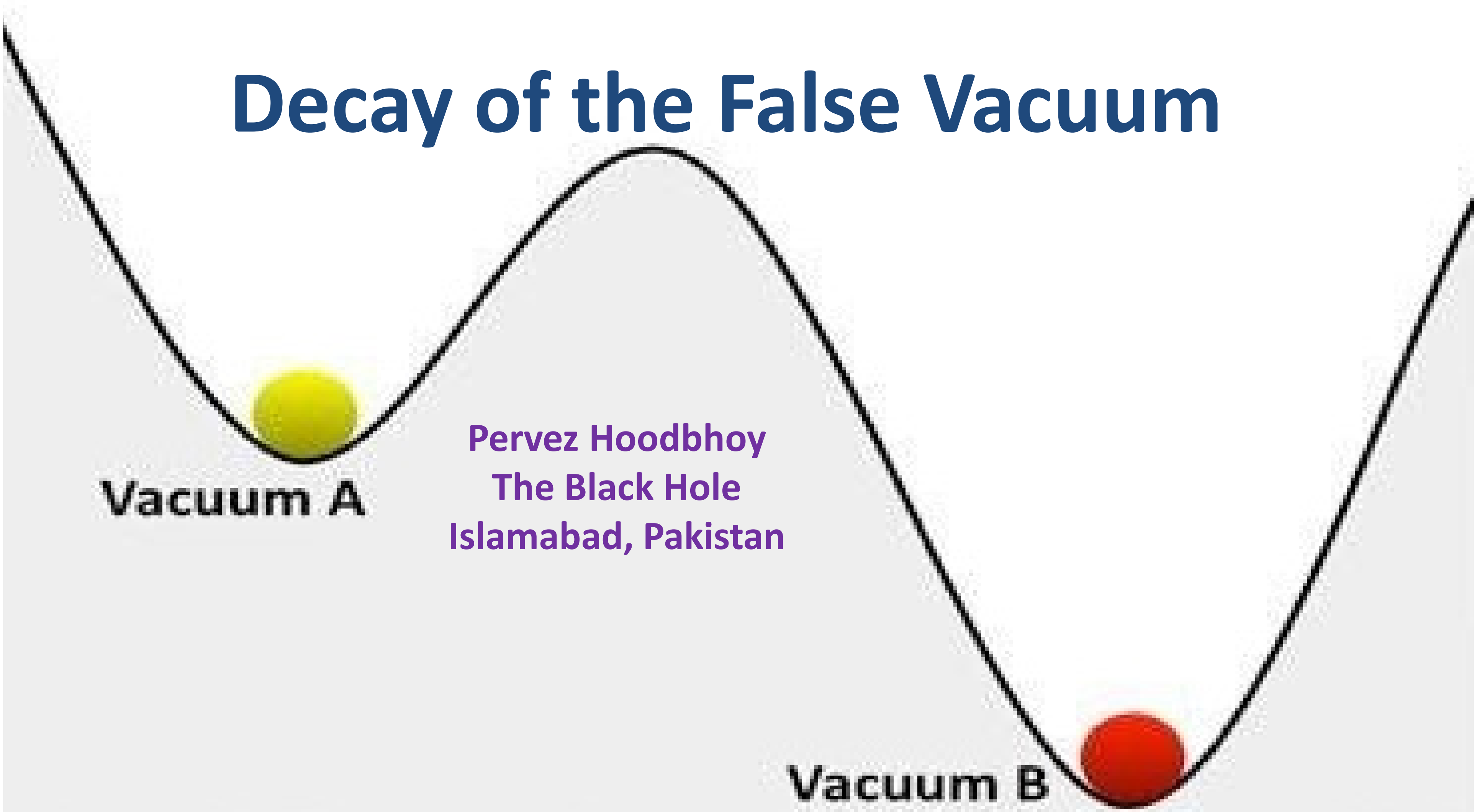
Decay of the False Vacuum



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Course Outline

Instantons in particle QM

- Intro to path integral ✓
- Imaginary time ✓
- Instantons in a symmetric double well ✓
- Decay of metastable states ✓
- The functional determinant ✓

Tunneling of quantum fields

- Basic QFT for a scalar field ✓
- Tunneling of field configurations ✓
- The O(4) instanton ✓
- How/when will the universe end? ✓
- Gauge fields and multiple vacuaa ✗
- Effective action

Instantons in Yang Mills gauge theories

Definition: An instanton is a configuration of a Yang-Mills field that solves the equation of motion and has finite action.

Review of differential forms

$A = A_\mu dx^\mu$ is called a one-form.

Under $x \to x'$, A doesn't change:

$$A = A_\mu dx^\mu = A_\mu \frac{dx^\mu}{dx'^\nu} dx'^\nu = A'_\mu dx'^\mu$$

$A = \frac{1}{2!} A_{\mu\nu} dx^\mu \wedge dx^\nu$ is called a two-form.

