10 Stellar Structure

In this chapter we cover the main equations describing the structure of a star. Furthermore, we trace the evolution of stars in the Hertzsprung-Russell diagram. The Voigt-Russell theorem states that the total stellar evolution is determined by the initial mass and the chemical composition.

10.1 Basic Physical Laws of Stellar Structure

Only a few equations are needed to describe the internal structure of a star. We start from the simplification that all physical parameters depend only on the distance r from the stellar center. So, for example, we can describe the temperature inside the star by the simple function $T(r)$. We therefore think of stars as homogeneous, isotropic, unflattened spheres of gas.

10.1.1 Hydrostatic Equilibrium

In a stable star gravity is balanced by the internal pressure:

> Gravity (acting inward) = internal pressure (acting outward).

This state of equilibrium is called *hydrostatic equilibrium.*

A. Hanslmeier, *Introduction to Astronomy and Astrophysics*,
https://doi.org/10.1007/978-3-662-64637-3_10

What happens if this equilibrium is not fulfilled? There are the two extreme cases:

- If the internal pressure falls away, the star collapses immediately as a result of gravity,
- without gravity, the star would expand.

We consider the simplest model of a star:

- Spherically symmetric gas sphere,
- homogeneous structure,
- the model should be static,
- no rotation,
- no magnetic fields.

From this follows: All physical parameters f depend only on the center distance r, thus become $f(r)$.

We briefly examine a case where such a simplification is not justified. The star *Vega* $(\alpha\, Lyr)$ is an example of a rapidly rotating star: $P_{\text{rot}} = 12.5$ Hours. Because of this rapid rotation, the temperature on Vega is different: at the equator, about 7600 K, at the poles about 10, 000 K. The star is oblate, and the poles are closer to the stellar center and therefore hotter.

The theorem of *Zeipel* states

$$T_{\text{eff}} \propto g_{\text{eff}}^{1/4} \tag{10.1}$$

where g_{eff} is the effective gravitational acceleration, i.e. the actual acceleration reduced by the effect of the centrifugal force .

Let r be the distance from the stellar center. Consider a thin shell of mass of thickness dr at the position r in the stellar interior. The mass per unit area is ρdr, the weight $-g\rho dr$. The weight is the inward gravitational force. The outward pressure is equal to the difference between the pressure P_i of the side of the mass shell facing the center and the pressure P_e:

$$P_i - P_e = -\frac{\partial P}{\partial r} dr \tag{10.2}$$

Thus we have (Fig. 10.1):

$$\frac{\partial P}{\partial r} = -g\rho \tag{10.3}$$

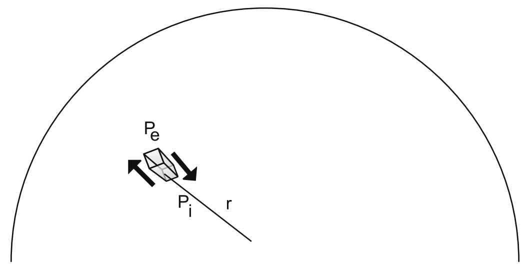


Fig. 10.1 Hydrostatic equilibrium

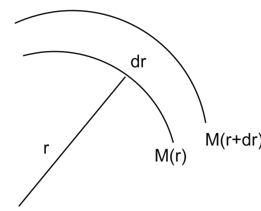


Fig. 10.2 On the derivation of the mass continuity equation

Let us set for $g = GM(r)/r^2$, then the condition for the *hydrostatic equilibrium* (Eulerian form):

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{10.4}$$

One can use instead of r is also the mass m inside the sphere with radius r as an independent variable, and one obtains then the Lagrangian Form:

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \tag{10.5}$$

where $m = M(r)$.

Within a shell of thickness dr the mass is $dM(r)$ (Fig. 10.2):

$$\frac{dM}{dr} = 4\pi r^2 \rho(r) \tag{10.6}$$

This is known as *Mass continuity equation* and represents our second fundamental equation describing stellar structure. If R is the radius of the star, then the total mass is:

$$M = 4\pi \int_{r=0}^{r=R} \rho(r) r^2 dr \tag{10.7}$$

In Lagrangian form, the equation is:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{10.8}$$

We estimate the central pressure P_c for the Sun. $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$; $M_\odot = 1.989 \times 10^{30}\,\mathrm{kg}$, $R_\odot = 6.96 \times 10^8\,\mathrm{m}$. Equation 10.4 can be written approximately as:

$$\frac{dP}{dr} \approx \frac{P_{\text{surface}} - P_c}{R}$$

and the pressure at the surface is $P_{\text{surface}} \approx 0$.

From this, the mean density of the Sun follows to:

$$< \rho_\odot > = 3M_\odot/4\pi R_\odot^3 = 1410\,\mathrm{kg/m^3}$$

Let's set $r = R_\odot$ and $M(r) = M_\odot$ in the two basic equations, then we get:

$$P_c \approx GM_\odot < \rho_\odot > /R_\odot \approx 10^{14}\mathrm{N/m^2}$$

Since $1\,\mathrm{atm} = 1.01 \times 10^5\,\mathrm{N/m^2}$, we find $P_c = 10^9\,\mathrm{atm}$. Since the actual central density of the Sun is greater, the central pressure must also be greater.

10.1.2 Equation of Motion with Spherical Symmetry

We again consider a thin shell of mass with *dm* at the distance *r* from the center. The force f_P per unit area results from *Pressure gradients:*

$$f_P = -\frac{\partial P}{\partial m} dm$$

The gravitational force per unit area is :

$$f_g = -\frac{g dm}{4\pi r^2} = -\frac{Gm}{r^2}\frac{dm}{4\pi r^2}$$

Now, if the sum of both forces is not zero, then there is an acceleration of the mass shell:

$$\frac{dm}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = f_P + f_g$$

and by substituting and dividing by dm:

$$\frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = -\frac{\partial P}{\partial m} - \frac{Gm}{4\pi r^4} \tag{10.9}$$

- If only pressure gradient → Outward acceleration , $\partial P/\partial m < 0$.
- If only gravity → Collapse.

From 10.9 hydrostatic equilibrium emerges if $\partial^2 r/\partial t^2 = 0$.

Now we investigate deviations from hydrostatic equilibrium:

- We assume the pressure vanishes. This gives us the so-called *free fall time* τ_{ff} . From the equations

$$\frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{4\pi r^4} \qquad g = Gm/r^2 \qquad |\partial^2 r/\partial t^2| \approx R/\tau_{\mathrm{ff}}^2$$

we get the *free fall time*:

$$\tau_{\mathrm{ff}} \approx \sqrt{\frac{R}{g\rho}} \tag{10.10}$$

- *Explosion time:* Here gravity is turned off, and we get:

$$\tau_{\mathrm{expl}} \approx R\sqrt{\frac{\rho}{P}} \tag{10.11}$$

The speed of sound is the typical speed at which disturbances propagate in the stellar interior. The *speed of sound* is given by:

$$v_c \approx \sqrt{\frac{P}{\rho}} \tag{10.12}$$

Therefore τ_{expl} is of the same order of magnitude as it takes a sound wave to travel from the stellar center to the stellar surface (given by $\tau_s \approx R/v_c$).

In general, the formula for calculating the speed of sound in an ideal gas with an adiabatic exponent κ and molar mass M is:

$$c = \sqrt{\kappa\frac{P}{\rho}} = \sqrt{\kappa\frac{\mathfrak{R}T}{M}} \tag{10.13}$$

Let us make a rough estimate of the speed of sound at the surface of the Sun, $T = 6 \times 10^3$ K. For the molar mass, we set the value 0.029 kg/mol (actually the value for air), $\kappa = 1.4$, gas constant $\mathfrak{R} = 8.31\,\mathrm{J\,mol^{-1}K^{-1}}$. We find 1500 km/s, which is very close to reality.

- The *hydrostatic time scale* is obtained by equating: $\tau_{\rm ff} = \tau_{\rm expl}$:

$$\tau_{\rm hydr} = \sqrt{\frac{R^3}{GM}} \approx \frac{1}{2}\frac{1}{\sqrt{G\bar{\rho}}} \tag{10.14}$$

For our sun $\tau_{\rm hydr} = 27$ min. For red giants with $M \approx M_\odot$, $R \approx 100\,R_\odot$ the hydrostatic time scale become much longer: $\tau_{\rm hydr} = 18$ d. For a compact White dwarf on the other hand with $M \approx M_\odot$, $R = R_\odot/50$ we get a very short time scale of $\tau_{\rm hydr} = 4.5$ s.

Since in most cases the stars change on timescales that are very large compared to $\tau_{\rm hydr}$ the assumption of a hydrostatic equilibrium is justified.

10.1.3 General Relativity

Effects of general relativity become important in the case of very strong gravitational fields: e.g. in the case of neutron stars. We will only sketch the derivation.

First, one starts from Einstein's field equations which shows the relation between matter (given by the so-called energy-momentum-tensor T_{ik}) and space curvature (given by the Ricci tensor R_{ik}) and the metric tensor g_{ik} which describes the distance between two points in space:

$$R_{ik} - \frac{1}{2}g_{ik}R = \frac{\kappa}{c^2}T_{ik} \qquad \kappa = 8\pi G/c^2 \tag{10.15}$$

Einstein's field equations:
Space-time geometry is on the left, energy/matter distribution is on the right.
Matter → Space curvature.

The metric tensor follows from the line element ds^2, which describes the distance between two points in space.

Example for line element in Euclidean space

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{10.16}$$

Since the line element follows in general from

$$ds^2 = g_{ik}dx_i dx_k \tag{10.17}$$

one has in the Euclidean case:

all $g_{ii} = 1$, and $g_{ij} = 0, if i \neq j$.

R is the curvature scalar and follows from R_{ik}. For an ideal gas the components of the energy-momentum tensor are

$$T_{00} = \rho c^2; \;\; T_{11} = T_{22} = T_{33} = P \tag{10.18}$$

Let us assume for the line element in polar coordinates:

$$ds^2 = \mathrm{e}^{\nu} c^2 dt^2 - \mathrm{e}^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{10.19}$$

From the line element we read the components of the metric tensor and from its derivatives one can calcclate the Ricci tensor. After a long calculation we find the *Tolman-Oppenheimer-Volkoff (TOV)* equation for hydrostatic equilibrium in general relativity:

$$\frac{\partial P}{\partial r} = -\frac{Gm}{r^2}\rho\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 P}{mc^2}\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1} \tag{10.20}$$

If we consider not too strong gravitational fields, then one can derive as approximation: One keeps in the expansion only terms which are linear $1/c^2$ are:

$$\frac{\partial P}{\partial r} = -\frac{Gm}{r^2}\rho\left(1 + \frac{P}{\rho c^2} + \frac{4\pi r^3 P}{mc^2} + \frac{2Gm}{rc^2}\right) \tag{10.21}$$

This is the *Post-Newtonian approximation.*

Fundamental to stable stars is the equation for hydrostatic equilibrium.

