^{2}\right] \mathbf{A}=\mu_{0} \mathbf{j}. \tag{9.118}$$

Using the definition (116), these equations may be merged to one:[47]

$$\square A^{\alpha}=\mu_{0} j^{\alpha}, \qquad \text{Maxwell equation for 4-potential} \tag{9.119}$$

where $\square$ is the d' Alembert operator,[48] which may be represented as either of two scalar products, D'Alembert operator $$\square \equiv \frac{\partial^2}{\partial(ct)^2} - \nabla^2 = \partial^\beta \partial_\beta = \partial_\beta \partial^\beta. \tag{9.120}$$

and hence is Lorentz-invariant. Because of that, and the fact that the Lorentz transform changes both 4-vectors A^α and j^α in a similar way, Eq. (119) does not depend on the reference frame choice. Thus we have arrived at a key point of this chapter: we see that the Maxwell equations are indeed form-invariant with respect to the Lorentz transform. As a by-product, the 4-vector form (119) of these equations (for potentials) is extremely simple – and beautiful!

However, as we have seen in Chapter 7, for many applications the Maxwell equations for the field vectors are more convenient; so let us represent them in the 4-form as well. For that, we may express all Cartesian components of the usual (3D) field vector vectors (6.7),

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{9.121}$$

via those of the potential 4-vector A^α. For example,

$$E_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} = -c\left(\frac{\partial}{\partial x}\frac{\phi}{c} + \frac{\partial A_x}{\partial(ct)}\right) \equiv -c\left(\partial^0 A^1 - \partial^1 A^0\right), \tag{9.122}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \equiv -\left(\partial^2 A^3 - \partial^3 A^2\right). \tag{9.123}$$

Completing similar calculations for other field components (or just generating them by appropriate index shifts), we find that the following antisymmetric, contravariant field-strength tensor,

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{9.124}$$

may be expressed via the field components as follows:[49]

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{9.125a}$$

Field-strength tensors

so that the covariant form of the tensor is

$$F_{\alpha\beta} \equiv g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{9.125b}$$

If Eq. (124) looks a bit too bulky, please note that as a reward, the pair of inhomogeneous Maxwell equations, i.e. two equations of the system (6.99), which in free space $(\mathbf{D} = \varepsilon_0 \mathbf{E}, \mathbf{B} = \mu_0 \mathbf{H})$ may be rewritten as

$$\nabla \cdot \frac{\mathbf{E}}{c} = \mu_0 c \rho, \quad \boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial}{\partial(ct)}\frac{\mathbf{E}}{c} = \mu_0 \mathbf{j}, \tag{9.126}$$

may now be expressed in a very simple (and manifestly form-invariant) way,

$$\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta, \qquad \text{Maxwell equation for tensor } F \tag{9.127}$$

which is comparable with Eq. (119) in its simplicity – and beauty. Somewhat counter-intuitively, the pair of homogeneous Maxwell equations of the system (6.99),

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{9.128}$$

look, in the 4-vector notation, a bit more complicated:[50]

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \tag{9.129}$$

Note, however, that Eqs. (128) may be also represented in a much simpler 4-form,

$$\partial_\alpha G^{\alpha\beta} = 0, \tag{9.130}$$

using the so-called dual tensor

$$G^{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}, \tag{9.131}$$

which may be obtained from $F^{\alpha\beta}$, given by Eq. (125a), by the following replacements:

$$\frac{\mathbf{E}}{c} \rightarrow -\mathbf{B}, \quad \mathbf{B} \rightarrow \frac{\mathbf{E}}{c}. \tag{9.132}$$

Besides the proof of the form-invariance of the Maxwell equations with respect to the Lorentz transform, the 4-vector formalism allows us to achieve our initial goal: find out how do the electric and magnetic field components change at the transfer between (inertial!) reference frames. For that, let us apply to the tensor $F^{\alpha\beta}$ the reciprocal Lorentz transform described by the second of Eqs. (109). Generally, it gives, for each field component, a sum of 16 terms, but since (for our choice of coordinates, shown in Fig. 1) there are many zeros in the Lorentz transform matrix, and the diagonal components of $F^{\gamma\delta}$ equal zero as well, the calculations are rather doable. Let us calculate, for example, $E'_x \equiv -cF'^{01}$. The only non-zero terms on the right-hand side are

$$E'_x = -cF'^{01} = -c\left(\frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^0}F^{10} + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^1}F^{01}\right) \equiv -c\gamma^2\left(\beta^2 - 1\right)\frac{E_x}{c} \equiv E_x. \tag{9.133}$$