Antisymmetry under multiplication: $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$

Under $x \to x'$, A also doesn't change:

$$A = \frac{1}{2!} A_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2!} A'_{\mu\nu} dx'^\mu \wedge dx'^\nu$$

Review of differential forms

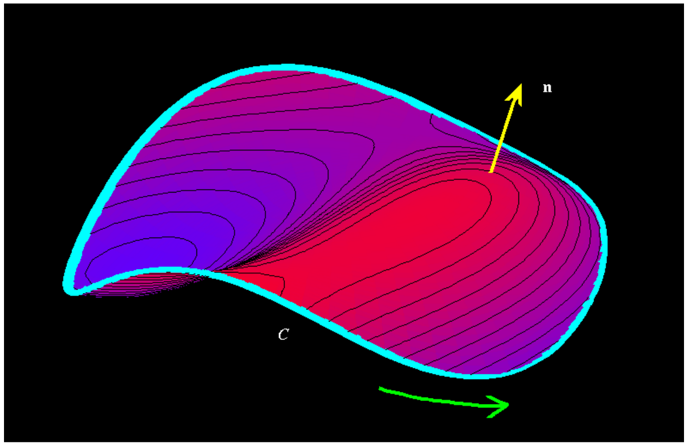
Define d (called the exterior derivative).

$$dA = \frac{1}{2!}\partial_\rho A_{\mu\nu}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$$ makes a 3-form if A is a 2-form

$$ddA = 0 \quad \Rightarrow \qquad d^2 = 0$$

Poincare Lemma: if $dH = 0$ then $H = dK$ locally.

Integration: $\int_M dA = \int_{\partial M} A$ $\quad A$ is a $(d-1)$ form



Constructing the 2-form for F

GOAL: Construct a two-form F that mirrors the $F_{\mu\nu}$ we already know from electromagnetism. We have in hand a one-form $A \equiv A_\mu dx^\mu$. How to make F?

Recall: $U = U(x)$ with $U^\dagger(x)U(x) = 1$ and $D_\mu\phi(x) = \partial_\mu\phi(x) - iA_\mu(x)\phi(x)$

Note: U is a zero-form so $dU = \partial_\mu U dx^\mu$ is a one-form.

Note: $d(U^\dagger U) = d(1) = 0 \rightarrow (dU^\dagger)U + U^\dagger dU = 0$

- Under $\phi(x) \rightarrow U\phi(x)$ we want $D_\mu\phi \rightarrow UD_\mu\phi$ and $(D^\mu\phi)^\dagger \rightarrow (D^\mu\phi^\dagger)U^\dagger$ so that, $(D_\mu\phi)^\dagger D^\mu\phi \rightarrow D_\mu\phi^\dagger U^\dagger U D^\mu\phi = D_\mu\phi^\dagger D^\mu\phi$
- For this to happen A_μ must transform as $A_\mu \rightarrow UA_\mu U^\dagger + iU\partial_\mu U^\dagger$. In the language of differential forms, $A_\mu dx^\mu \rightarrow UA_\mu U^\dagger dx^\mu + iU\partial_\mu U^\dagger dx^\mu$.

So, $A \rightarrow UAU^\dagger + UdU^\dagger$.

Convenient notation: hide the i by putting $-iA_\mu \equiv A_\mu$. This gives a nice, clean formula for the covariant derivative: $D_\mu = \partial_\mu + A_\mu$ or $D = d + A$.

Constructing the two-form F

Making F from $A \equiv A_\mu dx^\mu$. Next step?

First guess: make all the two-forms you can construct from A and add together.

- a) Look at $A^2 \equiv AA$: Under $A \to UAU^\dagger + UdU^\dagger$

$$A^2 \to (UAU^\dagger + UdU^\dagger)(UAU^\dagger + UdU^\dagger)$$
$$= UA^2U^\dagger + UAdU^\dagger + UdU^\dagger\, UAU^\dagger + UdU^\dagger\, UdU^\dagger$$

Use $(dU^\dagger)U + U^\dagger dU = 0$ $\quad = UA^2U^\dagger + UAdU^\dagger - dU\,AU^\dagger - dU\,dU^\dagger$

- b) Now look at dA and see how it transforms:
  Under $A \to UAU^\dagger + UdU^\dagger$, $dA \to d(UAU^\dagger + UdU^\dagger)$

$$= dU\,AU^\dagger + UdA\,U^\dagger + dU\; dU^\dagger + 0$$

- Define: $\boxed{F \equiv dA + AA = dA + A \wedge A}$ and see how it transforms:

$$F \to UdAU^\dagger + UAAU^\dagger = U(dA + AA)U^\dagger = UFU^\dagger \checkmark$$

Some properties of F $\quad F \equiv dA + AA$

- Remember that A is a one-form whereas $A^a_\mu(x)$ is a field that has $\mu = 1, \cdots 4$ and $a = 1, \cdots n^2 - 1$. The U's are $n \times n$ matrices of the group $SU(n)$.