10.1.4 Equation of State

Our set of equations to describe stellar structure needs to be complemented by an equation of state. Let us suppose that the gas in a star satisfies the laws for an *ideal gas:*

$$P(r) = n(r)kT(r) \tag{10.22}$$

The pressure $P(r)$ thus depends on the particle density $n(r)$ (number of particles per m^3). k is the Boltzmann constant $k = 1.381 \times 10^{-23}$ J/K. One can also write:

$$n(r) = \frac{\rho(r)}{\mu(r) m_{\mathrm{H}}}, \tag{10.23}$$

where $m_\mathrm{H} = 1.67 \times 10^{-27}$ kg is the mass of the hydrogen atom. The equation of state of ideal gases is valid in the interior of the star as long as the interaction of neighbouring particles is small compared to their thermal (= kinetic) energy. The *Molecular weight* μ is equal to the atomic weight divided by the number of all particles (nucleus + electrons). If we assume complete ionization, then the molecular weight μ becomes for :

- Hydrogen: number of particles 2 (one proton, p, and one electron, e$^-$), therefore $\mu_\mathrm{H} = 1/2$.
- For Helium: 3 particles ($2e^-$, nucleus), atomic weight = 4 (since in the nucleus 2 p and 2 n). Hence $\mu_\mathrm{He} = 4/3$.

Very often one denotes with X the fraction of hydrogen, Y the fraction of helium, and Z the fraction of elements heavier than helium (such elements are often called metals in astrophysics). The average molecular weight is then :

$$\mu = [2X + (3/4)Y + (1/2)Z]^{-1} \approx 1/2 \tag{10.24}$$

We therefore get following *equation of state of ideal gases:*

$$P(r) = \rho(r)kT(r)/\mu(r)m_\mathrm{H} \tag{10.25}$$

In the case of massive stars, the gas pressure is supplemented by the *Radiation pressure* (by momentum transfer of the photons):

$$P_\mathrm{rad}(r) = \frac{a}{3}T^4(r) \tag{10.26}$$

$a = 7.564 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^4}$.

We estimate the central temperature of the sun. Using the values P_c and $< \rho_\odot >$:

$$T_c \approx \frac{P_c \mu m_\mathrm{H}}{< \rho_\odot > k} = 12 \times 10^6\,\mathrm{K}$$

Modern computer models provide a central temperature of 14.7 million Kelvin. At such high temperatures, the gas behaves like a *plasma.* It consists of ions and electrons and is, on the whole neutral.

The total pressure is then:

$$P = P_g + P_\mathrm{rad} \tag{10.27}$$

Consider a gas in a volume dV, which is completely ionized by pressure. n_e let be the number of free electrons. The velocity distribution of the electrons is given by a *Boltzmann* *distribution*, their mean kinetic energy is:

$$\bar{E}_{\text{kin}} = \frac{3}{2}kT \tag{10.28}$$

If (p_x, p_y, p_z) are the coordinates in momentum space, then:

$$f(p)dpdV = n_e \frac{4\pi p^2}{(2\pi m_e kT)^{3/2}} \exp\left(-\frac{p^2}{2m_e kT}\right) dpdV \tag{10.29}$$

Let us now assume that, n_e remain constant and T decreases. Then the maximum of the distribution function ($p_{\text{max}} = (2m_e kT)^{1/2}$) shifts to smaller values of p, and the maximum $f(p)$ becomes larger, since $n_e = \int_0^\infty f(p)dp$.

10.1.5 Degeneracy

The *Fermions* include particles with half-integer spin, such as electrons, but also other elementary particles such as quarks and nucleons (protons, neutrons). For these particles the *Pauli principle states:*

Each quantum cell of a 6-dimensional phase space.

$$(x, y, z, p_x, p_y, p_z) \tag{10.30}$$

must not contain more than two fermions, in our case electrons.

The volume of such a quantum cell is:

$$h^3 = dp_x dp_y dp_z dV$$

So if we consider a shell $[p, p + dp]$ in momentum space, then there are $4\pi p^2 dV/h^3$ quantum cells that contain no more than $8\pi p^2 dpdV/h^3$ electrons; therefore from quantum mechanics follows the condition:

$$f(p)dpdV \leq 8\pi p^2 dpdV/h^3$$

The state in which all electrons have the lowest energy without violating the *Pauli exclusion principle* is that in which all phase space cells up to momentum p_F are occupied by two electrons; all other phase space cells $p > p_F$ are empty:

$$f(p) = \frac{8\pi p^2}{h^3} \qquad p \leq p_F \tag{10.31}$$

$$f(p) = 0 \qquad p > p_F \tag{10.32}$$

From this then to be derived:

$$n_e dV = dV \int_0^{p_F} \frac{8\pi p^2 dp}{h^3} = \frac{8\pi}{3\, h^3} p_F^3 dV \tag{10.33}$$

Thus, according to the Pauli exclusion principle, no more than two fermions differing in spin quantum number can occupy the same energy state. This is also called gas degeneracy. Because of the much lower mass, degeneracy occurs first in electrons. Thus, one may have the case where the electron gas is already fully degenerate, but the ion gas is not yet. In the case of degeneracy, the equation of state changes. In the case of complete degeneracy one distinguishes between

- *non-relativistic degeneracy,* $\rho < 2 \times 10^6\,\mathrm{g\,cm^{-3}}$,

$$P = K_1 \rho^{5/3} \tag{10.34}$$

  and

- *relativistic degeneracy,* $\rho > 2 \times 10^6\,\mathrm{g\,cm^{-3}}$:

$$P = K_2 \rho^{4/3} \tag{10.35}$$

K_1, K_2 depend on the chemical composition.

> In the case of degenerate stellar matter, the density now depends only on pressure and no longer also on temperature.

Degeneracy can be expected for certain stars and in some cases electrons in other cases neutrons are degenerated: :

- Degenerate electrons: Red giants, white dwarfs.
- Degenerate neutrons degenerate electrons: Neutron Stars.

Degeneracy occurs at very high densities. Repulsion between electrons (neutrons) is a consequence of quantum mechanics (Pauli exclusion principle) and not electrical repulsion. In the case of degeneracy, the equilibrium condition is: Gravity = degenerate pressure. If the star receives more matter (through accretion), gravity increases; however, the degenerate pressure increases only slightly, and therefore the star shrinks.

The greater the mass of a degenerate star, the smaller its volume and therefore its radius.

10.1.6 Summary: Equation of State

We summarize for which physical parameters one has to calculate with which equation of state:

- Photon gas—Radiation pressure: . $\rho < 3.0 \times 10^{-23}\mu T^3$; pressure $P = 2.521 \times 10^{-15}T^4$.
- Ideal nondegenerate gas:
  $3.0 \times 10^{-23}\mu T^3 < \rho < 2.4 \times 10^{-8}\mu_e T^{3/2}$; pressure $P = 8.31 \times 10^7 \rho T/\mu$.
- Non-relativistic fully degenerate electron gas:
  $2.4 \times 10^{-8}\mu_e T^{3/2} < \rho < 7.3 \times 10^6\mu_e$; Print $P = 1.004 \times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3}$
- Relativistic fully degenerate electron gas (white dwarfs):
  $7.3 \times 10^6\mu_e < \rho \leq 10^{11}$; Print $P = 1.244 \times 10^{15}\left(\frac{\rho}{\mu_e}\right)^{4/3}$.
- Degenerate neutron gas:
  $10^{11} \leq \rho \leq 10^{14}$; Pressure $P \approx 10^{10}(\rho)^{5/3}$.

10.2 Energy Transport

How is energy transferred inside a star? In principle, energy transport is possible by energy transport:

- thermal conduction,
- radiation,
- convection.

Heat conduction occurs through collisions. This type of energy transport works very well in solids (especially in metals); but less well in gases because their thermal conductivity is low.

10.2.1 Convection

Consider in Fig. 10.3 a bubble of gas that is supposed to move upward inside a star due to a random disturbance. As this travels a distance dr, its temperature changes from T_1 to T_2.

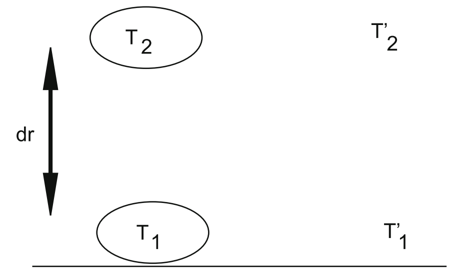


Fig. 10.3 Convection in a star. If the temperature change $(T_2 - T_1)$ of an upward moving gas bubble is smaller than that of the surrounding $(T_2' - T_1')$, it is lighter and rises further upwards, and one has convection

We assume here that during the ascent of the gas bubble there is no heat exchange with the environment. Then the temperature change of the gas bubble can be given by the *adiabatic temperature gradient:*

$$\left.\frac{dT}{dr}\right|_{\text{ad}} \tag{10.36}$$

The surrounding matter changes its temperature from T_1' to T_2', and the temperature gradient is to be described by :

$$\left.\frac{dT}{dr}\right|_{\text{rad}} \tag{10.37}$$

that is, by the radiation gradient. For the occurrence of convection in a star therefore applies the *Schwarzschild criterion.* If the temperature change of a volume element moving upwards due to a random perturbation is $|dT/dr|_{\text{ad}}$ and if the temperature gradient of the environment is equal to $|dT/dr|_{\text{rad}}$ then no convection occurs if holds:

$$\left|\frac{dT}{dr}\right|_{\text{rad}} < \left|\frac{dT}{dr}\right|_{\text{ad}} \tag{10.38}$$

In stars, convection occurs in different regions:

- Massive stars: the core is convective, the envelope is in equilibrium (Fig. 10.4, right). This is related to the extreme temperature dependence of energy production in the core, which implies a high radiation gradient. Convection in the core results in better mixing of the elements.
- Low-mass, cooler stars (e.g., the Sun): the core is in equilibrium, and the envelope becomes convective (Fig. 10.4, center). This is explained by the increasing number of layers from the surface inwards, in which elements such as hydrogen or helium are ionized, and therefore reduce the adiabatic gradient. In the case of the Sun, convection begins about 200,000 km below its surface.

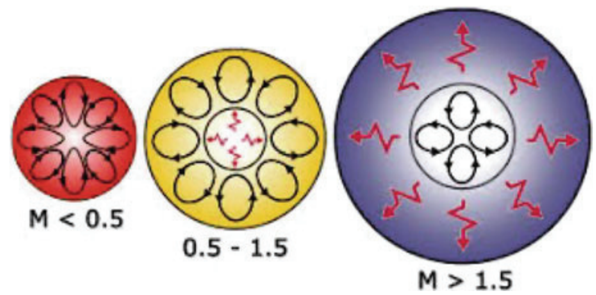


Fig. 10.4 Convection in stars; in stars with more than 1.5 solar masses ($M > 1.5$) convection occurs in the core region

The cooler the stars, the deeper the convection zone reaches! Stars with less than 0.5 solar masses are fully convective.

