Questions for Module 42

3.9. Show that the operator which performs a transformation from the Z basis to the X basis has the following matrix representation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

This operator is also known as the Hadamard operator and is very useful in quantum computation.

Verify that this operator is Hermitian. Show that it can be expressed as a linear combination of the Pauli matrices.

3.11. Given a unit vector $\hat{\boldsymbol{e}} = (e_x, e_y, e_z)$ in an arbitrary direction, we can define the component of spin along $\hat{\boldsymbol{e}}$ by

$$\sigma_e = e_x \sigma_x + e_y \sigma_y + e_z \sigma_z.$$

(a) Show that $\sigma_e^2 = \mathbb{1}$.

(b) Find the eigenvalues and eigenvectors of σ_e.

3.12. Define a "vector matrix" $\vec{\boldsymbol{\sigma}} = \hat{\boldsymbol{\imath}}\sigma_x + \hat{\boldsymbol{\jmath}}\sigma_y + \hat{\boldsymbol{k}}\sigma_z$. Show that

$$(\vec{\boldsymbol{a}}. \cdot \vec{\boldsymbol{\sigma}})(\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{\sigma}}) = (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}})\mathbb{1} + i(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \cdot \vec{\boldsymbol{\sigma}} \qquad (3.34)$$

for vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$.

3.13. Find the expectation value of σ_e in the state $|0\rangle$. Generalize this result to find the expectation value of σ_e in a state $|\hat{\boldsymbol{f}}+\rangle$ where $\hat{\boldsymbol{f}}$ is a general direction making angle θ with the $\hat{\boldsymbol{z}}$ axis.

4.1. Find out what the action of each of the σ_i operators is on the Bloch sphere by checking their effects on the eigenvectors $|Z\pm\rangle, |X\pm\rangle$ and $|Y\pm\rangle$.

4.2. Prove that the Bell states are mutually orthogonal and that they form a basis for $\mathcal{H}^2$. You must be able to express an arbitrary 2-qubit state $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ as a linear superposition of the Bell states. Find the coefficients in this superposition in terms of $a, b, c,$ and d.

4.3. Entanglement and basis change: suppose $|s_1\rangle$ and $|s_2\rangle$, linear combinations of the basis states $|0\rangle$ and $|1\rangle$ form an orthonormal basis for a spin Hilbert space. Show that the two-spin entangled "singlet" state

$$\frac{1}{\sqrt{2}}(|s_1\rangle \otimes |s_2\rangle - |s_2\rangle \otimes |s_1\rangle)$$

is equivalent to

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Check that this preservation of the *form* of entanglement does not hold for the other three Bell states in the transformed basis.

4.4. We found the directions $\hat{\boldsymbol{a}}, \hat{\boldsymbol{a}}', \hat{\boldsymbol{b}},$ and $\hat{\boldsymbol{b}}'$ of Stern–Gerlach machines for which the CHSH inequality is maximally violated for spin half particles. Translate this experiment to photon polarization measurements and find the corresponding directions for the axes of polarizers used by Alice and Bob that would maximally violate the CHSH inequality.