- We can get back space-time indices whenever we want:
  $F = dA_\mu dx^\mu + A_\mu A_\nu dx^\mu dx^\nu = \partial_\nu A_\mu dx^\nu dx^\mu + A_\mu A_\nu dx^\mu dx^\nu \equiv \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$
  $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

- Restoring group indices: suppose we write $A_\mu = A^a_\mu t^a$ where t^a are the $SU(n)$ generators obeying $[t^a, t^b] = f^{abc} t^c$. Then, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$.

- If we choose: $A = g dg^\dagger$ where $g^\dagger g = g g^\dagger = 1$ then $F = 0$. Note: this choice of A is called a "pure gauge".
  Proof: $F = d(g dg^\dagger) + (g dg^\dagger)(g dg^\dagger) = dg dg^\dagger - g g^\dagger dg dg^\dagger = 0$

- In 4-d we can make an invariant from $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ and $F_{\mu\nu}$ and write it as tr F^2. Obviously d tr $F^2 = 0$.

Instantons in non-Abelian gauge theories

Recall from lecture 2 we sent $t \to -i\tau$ and so KE and PE terms in the action S carry the same sign.

$$S[A] = \int d\tau \int d^3x \frac{1}{2g^2} \text{tr } F^2 = \int d^4x \frac{1}{2g^2} \text{tr} F^2$$

- Since $d^4x = r^3 dr d\Omega$ we know that for $r \to \infty$ to keep S finite we must have $F \sim \frac{1}{r^2} \to 0$ or $A \to g d g^\dagger$.
- $\text{tr } F^2$ can be rewritten as $\text{tr } F^2 = d\, \text{tr}(AdA + \frac{2}{3}A^3)$

Important because we can then use Gauss's theorem

Proof: $\text{RHS} = \text{tr } d(AdA) + \text{tr } \frac{2}{3} dA^3$

$$= \text{tr } dAdA + \text{tr } \frac{2}{3}(dAA^2 - AdA^2)$$

$$= \text{tr } dAdA + \text{tr } \frac{2}{3}(dAA^2 - AdAA + A^2dA) = \text{tr } (dAdA + 2dAA^2)$$

$$\text{LHS} = \text{tr } (dA + A^2)(dA + A^2) = \text{tr } (dAdA + A^2dA + dAA^2 + A^4)$$

But $\text{tr } A^4 = \text{tr } A^3 A = -\text{tr } AA^3 = -\text{tr } A^4 \therefore \text{tr } A^4 = 0$

$$= \text{tr } (dAdA + 2dAA^2) = \text{tr } F^2$$ ✓

Instantons in non-Abelian gauge theories

$$\int_{R^4} \text{tr } F^2 = \int_{R^4} d\,\text{tr}(AdA + \tfrac{2}{3}A^3) = \text{tr}\int_{S^3}(AdA + \tfrac{2}{3}A^3) \quad = \text{tr}\int_{S^3}[A(F - A^2) + \tfrac{2}{3}A^3]$$

Now use $F = 0$ on S^3 as $r \to \infty$: $\quad = -\tfrac{1}{3}\text{tr}\int_{S^3} A^3$

Hence $\int_{R^4} \text{tr } F^2 = -\tfrac{1}{3}\text{tr}\int_{S^3}(gdg^\dagger)^3$ for any matrix g in $SU(n)$.

- $SU(2)$ example: $g = \tau\hat{1} + i\vec{x}\cdot\vec{\sigma} = \begin{pmatrix} \tau + ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & \tau - ix_3 \end{pmatrix}$ has det $g = 1$.

  For this $x_1^2 + x_2^2 + x_3^2 + \tau^2 = 1$. (The group manifold of $SU(2)$ is S_3.) Also, note: $g^\dagger g = 1$.

- What if we take g^n instead of n? This is also valid because det $g^n = 1$. Different n's correspond to different vacuum states of the same energy.

- What if we smoothly deform S_3 into an ellipsoid or something else? Answer: $\int \text{tr } F^2$ does not change. It is a topological invariant.

Issues for future discussion

- The EOM $D_\mu F_{\mu\nu} = 0$ is a highly non-linear and complicated pde. How to solve it?
- The imaginary time instanton connects the n vacuum to the $n+1$ vacuum. How?
- Yang-Mills theory has an infinite number of vacuaa. The true vacuum comes from taking an infinite superposition:

$$|\theta\rangle = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta}|\nu\rangle$$

What are the implications for solving QCD?

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