In the Centre of the Sun the temperature is about 15 million Kelvin, and at the surface it is about 6000 Kelvin. In the radiative zone of the Sun, energy transport is by radiation. The formula for this can be derived by assuming a diffusion approximation. The free path length of the Photons l_{Ph} is given by:

$$l_{\mathrm{Ph}} = \frac{1}{\kappa\rho} \tag{10.39}$$

It is only a few centimeters, so the diffusion approximation is a reasonable assumption ($l_{\mathrm{Ph}} \ll R_{\odot}$).

The following estimate shows how long it takes for a photon to travel from the interior of the Sun to the surface and be emitted: The total solar radius is made up of all the partial distances $\mathbf{l_i}$, therefore:

$$\mathbf{L} = \sum_i \mathbf{l_i} = \mathbf{l_1} + \mathbf{l_2} + \ldots$$

Since the partial distances are vectors pointing in arbitrary directions, holds:

$$< L^2 > = < l_1^2 > + < l_2^2 > + \ldots + < L_N^2 >$$

therefore we get:

$$< L^2 > = N < l^2 >$$

Now $l \approx 1\,\mathrm{cm}$, $< L >= R_{\odot} = 7 \times 10^{10}\,\mathrm{cm}$ and $< L^2 > \approx 10^{22}\,\mathrm{cm}$ therefore $N \approx 10^{22}$ and from $t = < L^2 > /c = 3 \times 10^{12}\,\mathrm{s} = 10^5$ years.

Therefore, we now see photons (i.e., radiated energy) from the Sun produced by nuclear fusion in its interior about 100,000 years ago!

Let us consider diffusion in general: a concentration n of particles is said to depend on r. Let the mean free path length be l and its mean velocity v. Let j describe the diffusion flux of particles from sites of high concentration to sites of low concentration:

$$j = -\frac{1}{3} v l \frac{dn}{dr} \tag{10.40}$$

Now we put into this equation the parameters that describe our radiation field:

1. $n \to u = aT^4$, radiation density,
2. $v \to c$,
3. $l \to l_{\mathrm{Ph}}$,
4. $j \to L_r/4\pi r^2$, radiation flux.

One can immediately see $dn/dr \to du/dr = 4aT^3 dT/dr$; $a = 4\sigma/c$, and resolved by the temperature gradient gives the diffusion approximation:

$$\frac{dT}{dr} = -\frac{3}{64\pi\sigma} \frac{\kappa \rho L_r}{r^2 T^3} \tag{10.41}$$

One can also derive this equation by the following reasoning: Take a thin shell, where the radiative flux is given by $F(r) = \sigma T^4(r)$. At the point $r + dr$ one has a temperature $T + dT$ and the flux is $F + dF = \sigma(T + dT)^4 \approx \sigma(T^4 + 4T^3 dT)$. dT is negative because the shell of mass is cooler on the outside than on the inside. The flux absorbed inside the shell is then:

$$dF = 4\sigma T^3(r) dT \tag{10.42}$$

This absorption comes from the *Opacity* (it describes quasi the transparency) of the stellar material:

$$dF = -\kappa(r)\rho(r)F(r)dr \tag{10.43}$$

On the other hand, the luminosity of the star is :

$$L(r) = 4\pi r^2 F(r) \tag{10.44}$$

and therefore

$$L(r) = -[16\pi\sigma r^2 T^3(r)/\kappa(r)\rho(r)](dT/dr) \tag{10.45}$$

An exact treatment still gives the factor 4/3:

$$L(r) = -\frac{64\pi\sigma r^2 T^3(r)}{3\kappa(r)\rho(r)}\frac{dT}{dr} \tag{10.46}$$

If one has a high opacity, then convection becomes dominant. The gradient is then:

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right)\frac{T(r)}{P(r)}\frac{dP}{dr} \tag{10.47}$$

where $\gamma = c_p/c_v$ where c_p, c_V are the specific heat capacities at constant pressure and constant volume, respectively.

Let us determine the luminosity of the Sun from radiative transfer. For this we approximate $dT/dr \to -T_c/R_\odot$ and thus find the gradient -2×10^{-2} K/m. In the above equation we then set $r \to R_\odot$, $T(r) \to T_c$, $\rho(r) \to \rho_\odot$:

$$L_\odot = \frac{9.5 \times 10^{29}}{\kappa} \text{ J/s}$$

Here we still have to define a suitable value for the opacity κ. κ is the effective cross-section per gas particle multiplied by the number of particles in 1 kg. A mass of 1 kg of completely ionized hydrogen contains 6×10^{26} protons and as many electrons. For electron scattering, the effective cross section is 10^{-30} m^2 for photoionization of hydrogen 10^{-20} m^2; photoionization as a source of opacity predominates in the solar interior, one has roughly:

$$10^{-3} \ll \kappa \leq 10^7$$

Thus our estimate is:

$$10^{22} \leq L_\odot \ll 10^{32} \text{ J/s}$$

The mean value of 10^{27} J/s fits well to the measured value of 3.9×10^{26} J/s. This corresponds to an opacity of 2.4×10^3.

10.2.2 Opacity

The Opacity is a measure of the absorptivity in the stellar matter and therefore essential for energy transport.

It is composed of several components, which will be treated briefly.

Electron Scattering Once an electromagnetic wave passes an electron, the electron begins to oscillate and radiates. So the original radiation is attenuated, energy is transferred to the oscillating electron, and there is → Absorption.

The equation of motion of an electron subjected to an electric field $\mathbf{E}$ is:

$$m_e \left(\frac{d^2\mathbf{x}}{dt^2} + \gamma \frac{d\mathbf{x}}{dt} + \omega_0 \mathbf{x}^2 \right) = -e\mathbf{E} \tag{10.48}$$

γ is the damping constant. Electric field → excites electrons to oscillate.

Two limiting cases are obtained as solutions:

- $\omega \gg \omega_0, \gamma \rightarrow$ Electron moves like a free electron, *Thomson effective cross section* σ_T

$$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2} \right)^2 \tag{10.49}$$

- $\omega_0 \gg \omega, \gamma$: *Rayleigh scattering*

$$\sigma_R = \sigma_T \left(\frac{\omega}{\omega_0} \right)^4 \tag{10.50}$$

so scattering $\propto \omega^4$ or $\propto \lambda^{-4}$. In the Earth's atmosphere blue light is scattered more than red light. When starlight passes through an interstellar cloud, it becomes reddened.

The Thomson effective cross section is obtained as $\sigma_T = 6.65 \times 10^{-29}\,\mathrm{m}^2$, and when there are n_e electrons in the unit volume, then the absorption coefficient is:

$$n_e \sigma_T \tag{10.51}$$

It is found:

$$\kappa_\nu = 0.20(1 + X) \tag{10.52}$$

This *Thomson scattering* is independent of frequency, X is the fraction of hydrogen. This treatment neglects the momentum exchange between electrons and radiation, but this only becomes effective at very high temperatures. The momentum of the photons is $h\nu/c$, this is then partially transferred to the electrons after scattering $m_e v \approx h\nu/c$. The relativistic correction *(Compton scattering)* is effective when $v_e \approx 0.1c$.

Free-Free Transitions The electron are in thermal motion due to the temperatures. If such an electron now passes an ion, then the two charged particles can absorb radiation. The thermal velocity of the free electrons is $v \approx T^{1/2}$, and the time during which they absorb or emit, $\propto 1/v \approx T^{-1/2}$. *Kramers* derived the following relation:

$$\kappa_\nu \propto Z^2 \rho T^{-1/2} \nu^{-3} \tag{10.53}$$

Using Rosseland's mean, one obtains:

$$\kappa_{\text{ff}} = \rho T^{-7/2} \tag{10.54}$$

If one has a fully ionized gas (stellar center), then it holds *(Kramers):*

$$\kappa_{\text{ff}} = 3.8 \times 10^{22}(1+X)[(X+Y)+B]\rho T^{-7/2} \text{ cm}^2\,\text{g}^{-1}, \tag{10.55}$$

where $B = \sum_i X_i Z_i^2/A_i$ and A_i the atomic Mass numbers are.

Bound-Free Transitions Let us first consider bound-free transitions of a neutral hydrogen atom. In the ground state, the ionization energy is χ_0 ; it is ionized by a photon of energy $h\nu > \chi_0$. It follows, then:

$$h\nu = \chi_0 + \frac{1}{2} m_e v^2, \tag{10.56}$$

where v is the velocity of the released electron. If a_ν is the absorption coefficient per ion, $a_\nu = \kappa_\nu \rho / n_{\text{ion}}$ then $a_\nu = 0,\ \nu < \chi_0/h$ and $a_\nu > 0,\ \nu \geq \chi_0/h$. One obtains $a_\nu \propto \nu^{-3}$ for $\nu \geq \chi_0/h$. The so-called Gaunt *factor* is a quantum mechanical correction and occurs when the problem is treated exactly. Analogously, the matter continues for the first excited state, $a_\nu = 0,\ h\nu < \chi_1$ and $a_\nu \propto \nu^{-3}$ for $h\nu \geq \chi_1$ where χ_1 is the energy required to ionize a hydrogen atom from the first excited state. This is why the jagged shape of the absorption coefficient occurs.

Heat Conduction Like all particles, electrons can also transport energy by thermal conduction. Normally, their contribution to the total energy transport is negligible. The thermal conduction is proportional to the mean free path length l and in the non-degenerate case $l_{\text{photon}} \gg l_{\text{particle}}$.

Heat conduction becomes important for degenerate stellar material, i.e. in the interiors of far evolved stars as well as in white dwarfs. Here all quantum cells below the Fermi momentum p_F are occupied.

10.3 Energy Sources

Our sun has been radiating with almost unchanged luminosity for about 4.5 billion years. The question arose where this energy comes from. We will first cover classical sources of energy, such as the gravitational energy released during contraction, then nuclear fusion, which is the source of energy for most of a star's evolution.

A star continuously radiates energy. Therefore, stellar models are not static in the strict sense. Stars evolve. Let $\epsilon(r)$ be the rate of energy production related to the unit mass (J/s kg). In a strict sense ϵ also depends on *T*, *P* and the density, but for simplicity we write $\epsilon(r)$. For stellar structure, we assume : $\epsilon = 0$, except in the central regions where the energy is produced by thermonuclear fusion.

For the Sun, we obtain:

$$\epsilon_\odot \approx L_\odot / M_\odot = 2.0 \times 10^{-7}\,\mathrm{J/s\,kg} \tag{10.57}$$

Inside a shell *dr* changes the luminosity by:

$$dL = 4\pi r^2 \rho(r)\epsilon(r) dr \tag{10.58}$$

In the following, we consider the various ways in which energy is generated.

Energy is released when the star contracts. Analogy: consider a rock falling to earth → Gravitational energy is converted into kinetic energy.