Repeating the calculation for the other five components of the fields, we get very important relations

$$\begin{aligned} E'_x &= E_x, & B'_x &= B_x \\ E'_y &= \gamma\left(E_y - \nu B_z\right), & B'_y &= \gamma\left(B_y + \nu E_z/c^2\right), \\ E'_z &= \gamma\left(E_z + \nu B_y\right), & B'_z &= \gamma\left(B_z - \nu E_y/c^2\right), \end{aligned} \tag{9.134}$$

whose more compact "semi-vector" form is

Lorentz transform of field components

$$\begin{aligned} \mathbf{E}'_{||} &= \mathbf{E}_{||}, & \mathbf{B}'_{||} &= \mathbf{B}_{||}, \\ \mathbf{E}'_{\perp} &= \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B})_{\perp}, & \mathbf{B}'_{\perp} &= \gamma\left(\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2\right)_{\perp}, \end{aligned} \tag{9.135}$$

where the indices $||$ and $\perp$ stand, respectively, for the field components parallel and perpendicular to the relative velocity $\mathbf{v}$ of the two reference frames. In the non-relativistic limit, the Lorentz factor γ tends to 1, and Eqs. (135) acquire an even simpler form

$$\mathbf{E}' \rightarrow \mathbf{E} + \mathbf{v}\times\mathbf{B}, \quad \mathbf{B}' \rightarrow \mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}. \tag{9.136}$$

Thus we see that the electric and magnetic fields are transformed to each other even in the first order of the ν/c ratio. For example, if we fly across the field lines of a uniform, static, purely electric field $\mathbf{E}$ (e.g., the one in a plane capacitor) we will see not only the electric field's renormalization (in the second order of the ν/c ratio), but also a non-zero dc magnetic field $\mathbf{B}'$ perpendicular to both the vector $\mathbf{E}$ and the vector $\mathbf{v}$, i.e. to the direction of our motion. This is of course what might be expected from the relativity principle: from the point of view of the moving observer (which is as legitimate as that of a stationary observer), the surface charges of the capacitor's plates, which create the field $\mathbf{E}$, move back creating the dc currents (114), which induce the magnetic field $\mathbf{B}'$. Similarly, motion across a magnetic field creates, from the point of view of the moving observer, an electric field.

This fact is very important philosophically. One may say there is no such thing in Mother Nature as an electric field (or a magnetic field) all by itself. Not only can the electric field induce the magnetic field (and vice versa) in dynamics, but even in an apparently static configuration, what exactly we measure depends on our speed relative to the field sources – hence the very appropriate term for the whole field we are studying: the electromagnetism.

Another simple but very important application of Eqs. (134)-(135) is the calculation of the fields created by a charged particle moving in free space by inertia, i.e. along a straight line with constant velocity $\mathbf{u}$, at the impact parameter[51] (the closest distance) b from the observer. Selecting the reference frame 0' to move with the particle in its origin, and the frame 0 to reside in the "lab" in that the fields $\mathbf{E}$ and $\mathbf{B}$ are measured, we can use the above formulas with $\mathbf{v}=\mathbf{u}$. In this case the fields $\mathbf{E}'$ and $\mathbf{B}'$ may be calculated from, respectively, electro- and magnetostatics:

$$\mathbf{E}'=\frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}'}{r'^3},\quad \mathbf{B}'=0, \tag{9.137}$$

because in frame 0', the particle does not move. Selecting the coordinate axes so that at the measurement point $x=0, y=b, z=0$ (Fig. 11a), for this point we may write $x'=-ut', y'=b, z'=0$, so that $r'=\left(u^2t'^2+b^2\right)^{1/2}$, and the Cartesian components of the fields (137) are:

$$\begin{gathered}E'_x=-\frac{q}{4\pi\varepsilon_0}\frac{ut'}{\left(u^2t'^2+b^2\right)^{3/2}},\quad E'_y=\frac{q}{4\pi\varepsilon_0}\frac{b}{\left(u^2t'^2+b^2\right)^{3/2}},\quad E'_z=0,\\ B'_x=B'_y=B'_z=0.\end{gathered} \tag{9.138}$$