Virial Theorem

When a star slowly contracts, gravitational energy is released

- half of which heats the star,
- the other half is radiated.

Suppose we add the fraction *dM*(*r*) to a mass *M*(*r*), then the change in gravitational energy is:

$$dU = -\frac{GM(r)dM(r)}{r} \tag{10.59}$$

Let us integrate over all mass shells:

$$U = -\int_0^M G\frac{M(r)dM(r)}{r} = -q(GM^2/R) \tag{10.60}$$

Here q depends on the mass distribution in the sphere. If one has a uniform density, then $q = 3/5$. For most main sequence stars we can use $q = 1.5$.

Let us do an exercise. How long can the sun shine by contraction?

We calculate with

$$E \approx \frac{GM^2}{R} = \frac{6.67 \times 10^{-11}(2 \times 10^{30})^2}{7 \times 10^8} = 4 \times 10^{41}\ \mathrm{J}$$

By comparison with Eq. 10.57: The Sun can only radiate at present luminosity for about 30 million years by releasing gravitational energy.

10.3.1 Thermonuclear Energy Production

Physical Preconditions

Prior to nuclear fusion process, the mass of the i nuclei involved would be $\sum M_i$. After fusion the resulting nucleus has a total mass $\sum M_p$. The mass of the fused nuclei is less than the mass of the original nuclei, and the missing amount, *Mass defect,* ΔM, is

$$\Delta M = \sum_i M_i - M_p \tag{10.61}$$

This missing mass is converted into energy according to Einstein's well-known formula:

$$E = \Delta M c^2. \tag{10.62}$$

If we consider as an example the *Hydrogen burning*. Here a helium nucleus is produced from four hydrogen nuclei:

$$4\,{}^1\mathrm{H} \rightarrow {}^4\mathrm{He} \tag{10.63}$$

- The total mass of $4\,{}^1\mathrm{H}$ amounts to: $4 \times 1.0081\mathrm{m_u}$.
- The total mass of a ${}^4\mathrm{He}$-Kerns amounts to: $4.0029\ \mathrm{m_u}$.

Therefore $\Delta m = 2.85 \times 10^{-2}\,\mathrm{m_u}$ or 0.7 % of the total mass is converted, which corresponds to an energy of 26.5 MeV. The following conversions are practical:

$$1\,\mathrm{keV} \approx 1.16 \times 10^{7}\,\mathrm{K} \tag{10.64}$$

$$931.1\,\mathrm{MeV} \approx 1\,\mathrm{m_u} \tag{10.65}$$

Let us apply this to the Sun. The Mass loss rate is $L_\odot/c^2 = 4.25 \times 10^{12}\,\mathrm{g\,s^{-1}}$. So the sun loses 4 million tons per second. Let's assume that 1 $M_\odot$ is converted into He, then 0.7 % corresponds to 1.4×10^{31} g that is converted into energy and → the sun could life $3 \times 10^{18}\,\mathrm{s} \approx 10^{11}$ a.

Consider an atomic nucleus of mass M_{nuc}, mass number A, which contains Z protons of mass m_p and contains $(A - Z)$ neutrons of mass m_n. The binding energy E_B is then:

$$E_B = [(A - Z)m_n + Zm_p - M_{\mathrm{nuc}}]c^2 \tag{10.66}$$

And the mean *binding energy* per nucleon f:

$$f = \frac{E_B}{A} \tag{10.67}$$

If one plots f vs. A, then one sees the following curve behavior:

- steep slope at the fusion of light elements,
- then flat slope up to the element ^{56}Fe,
- flat decrease from the element ^{56}Fe.

Accordingly, there are two ways to gain energy. Both have in common that the final product after fusion has a higher binding energy per nucleon than the initial products:

- up to the element iron: by fusion,
- elements heavier than Fe: by fission.

In the fusion of hydrogen to helium, only 0.7% of the initial mass is converted to energy.

Let's take a closer look at fusion. Between two particles, which are charged with the same sign with the charges $Z_1 e$, $Z_2 e$ (Z denotes the *Nuclear charge number,* the number of protons in the nucleus) there is a *Coulomb repulsion:*

$$E_{\mathrm{Coul}} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r} \tag{10.68}$$

One must bring the particles so close to each other that the short-range strong *Nuclear Forces* dominate over the long-range but weaker Coulomb repulsion forces. In this case, the interaction radius is:

$$r_0 \approx A^{1/3} 1.44 \times 10^{-15}\,\mathrm{m} \tag{10.69}$$

And the *Coulomb barrier* then results to

$$E(\mathrm{Coul}) \approx Z_1 Z_2\,\mathrm{MeV} \tag{10.70}$$

Inside the Sun, the temperature near the center is 10^7 K which corresponds to an energy of about one keV. Classically, nuclear fusion would thus be impossible in the stellar interior because of the too low temperatures. However, due to the *Tunnelling effect*[1] particles of lower energy can tunnel through the Coulomb barrier → fusion.

> Nuclear fusion in the stellar interior can only be explained by the quantum mechanical tunnel effect.

The probability that a particle tunnels through the Coulomb barrier is:

$$P(v) = e^{-2\pi\eta} \qquad \eta = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\hbar v} \tag{10.71}$$

→ strong temperature dependence of thermonuclear reactions!

The thermonuclear reaction rates depend on: Number of particles n_j, n_k, cross section σ. The number of reactions per second is then $n_k \sigma v$ and if there are n_j particles are in the volume, it is :

$$r_{jk} = n_j n_k \sigma v \tag{10.72}$$

[1] *G. Gamow,* 1928.

The *Cross section* depends on v . Under normal conditions, particle velocities are distributed according to Maxwell-Boltzmann. Let the energy be

$$E = \frac{1}{2}mv^2 \tag{10.73}$$

and $m = m_j m_k/(m_j + m_k) \dots$ reduced mass. In the interval $[E, E + dE]$ we have thus:

$$f(E)dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} \exp^{-E/kT} dE \tag{10.74}$$

The averaged reaction probability is:

$$< \sigma v >= \int_0^\infty \sigma(E) v f(E) dE \tag{10.75}$$

Let X_i is the fractional mass of the particles, i.e.

$$X_i \rho = n_i m_i$$

and Q is the energy released per reaction, then the *energy generation rate:*

$$\epsilon_{jk} = \frac{1}{1 + \delta_{jk}} \frac{Q}{m_j m_k} \rho X_j X_k < \sigma v > \tag{10.76}$$

where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$.

Another effect is shielding by free electrons. Beyond a certain distance, the incoming particle senses a neutral conglomerate of a positively charged nucleus surrounded by a cloud of free electrons. A nucleus of charge *Ze* causes polarization in its environment: electrons of charge $-e$ are attracted, and their density n_e in the vicinity of the nucleus is greater. The other ions are repelled, and their density n_i is lower. We have therefore deviations of the n_e, n_i from the mean values $\bar{n}_e, \bar{n}_i$. For the potential Φ we find:

$$\Phi = \frac{Ze}{r} \mathrm{e}^{-r/r_D} \tag{10.77}$$

Here r_D the *Debye radius,* which indicates the point at which the electrons start to shield the potential of the core. If $r \rightarrow 0$ then this potential changes to *Ze/r*. This also leads to a reduction of the Coulomb interaction and increases the probability of tunneling through the Coulomb barrier.

We now discuss the most important reactions. The superscript for the elements indicates the mass number, i.e., the number of protons and neutrons in the nucleus. ^{2}H is deuterium, i.e. a nucleus with one proton and one neutron, an isotope of hydrogen.

Hydrogen Burning The two hydrogen burning basic reactions are:

$$^{1}\mathrm{H} + {}^{1}\mathrm{H} \rightarrow {}^{2}\mathrm{H} + e^{+} + \nu \tag{10.78}$$

$$^{2}\mathrm{H} + {}^{1}\mathrm{H} \rightarrow {}^{3}\mathrm{He} + \gamma \tag{10.79}$$

From here on, there are branching reactions:

- pp1:

$$^{3}\mathrm{He} + {}^{3}\mathrm{He} \rightarrow {}^{4}\mathrm{He} + 2\,{}^{1}\mathrm{H} \tag{10.80}$$

- Further:

$$^{3}\mathrm{He} + {}^{4}\mathrm{He} \rightarrow {}^{7}\mathrm{Be} + \gamma \tag{10.81}$$

And from here, the branches:

- pp2:

$$^{7}\mathrm{Be} + e^{-} \rightarrow {}^{7}\mathrm{Li} + \nu \tag{10.82}$$

$$^{7}\mathrm{Li} + {}^{1}\mathrm{H} \rightarrow {}^{4}\mathrm{He} + {}^{4}\mathrm{He} \tag{10.83}$$

- pp3:

$$^{7}\mathrm{Be} + {}^{1}\mathrm{H} \rightarrow {}^{8}\mathrm{B} + \gamma \tag{10.84}$$

$$^{8}\mathrm{B} \rightarrow {}^{8}\mathrm{Be} + e^{+} + \nu \tag{10.85}$$

$$^{8}\mathrm{Be} \rightarrow {}^{4}\mathrm{He} + {}^{4}\mathrm{He} \tag{10.86}$$

The Energy Production Rate ϵ of the pp process Is strongly temperature dependent and given by :

$$\epsilon \propto \rho T^{5} \tag{10.87}$$

Hydrogen burning dominates at temperatures between 5 and 15 $\times\ 10^{6}$ K.

CNO Cycle

Here, the carbon serves only as a catalyst. One has the following six reaction steps:

$$^{12}\mathrm{C} + {}^{1}\mathrm{H} \rightarrow {}^{13}\mathrm{N} + \gamma \tag{10.88}$$

$$^{13}\mathrm{N} \rightarrow {}^{13}\mathrm{C} + e^{+} + \nu \tag{10.89}$$

$$^{13}\mathrm{C} + {}^{1}\mathrm{H} \rightarrow {}^{14}\mathrm{N} + \gamma \tag{10.90}$$

$$^{14}\mathrm{N} + {}^{1}\mathrm{H} \rightarrow {}^{15}\mathrm{O} + \gamma \tag{10.91}$$

$$^{15}\mathrm{O} \rightarrow {}^{15}\mathrm{N} + e^{+} + \nu \tag{10.92}$$

$$^{15}\mathrm{N} + {}^{1}\mathrm{H} \rightarrow {}^{12}\mathrm{C} + {}^{4}\mathrm{He} \tag{10.93}$$

Here the energy production rate is even more dependent on the temperature:

$$\epsilon_{\mathrm{CNO}} \propto \rho T^{12\ldots18} \tag{10.94}$$

At lower temperatures, the pp chain predominates, and at higher temperatures, the CNO cycle predominates (Fig. 10.5).

The nascent Positron e^{+} immediately annihilate with the electrons and radiate forming γ quanta. The neutrinos ν have a very small interaction cross section and can pass the star practically unhindered after their formation. In the process, they dissipate energy. In the interior of the sun, the fusion of a ^{4}He nucleus produces two Neutrinos, and a solar neutrino flux on Earth results. One measures a neutrino flux of 10^{15} neutrinos per m^{2} per second.

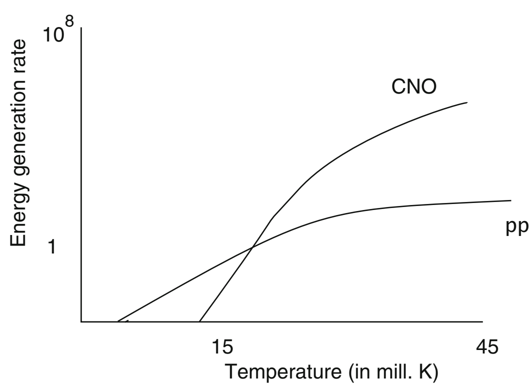


Fig. 10.5 Energy production rates of pp and CNO reactions. The energy production by the CNO cycle predominates from about 18 Million K onwards

Helium Burning As soon as in the central region of the sun all hydrogen has fused into helium, the thermonuclear reactions cease. The temperature is still too low for further reactions to ignite. Only when it reaches 10^8, K by contraction (cf. virial theorem), helium burning begins:

$$^4\mathrm{He} + {}^4\mathrm{He} \rightleftharpoons {}^8\mathrm{Be} \tag{10.95}$$

$$^8\mathrm{Be} + {}^4\mathrm{He} \rightarrow {}^{12}\mathrm{C} + \gamma \tag{10.96}$$

$$^{12}\mathrm{C} + {}^4\mathrm{He} \rightarrow {}^{16}\mathrm{O} + \gamma \tag{10.97}$$

The first reaction produces a ^{8}Be nucleus, which is unstable and decays after 10^{-1}s . Only if it reacts with a third ^{4}He nucleus within this short lifetime, a stable nucleus ^{12}C nucleus is formed. High densities are necessary for this. Some ^{16}O nuclei still react with ^{4}He and form ^{20}Ne.

The energy production rate is:

$$\epsilon_{\mathrm{He}} \propto \rho^2 T^\nu \tag{10.98}$$

with $\nu = 20 \ldots 30$.

Carbon Burning Once He is burned and the temperature has increased high enough, carbon burning sets in at $5 \times 10^8 \ldots 10^9$ K and the following reactions occur:

$$^{12}\mathrm{C} + {}^{12}\mathrm{C} \rightarrow {}^{24}\mathrm{Mg} + \gamma \tag{10.99}$$

$$\rightarrow {}^{23}\mathrm{Mg} + n \tag{10.100}$$

$$\rightarrow {}^{23}\mathrm{Na} + {}^1\mathrm{H} \tag{10.101}$$

$$\rightarrow {}^{20}\mathrm{Ne} + {}^4\mathrm{He} \tag{10.102}$$

Oxygen Burning This starts at $T = 1.4 \times 10^9$ K:

$$^{16}\mathrm{O} + {}^{16}\mathrm{O} \rightarrow {}^{32}\mathrm{S} + \gamma \tag{10.103}$$

$$\rightarrow {}^{31}\mathrm{S} + n \tag{10.104}$$

$$\rightarrow {}^{31}\mathrm{P} + {}^1\mathrm{H} \tag{10.105}$$

$$\rightarrow {}^{28}\mathrm{Si} + {}^4\mathrm{He} \tag{10.106}$$

There are further reactions due to the capture of ^{1}H and ^{4}He.

Silicon Burning From $T \approx 2 \times 10^9$ K many reactions occur, the most important being the buildup of iron:

$$^{28}\mathrm{Si} + {}^{28}\mathrm{Si} \rightarrow {}^{56}\mathrm{Fe} \tag{10.107}$$

This brings us to the end of the nuclear fusion chain. Further fusions no longer release energy, but consume it.

We can explain the formation of all elements up to Fe by thermonuclear fusion processes in stellar interiors.

10.3.2 Neutrinos

As already mentioned, the cross section of neutrinos σ_ν with matter is very small. With an energy of E_ν one has

$$\sigma_\nu = (E_\nu/m_e c^2)^2 \times 10^{-44}\,\mathrm{cm}^2$$

For neutrinos in the MeV range $\sigma_\nu \approx 10^{-44}\,\mathrm{cm}^2$. This is by a factor 10^{18} smaller than the cross section for interactions between photons and matter. For a density of $\rho = n\mu m_u$[2] (let mean molecular weight be equal to 1), the mean free path length is:

$$l_\nu = \frac{1}{n\sigma_\nu} \approx \frac{2 \times 10^{20}\,\mathrm{cm}}{\rho} \tag{10.108}$$

It follows:

- Normal stars: $\rho \approx 1\,\mathrm{g\,cm^{-3}}$ and $l_\nu = 100\,\mathrm{pc}$. Even if the density $\rho = 10^6\,\mathrm{g\,cm^{-3}}$ would be $l_\nu = 3000\,\mathrm{R}_\odot$.
- However, in a stellar collapse at the end of stellar evolution, the density can reach nuclear values, $\rho = 10^{14}\,\mathrm{gcm^{-3}}$ and $l_\nu = 20\,\mathrm{km}$. Some of the neutrinos are then reabsorbed in the star, and one must take into account the energy transport of the neutrinos.

[2] $m_u = 1.67 \times 10^{-24}$ g.

Table 10.1 Thermonuclear processes in which neutrinos are released

$^{1}H + {}^{1}H \rightarrow {}^{2}H + e^{+} + \nu$	pp	0.263 MeV
$^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu$	pp2	0.80 MeV
$^{8}B \rightarrow {}^{8}Be + e^{+} + \nu$	pp3	7.2 MeV
$^{13}N \rightarrow {}^{13}C + e^{+} + \nu$	CNO	0.71 MeV
$^{15}O \rightarrow {}^{15}N + e^{+} + \nu$	CNO	1.0 MeV

Neutrinos can occur in different flavours (electron-, muon- and tau-neutrinos), in nuclear fusion only the electron neutrinos are are important. A list of thermonuclear processes in which neutrinos are released is given in Table 10.1.

Furthermore, there are other processes that lead to the production of neutrinos:

1. Capture of electrons by protons – this happens at extremely high densities; let Z is the atomic number and A the atomic weight, then you have:

$$e^{-} + (Z, A) \rightarrow (Z - 1, A) + \nu$$

(Z, A) means an atom with charge number Z and the atomic weight A.

2. *Urca* process: electron capture and beta decay occur:

$$(Z, A) + e^{-} \rightarrow (Z - 1, A) + \nu$$
$$(Z - 1, A) \rightarrow (Z, A) + e^{-} + \bar{\nu}$$

3. Neutrinos by pair annihilation:

$$e^{-} + e^{+} \rightarrow \nu + \bar{\nu}$$

This requires temperatures above 10^9 K.

4. Photoneutrinos:

$$\gamma + e^{-} \rightarrow e^{-} + \nu + \bar{\nu}$$

The analogue would be the scattering of a photon by an electron (Compton scattering).

5. Plasma neutrinos:

$$\gamma_{\text{plasm}} \rightarrow \nu + \bar{\nu}$$

Decay of a plasmon into a neutrino-antineutrino pair. The frequency of a plasma depends on whether it is degenerate or not. For non-degenerate plasma one has:

$$\omega_0^2 \frac{m_e}{4\pi e^2 n_e} = 1 \tag{10.109}$$

and for degenerate plasma:

$$\omega_0^2 \frac{m_e}{4\pi e^2 n_e} = \left[1 + \left(\frac{\hbar}{m_e c}\right)^2 (3\pi^2 n_e)^{2/3}\right]^{-1/2} \tag{10.110}$$

If an electromagnetic wave of frequency ω passes through a plasma and K is the wavenumber, then one has the following dispersion relation:

$$\omega^2 = K^2 c^2 + \omega_0^2 \tag{10.111}$$

The wave is thus coupled to the collective motions of the electrons, and only waves with $\omega < \omega_0$ can propagate. Multiplying the above equation by $h/2\pi$ then you have the square energy of a quantum, which behaves like a relativistic particle of rest energy $h/2\pi\omega_0$ called a *Plasmon*.

6. Neutrinos due to Bremsstrahlung. When an electron is decelerated in the Coulomb field of a nucleus, there occurs emission of a photon, which in turn can decay into a neutrino-antineutrino pair.

Nuclear fusion provides energy up to the element iron. These elements were formed inside the stars. All elements heavier than iron were formed by other processes, such as a supernova explosion.

10.4 Special Stellar Models

We consider two examples of simple stellar models, which help to simplify the extensive numerical calculations help.

10.4.1 Polytropic Models

In these models we set the following relation between pressure P and density ρ:

$$P = K\rho^\gamma \tag{10.112}$$

K is the polytropic constant and γ the polytropic exponent. Often one uses the *Polytropic index n*:

$$n = \frac{1}{\gamma - 1} \tag{10.113}$$

For a completely degenerate gas, this condition is satisfied by $\gamma = 5/3$, $n = 3/2$. Here one can calculate K, in other cases this is a free parameter.
Another special case would be a star with constant temperature (isothermal star):

$$\rho = \mu P/(\Re T_0)$$

Further special case: completely convective star. Here $\nabla = \nabla_{\text{ad}} = 2/5(\nabla$ stands for temperature gradient), if one can neglect the radiation pressure and the star is completely ionized. Thus

$$T \approx P^{2/5}$$

and for an ideal gas with constant molecular weight $T \approx P/\rho$. Thus one has $\gamma = 5/3$, and K is also fixed again.

The first basic equation of the stellar structure can also be written as:

$$\frac{dP}{dr} = -\frac{d\Phi}{dr}\rho \tag{10.114}$$

where Φ is the gravitational potential. Furthermore one has *Poisson's equation:*

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi G\rho \tag{10.115}$$

If we use our relation for polytropic stars, then the basic equation is:

$$\frac{d\Phi}{dr} = -\gamma K\rho^{\gamma-2}\frac{d\rho}{dr} \tag{10.116}$$

If $\gamma \neq 1$then one can integrate this equation:

$$\rho = \left(\frac{-\Phi}{(n+1)K}\right)^n \tag{10.117}$$

At the surface $\Phi = 0$, $\rho = 0$. Substituting this into the Poisson equation, we have:

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G\left(\frac{-\Phi}{(n+1)K}\right)^n \tag{10.118}$$

Now define:

$$z = Ar \qquad A^2 = \frac{4\pi G}{(n+1)^n K^n}(-\Phi_c)^{n-1}$$
$$w = \frac{\Phi}{\Phi_c} = \left(\frac{\rho}{\rho_c}^{1/n}\right)$$

The Poisson equation then goes into the *Lane-Emden* equations:

$$\frac{d^2w}{dz^2} + \frac{2}{z}\frac{dw}{dz} + w^n = 0 \tag{10.119}$$

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dw}{dz}\right) + w^n = 0 \tag{10.120}$$

From this we find the radial distribution of density:

$$\rho(r) = \rho_c w^n \qquad \rho_c = \left[\frac{-\Phi_c}{(n+1)K}\right]^n \tag{10.121}$$

For pressure we find:

$$P(r) = P_c w^{n+1} \qquad P_c = K\rho_c^\gamma \tag{10.122}$$

One can find a power series for $w(z)$ and finds:

$$w(z) = 1 - \frac{1}{6}z^2 + \frac{n}{120}z^4 + \ldots \tag{10.123}$$

For the following cases there is an analytical solution:

- $n = 0$

$$w(z) = 1 - \frac{1}{6}z^2 \tag{10.124}$$

- $n = 1$

$$w(z) = \frac{\sin z}{z} \tag{10.125}$$

- $n = 5$

$$w(z) = \frac{1}{(1 + z^2/3)^{1/2}} \tag{10.126}$$

Otherwise, one has to solve the Lane-Emden equation numerically.