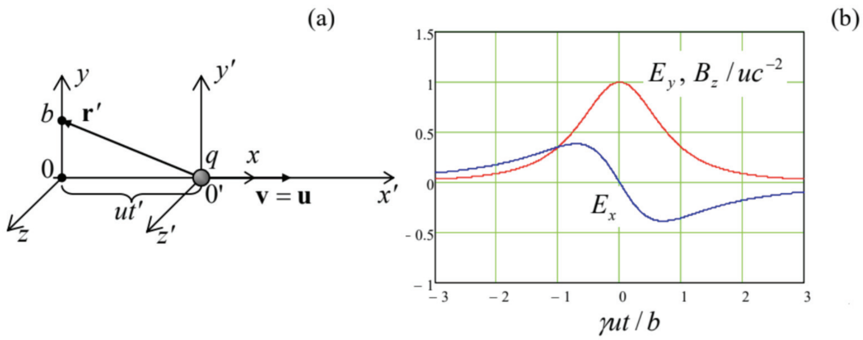

Fig. 9.11. The field pulses induced by a uniformly moving charge.

Now using the last of Eqs. (19b) with $x=0$, giving $t'=\gamma t$, and the relations reciprocal to Eqs. (134) for the field transform (it is evident that they are similar to the direct transform, with ν replaced with $-\nu=-u$), in the lab frame we get

$$E_x=E'_x=-\frac{q}{4\pi\varepsilon_0}\frac{u\gamma t}{\left(u^2\gamma^2t^2+b^2\right)^{3/2}},\quad E_y=\gamma E'_y=\frac{q}{4\pi\varepsilon_0}\frac{\gamma b}{\left(u^2\gamma^2t^2+b^2\right)^{3/2}},\quad E_z=0, \tag{9.139}$$

$$B_x=0,\quad B_y=0,\quad B_z=\frac{nu}{c^2}E'_y=\frac{u}{c^2}\frac{q}{4\pi\varepsilon_0}\frac{\gamma b}{\left(u^2\gamma^2t^2+b^2\right)^{3/2}}\equiv\frac{u}{c^2}E_y. \tag{9.140}$$

These results,[52] plotted in Fig. 11b in the units of $\gamma q^2/4\pi\varepsilon_0 b^2$, reveal two major effects. First, the charge passage by the observer generates not only an electric field pulse but also a magnetic field pulse. This is natural, because, as was repeatedly discussed in Chapter 5, any charge motion is essentially an electric current.[53] Second, Eqs. (139)-(140) show that the pulse duration scale is

$$\Delta t=\frac{b}{nu}=\frac{b}{u}\left(1-\frac{u^2}{c^2}\right)^{1/2}, \tag{9.141}$$

i.e. shrinks to virtually zero as the charge's velocity u approaches the speed of light. This is of course a direct corollary of the relativistic length contraction: in the frame 0' moving with the charge, the longitudinal spread of its electric field at distance b from the motion line is of the order of $\Delta x'=b$. When observed from the lab frame 0, this interval, in accordance with Eq. (20), shrinks to $\Delta x=\Delta x'/\gamma=b/\gamma$, and hence so does the pulse duration scale $\Delta t=\Delta x/u=b/\gamma u$.

Reference

[45] In some texts, the equations preserving their form at a transform are called "covariant", creating a possibliliy for confusion with the covariant vectors and tensors. On the other hand, calling such equations "invariant" would not distinguish them properly from invariant quantities, such as the scalar products of 4-vectors.

[46] In the Gaussian units, the scalar potential should not be divided by c in this relation.

47 In the Gaussian units, the coefficient μ_0 in Eq. (119) should be replaced, as usual, with $4\pi/c$.

48 Named after Jean-Baptiste le Rond d'Alembert (1717-1783), who has made several pioneering contributions to the general theory of waves – see, e.g., CM Chapter 6. (Some older textbooks use notation $\square^2$ for this operator.)

49 In Gaussian units, this formula, as well as Eq. (131) for $G^{\alpha\beta}$, do not have the factors c in all the denominators.

50 To be fair, note that just as Eq. (127), Eq. (129) this is also a set of four scalar equations – in the latter case with the indices α, β, and γ taking any three different values of the set $\{0, 1, 2, 3\}$.

51 This term is very popular in the theory of particle scattering – see, e.g., CM Sec. 3.7.

52 In the next chapter, we will re-derive them in a different way.

53 It is straightforward to use Eq. (140) and the linear superposition principle to calculate, for example, the magnetic field of a string of charges moving along the same line and separated by equal distances $\Delta x = a$ (so that the average current, as measured in frame 0, is qu/a), and to show that the time-average of the magnetic field is given by the familiar Eq. (5.20) of magnetostatics, with b instead of ρ.