We consider a polytropic model with index 3 for the Sun, $M = 1.98 \times 10^{33}$ g, $R = 6.96 \times 10^{10}$ cm. From a table one takes for $n = 3$: $z_3 = 6.897$ and $\rho_c/\bar{\rho} = 54.18$. With $\bar{\rho} = 1.41\,\mathrm{g\,cm^{-3}}$ it follows for the central density $\rho_c = 76.39\,\mathrm{g\,cm^{-3}}$. Further more $A = z_3/R = 9.91 \times 10^{-11}$. From the relation

$$A^2 = \frac{4\pi G}{(n+1)K} \rho_c^{(n-1)/n}$$

follows $K = 3.85 \times 10^{14}$. Then from

$$P_c = K\rho_c^{\gamma}$$

the central pressure to $P_c = 1.24 \times 10^{17}\,\mathrm{dyn/cm^2}$. We assume the following chemical composition: $X \approx 0.7$, $Y \approx 0.3$. This gives an average molecular weight of $\mu = 0.62$. The central temperature follows from the ideal gas equation with $T_c = 1.2 \times 10^7$ K. The mass distribution is calculated from :

$$m(r) = \int_0^r 4\pi\rho r^2 dr = 4\pi\rho_c \int_0^r w^n r^2 dr \tag{10.127}$$

10.4.2 Homologous Equations

Very often one can solve problems in physics by starting from a known solution and then performing a transformation. Here we compare different stellar models with masses M, M', radii R, R' and consider so-called homologous points at which holds:

$$r/R = r'/R' \tag{10.128}$$

One speaks of homologous stars if holds $m/M = m'/M' = \xi$. The condition is then:

$$\frac{r(\xi)}{r'(\xi)} = \frac{R}{R'} \tag{10.129}$$

One introduces the following parameters: $x = M/M'$; $y = \mu/\mu'$; $z = r/r' = R/R'$; $p = P/P' = P_c/P_c'$; $t = T/T' = T_c/T_c'$; $s = l/l' = L/L'$. Now one can construct homologous main sequence stars:

$$\begin{aligned}
\frac{dr}{d\xi} &= c_1 \frac{M}{r^2\rho} \\
c_1 &= \frac{1}{4\pi} \\
\frac{dP}{d\xi} &= c_2 \frac{\xi M^2}{r^4} \\
c_2 &= -\frac{g}{4\pi} \\
\frac{dl}{d\xi} &= \epsilon M \\
\frac{dT}{d\xi} &= c_4 \frac{\kappa l M}{r^4 T^3} \\
c_4 &= -\frac{3}{64\pi^2 ac}
\end{aligned}$$

and:

$$\begin{aligned}
\frac{dr'}{d\xi} &= c_1 \frac{M'}{r'2\rho'} \left[\frac{x}{z^3 d}\right] \\
\frac{dP'}{d\xi} &= c_2 \frac{\xi M'2}{r'4} \left[\frac{x^2}{z^4 p}\right] \\
\frac{dl'}{d\xi} &= \epsilon' M' \left[\frac{ex}{s}\right] \\
\frac{dT'}{d\xi} &= c_4 \frac{\kappa' l' M'}{r'4T'3} \left[\frac{ksx}{z^4 t^4}\right]
\end{aligned}$$

where $\rho/\rho' = d$; $\epsilon/\epsilon' = e$; $\kappa/\kappa' = k$. Thus, solving these equations yields multiple stellar models at once.

10.5 Further Literature

We give a small selection of recommended further literature.

Stellar Structure and Evolution, R. Kippenhahn, A. Weigert, Springer, 1996
Stars and Stellar Processes, M. Guidry, Cambridge Univ. Press, 2019

Stellar Interiors, V. Trimble, Springer, 2004
Introduction to Stellar Structure, W. J. Marciel, Springer, 2015

Tasks

10.1 At which point in the HRD are stars in hydrostatic equilibrium the longest?

Solution
Main sequence

10.2 Discuss why simplifications in stellar models are justified or what a consideration of rotation, magnetic field would change in the models!

Solution
Stars evolve very slowly, so stat. Model; Rotation $\rightarrow$ Flattening, magnetic field $\rightarrow$ Anisotropy, ...

10.3 Derive the classical condition for hydrostatic equilibrium from the TOV equation!

Solution
The solution is very simple: $c^2 \rightarrow \infty$.

10.4 Calculate the escape velocity of a white dwarf of 0.5 solar masses!

Solution
The radius of the object is 1.5 Earth radii. The escape velocity is defined as $v_e = \sqrt{2GM/R}$ and inserting the values yields $v_e = 3.7 \times 10^6$ m/s, i.e. about 1/100 of the speed of light!

10.5 At what temperature does Compton scattering reduce opacity?

Solution
We first consider Wien's law: $h\nu = 4.965\,kT$, and as soon as $T > 0.1 m_e c^2/(4.965k)$, i.e. for $T > 10^8$ K the Compton scattering becomes important.

11 Stellar Evolution

In principle, there are the following stages in stellar evolution:

- Protostar,
- pre-main sequence evolution,
- main sequence,
- post-main-sequence existence.

The most important physical quantity that characterizes stellar evolution is the mass. In addition, the chemical composition plays a role. We have already mentioned the difference between stars of populations I and II. Population II stars contain significantly fewer metals (all elements heavier than He) than Population I stars.

11.1 Star Formation and Evolution

In this section we investigate the conditions under which a gas-dust cloud collapses. The evolution towards a protostar is then shown. Furthermore, we discuss the evolution of our Sun.

11.1.1 Protostars

Stars are formed by contraction from interstellar clouds consisting of gas and dust.

A. Hanslmeier, *Introduction to Astronomy and Astrophysics*,
https://doi.org/10.1007/978-3-662-64637-3_11

Based on the virial theorem mentioned above, we know that half of the potential energy released during contraction is converted into thermal energy, i.e. it heats up the star, and the other half is radiated.

Let us consider the collapse of such a gas cloud. Let the mass of the cloud be M, the radius R, the total number of particles N, the mean particle mass $\overline{m}$ and the temperature T. The gravitational potential energy is thus:

$$U = -\text{const}\frac{GMN\overline{m}}{R} \tag{11.1}$$

The value of the constant depends on the internal matter distribution in the cloud. The kinetic energy per particle is on average:

$$E_{\text{kin}} = \frac{3}{2}kT \tag{11.2}$$

and per unit mass

$$E_{\text{kin}} = \frac{3}{2}\frac{kT}{\mu m_u}. \tag{11.3}$$

For a cloud of mass M is the kinetic energy thus given by Eq. 11.3, multiplied by the mass M or Eq. 11.2, multiplied by the number of particles N; this cloud will then contract when

$$U > E_{\text{kin}}. \tag{11.4}$$

This is called the *Jeans criterion*. The Jeans mass is:

$$M_J = \frac{3}{2}\frac{kT}{G\overline{m}}R \tag{11.5}$$

and from that the jeans density is obtained.

So only large masses can become gravitationally unstable.

We also see that especially cool regions of interstellar matter are relevant for star formation, otherwise the kinetic energy becomes too large.

In Fig. 11.1 protoplanetary disks in the Orion Nebula are shown.

One can also estimate how long it takes for a gas envelope that is not in hydrostatic equilibrium to collapse. Let us assume $\Delta E_{\text{kin}} = \Delta U$ (Virial theorem) and

$$1/2(dr/dt)^2 = Gm_0/r - Gm_0/r_0,$$

Fig. 11.1 Formation of protoplanetary disks in the Orion Nebula

then we find for *free fall time:*

$$
\begin{aligned}
t_{\text{ff}} &= \int_{r_0}^{0} (dt/dr)dr = -\int_{r_0}^{0} \frac{dr}{\sqrt{Gm_0/r - Gm_0/r_0}} \\
x &= r/r_0 \\
t_{\text{ff}} &= [r_0^3/(2Gm_0)]^{1/2} \int_0^1 [x/(1-x)]^{1/2} dx \\
x &= \sin^2 \Theta \\
t_{\text{ff}} &= \sqrt{\frac{3\pi}{32\, G\rho}}
\end{aligned}
\tag{11.6}
$$

Before the temperature in the interior of a star is great enough to ignite nuclear reactions, one speaks of a *Protostar* or pre-main sequence evolution. The evolutionary paths for protostars in the HRD depend on their mass. There are four stages:

1. Collapse in free fall—particles do not collide with each other during free fall, internal pressure is zero.
2. The core regions collapse faster than the outer parts.
3. Once the core is formed, accretion from the shell occurs.
4. Only when the material surrounding the core is gone (due to radiation pressure) does the star become visible.

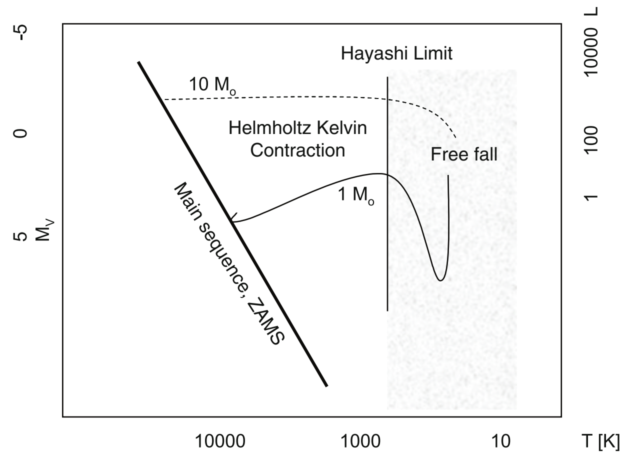


Fig. 11.2 Evolution of two protostars with (a) 1 $M_\odot$ and (b) 10 $M_\odot$ (dashed) to the main sequence. For case (a) the evolution takes 50 million years, for case (b) only 200,000 years

11.1.2 Collapse of a Sun-Like Star

We assume an interstellar cloud of sufficiently high mass. Collisions occur between the molecules during the collapse phase, and the dust radiates in the IR. This radiation can escape, and the cloud initially remains cool. But when the density of the core exceeds a critical value, then it becomes opaque (optical depth greater than 1)—the collapse of the core slows down, you have hydrostatic equilibrium, and a pre-main sequence star evolves. It takes a million years to get to this point. The luminosity of a star is given by:

$$L = 4\pi R^2 \sigma T^4 \tag{11.7}$$

In Fig. 11.2 the evolution is shown: Until the so-called Hayashi limit is reached, the star is fully convective, but gains luminosity by collapsing in free fall. Above a temperature of 2000 K dissociates H_2, so energy is consumed for this process, the luminosity of the star decreases. Above 10^4 K, hydrogen is ionized and the star becomes optically thick. Temperature and gas pressure increase, and after crossing the Hayashi limit, the contraction enters a Helmholtz-Kelvin phase (conversion of released gravitational energy into heat).

> The Hayashi limit separates the region of fully convective stars from no longer fully convective ones in the HRD.

In general: the luminosity is high because the radius R is still large. A Pre main sequence star glows due to contraction and accretion, the temperature hardly increases at first, and the luminosity decreases. Subsequently, the nucleus heats up and the opacity increases. A zone of radiative transfer develops, slowly moving from the inside to the outside. Then the evolution turns to the left in the HRD. If the temperature is high enough, the thermonuclear reactions ignite, one speaks of a *Zero Age Main Sequence Star, ZAMS*. Here only the outer envelope is still convective.

How long can a star stay on the main sequence? In principle, about 80% of its total lifetime. If the hydrogen content decreases in the interior due to nuclear fusion, then the temperature and density increase, and the star expands. Thus, the luminosity of the star also increases, and it evolves upward away from the main sequence. This can be estimated as follows: For luminosity, we have the mass-luminosity relation:[1]

$$L_*/L_\odot = (M_*/M_\odot)^{3.3} \tag{11.8}$$

The lifetime[2] of a star is given by:

$$t_*/t_\odot = (M_*/M_\odot)/(L_*/L_\odot) = (M_*/M_\odot)^{-2.3} \tag{11.9}$$

where $t_\odot = 10^{10}$, the main sequence lifetime of the Sun. Compare the main sequence lifetime of our Sun to that of a star of ten solar masses!

11.1.3 The Age of Stars

Stars in a cluster form at about the same time. Since massive stars evolve faster than low-mass stars (cf. Eq. 11.9), the main sequence of older star clusters clusters will no longer be fully occupied.

- In the HRD, the hot, massive, luminous stars are in the upper left.
- Hot, luminous massive stars have already moved away from the main sequence in older clusters.

> Therefore, one can infer the age of a cluster, and thus of the stars in it, from the location of the turn off point from the main sequence.

In Fig. 11.3 one can see an HRD of two star clusters. It is clear that the star cluster M 67 must be somewhat younger,[3] because here are hotter stars still on the main sequence than

[1] Valid only for stars of the main sequence!

[2] Actually, the main sequence lifetime.

[3] Its age is given as about four billion years.

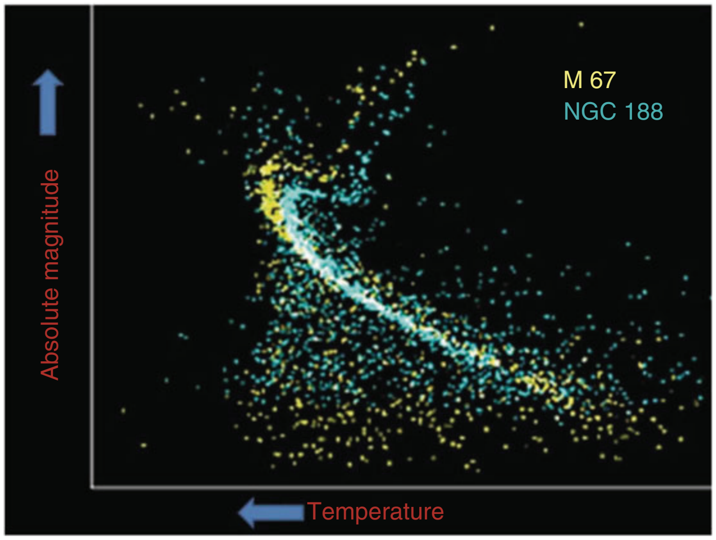


Fig. 11.3 Comparison of the HRD of two star clusters. From the position of the turn off point the age follows

in NGC 188. The star cluster NGC 188 is older than M 67 by a billion years and is one of the oldest open star clusters.

11.1.4 Evolution of a Star with One Solar Mass

As already mentioned above, a star with one solar mass reaches the zero-age main sequence, ZAMS, as soon as the pp chain ignites. After about ten billion years, its main-sequence existence ends, almost all the hydrogen in the core has been converted to helium, and the star expands slightly, increasing energy production as its temperature increases in the interior, and also increasing luminosity as its surface area increases. The nuclear reactions in the center eventually die out, but the fusion of hydrogen to helium continues in a shell around the core (shell burning). The radius of the star now increases considerably: the core contracts, so heat is produced (cf. virial theorem), and the hydrogen-burning shell heats up—more energy is produced, and the star expands. But this lowers the surface temperature, and thus the opacity increases, which leads to an increase in convection, which in turn is important for the mixing of the elements. The star evolves into a red giant and moves obliquely upward to the right in the HRD.

A *red giant* has the following structure: small dense core with $T \sim 50 \times 10^6$ K, degenerate electron gas in the core. So the gas pressure depends only on the density and not on T, the nucleus can thus resist the gravitational force even though there is no more fusion there. The contraction raises the temperature to 10^8 K and the triple-alpha process ignites.

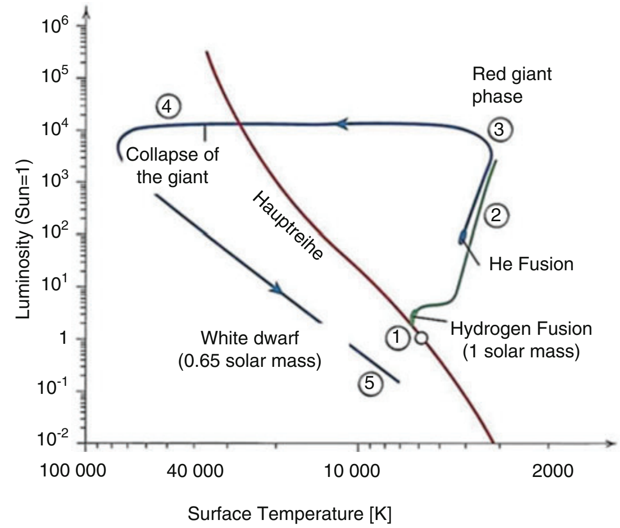


Fig. 11.4 Evolution of a Sun-like star in the HRD from main sequence star to red giant and white dwarf. The main sequence stage (1) takes about 9 billion years, stages 2 to 4 are passed in a few 100 million years

If this happens, then the heat spreads out through the very effective thermal conduction of the degenerate electrons. The entire nucleus then ignites, but since the matter is degenerate, only the temperature increases, the pressure remains constant, and the nucleus does not expand. This is called a *helium flash.* Only when the core temperature has reached 350 million K, the electrons are not degenerate and the core can expand and cool down. After this He flash, the stellar radius and thus its luminosity decreases slightly, and the star moves down and to the left in the HRD. When the He is consumed by the triple alpha process, its fusion occurs in a shell, and the star expands again. The electrons are degenerate again, and this time the core is enriched in carbon (Fig. 11.4).

The triple-alpha process is very strongly dependent on temperature, and thermal pulses occur that are in fact giant thermonuclear explosions. Such explosions happen every approx. 10^3 years and lead to luminosity changes of the star by up to 50% during some years. The star is located at the asymptotic giant branch in the HRD *(asymptotic giant branch, AGB).* During these phases there are also very strong *stellar winds,* and in a few 1000 years the envelope is completely blown away; an expanding envelope forms around the star, heated by the hot core and excited to glow; this is called a *planetary nebula.* A very well-known example is the Ring Nebula, M57 (Fig. 11.5). The Ring Nebula is about 2000 light-years away and about 20,000 years old. The central star is a white dwarf, the luminosity is $15\overset{\mathrm{m}}{.}8$.

If the mass of a star is less than about 1 $M_\odot$, then the core temperature is not enough to start carbon burning. Within about 100,000 years, a *White Dwarf evolves.*

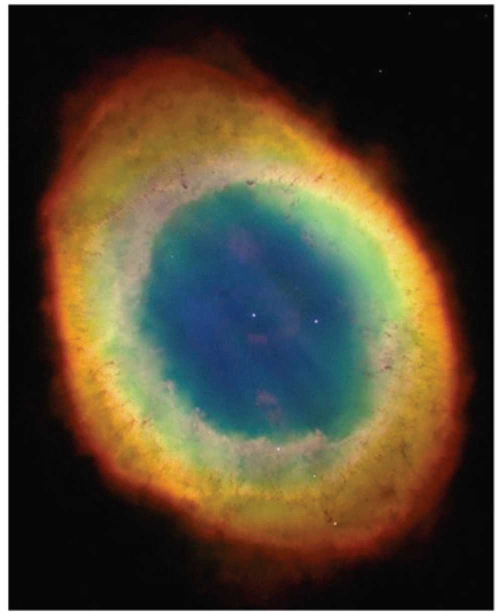


Fig. 11.5 The ring nebula M57. The central star, a white dwarf, can be seen in the center (HST/NASA)

So the fate of our sun is as follows:

Evolution of our sun:
Main sequence star (about 10 billion years total) $\rightarrow$ Red giant (approx. 10^8 years)$\rightarrow$ White dwarf.

11.2 Comparison of Stellar Evolution

An O5 star can reach a total age of about 5 million years, whereas an M0 star can reach about 30 billion years if its mass is only 1/2 solar masses.

There are the following final stages in stellar evolution:

- White dwarfs,
- neutron stars, pulsars,
- black holes.

A detailed description of the final stages follows in the next sections.

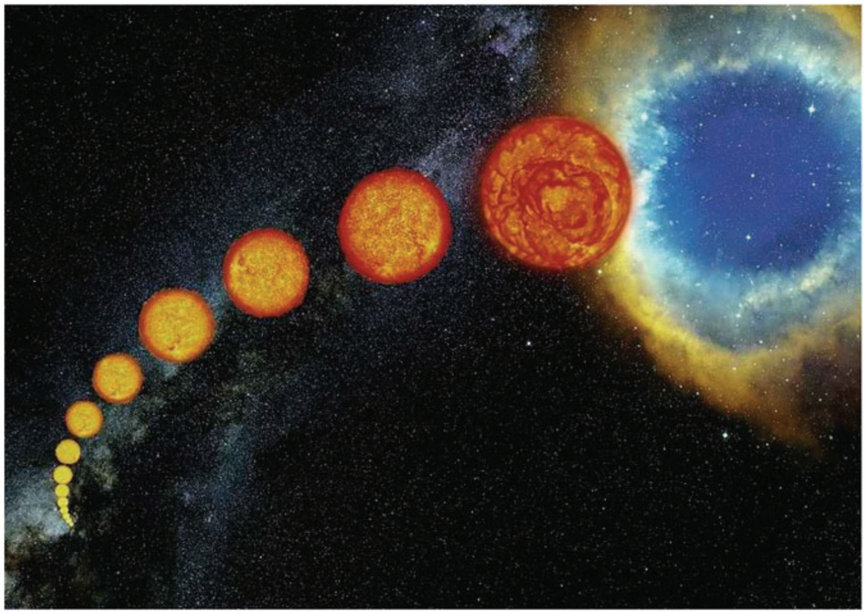


Fig. 11.6 Sketch of the evolution of low-mass stars from main sequence star (lower left) to planetary nebula (upper right). (ESO/Steinhöfel)

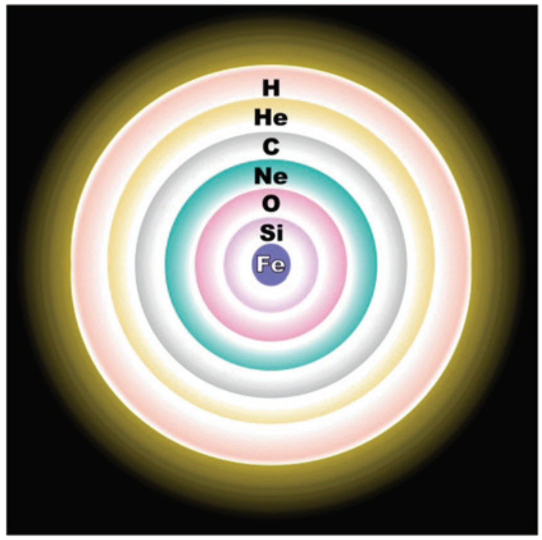
11.2.1 Low-Mass Stars

The evolution of low-mass stars is sketched in Fig. 11.6. At the end of the main sequence existence, the stars evolve into red giants and finally into a white dwarf. As they do so, the outer envelopes are slowly ejected; these envelopes can be seen glowing as planetary nebulae for a few thousand years.

11.2.2 Massive Stars

Massive stars burn elements down to iron. At the end of its evolution, the star has a shell-like structure (Fig. 11.7) with an iron core in the middle, followed by a shell of silicon burning, and so on. Once the mass of the iron core exceeds the Chandrasekhar limit mass ($1.4\,M_\odot$), there is an implosion of the core combined with the repulsion of the outer layers, which greatly enlarges the surface of the exploding star and therefore makes it very bright; a *Supernova* lights up.

Fig. 11.7 Structure of a massive star at the end of its evolution. As soon as the iron core exceeds the Chandrasekhar mass, a supernova lights up



11.3 White Dwarfs

11.3.1 General Properties

They develop from Red Giants. As we have seen, in the final stages of stellar evolution pulsations occur in which parts of the outer envelope are ejected. This results in the formation of a planetary nebula (e.g. M57, Fig. 11.5).

Depending on the initial mass (the final mass leading to the formation of white dwarfs is always below 1.4 solar masses) a distinction is made:

- Stars with $\leq 0.5\,M_\odot$ form He-white dwarfs, because the core temperature too low to ignite the helium.
- Stars with masses between 0.5–5.0 $M_\odot$ leave C-O stars behind.
- Stars with masses between 5–7 $M_\odot$ form O-Ne-Mg-rich white dwarfs.

The mass values given here refer to the initial mass of the star!

In the case of white dwarfs (WD, *white dwarfs*) the matter is so densely packed that the electrons can no longer move freely, but form a degenerate electron gas.

Equilibrium state (hydrostatic equilibrium): The gravity of a white dwarf is compensated by the pressure of the degenerate electrons.

$\rightarrow$ However, this only goes up to 1.4 $M_\odot$ $\rightarrow$ *Chandrasekhar limiting mass.*

Chandrasekhar limiting mass: the final evolutionary stages of stars up to a mass of about 1.4 solar masses are white dwarfs.

One can easily establish a relationship between the Chandrasekhar limit mass, the radius and the mass of white dwarfs. In the case of complete non-relativistic degeneracy is:

$$P = K\rho^{5/3} \tag{11.10}$$

The condition for hydrostatic equilibrium gives:

$$P \approx M^2/R^4 \tag{11.11}$$

the density $\rho \approx M/R^3$ and therefore $P \approx M^{5/3}R^5$. One obtains:

$$R = \frac{4\pi K}{G(4/3\pi)^{5/3}M^{1/3}} \qquad R_{\mathrm{WD}} \approx \frac{1}{M^{1/3}} \tag{11.12}$$

$\rightarrow$ Therefore, the larger the mass of a white dwarf, the smaller its radius R.

White dwarfs glow by cooling, the thermal energy is given by:

$$E_{\mathrm{th}} = \frac{3}{2}\frac{kT}{\mu m_u}M. \tag{11.13}$$

Consider a star with 0.8 $M_\odot$ and a temperature of 10^7 K then the thermal energy is 4×10^{40} J and we assume a luminosity of 10^{-3} $L_\odot$,[4] then the cooling time is τ_c:

$$\tau_c = E_{\mathrm{th}}/L \approx 4 \times 10^{40}\mathrm{J}/[(10^{-3})(3.8 \times 10^{26}\,\mathrm{J/s})] \approx 3 \times 10^9\,\mathrm{a}$$

The first white dwarf was found in 1862 by *A. Clark*: *Sirius* B, a hot but inconspicuous companion of Sirius (Fig. 11.8).

White dwarfs are divided into:

- DA: D stands for Dwarf, and A means a spectrum similar to that of an A star, i.e., hydrogen-rich.
- DB: spectrum with nebular lines.
- DC: predominantly continuous spectrum.

[4] Solar luminosity: $L_\odot \approx 3.86 \times 10^{26}$ J/s.

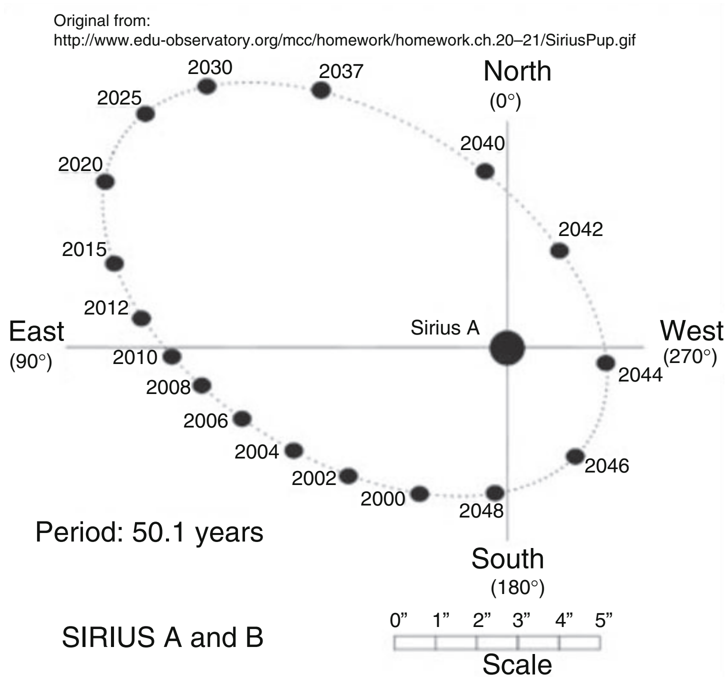


Fig. 11.8 Sirius with companion

The most reliable data are for DA stars, they lie shifted to the left parallel to the main sequence. The mean radius is 0.013 $R_\odot$, the mean mass 0.7 $M_\odot$ and density 10^9 kg/m^3.

11.3.2 General Relativity and White Dwarfs

In the case of white dwarfs, the contribution of gravitational red shift becomes important, and the spectral lines appear shifted to the red. Photons of energy E have an equivalent mass of $E = mc^2$, and the gravitational field acts on this mass, leading to a decrease in its energy, or red shift, since photons of higher energy have a shorter wavelength than photons of lower energy. Let us first consider the classical case: The change of energy in the gravitational field is according to *Newton:*

$$\Delta(h\nu) = -GmM/R \tag{11.14}$$

Since the mass of the photon $m = E/c^2$ with $E = h\nu$, we obtain by substitution:

$$\Delta\nu/\nu_0 = -GM/c^2R \tag{11.15}$$

Let us briefly consider the relativistic Doppler effect:

$$\frac{\Delta\nu}{\nu} = 1 - \frac{1}{\sqrt{1 - R_S/R}} \approx -\frac{GM}{Rc^2} \qquad R_S = \frac{2GM}{c^2} \tag{11.16}$$

Where R_S is the Schwarzschild radius.

Thus, the relativistic Doppler effect depends on the ratio of the Schwarzschild radius R_S to the radius of the star *R*, and the effect increases for

- large masses,
- small compact objects (small value for *R*).

11.3.3 Magnetic Fields

In the formation of a white dwarf, the conservation of magnetic flux must be considered. This is the number of magnetic field lines multiplied by the area they penetrate. If now the star is compressed in the course of its evolution, the number of field lines remains the same, of course, but the surface area decreases, and therefore the magnetic field strength increases. One can show that the magnetic field strength of a White dwarf compared to the field strength of the Sun is:

$$B_{\mathrm{WD}}/B_{\odot} = (R_{\odot}/R_{\mathrm{WD}})^2 \tag{11.17}$$

$\rightarrow$ Extreme amplification of the magnetic field by compression of the star!

11.3.4 Brown Dwarfs

Unlike white dwarfs, brown dwarfs are not at the end of stellar evolution, but in their case hydrogen burning never ignited because the central temperatures were too low. The limits are not exactly definable, but one speaks of:

- Planets when $M < 0.002\, M_{\odot}$;
- Brown dwarfs, if $0.002\, M_{\odot} < M < 0.08\, M_{\odot}$.

For brown dwarfs near the 0.08 solar mass limit, there is a phase of deuterium burning (some 10,000 years).

The Hubble Space Telescope has been used to search for brown dwarfs. In this context, the object *Gliese 229* should be mentioned. The star is a double system, the main star a red dwarf and Gliese 229B a brown dwarf with more than 20 Jupiter masses. The companion is located at a distance of 40 AU from the main star.

11.4 Neutron Stars

11.4.1 Formation of Neutron Stars

For a contracting star at the end of stellar evolution whose mass is larger than the Chandrasekhar limit mass of 1.4 solar masses, the pressure of the degenerate electrons is no longer sufficient to resist the strong gravity. Matter is compressed to extremely high densities, and the inverse beta decay starts:

$$p^{+} + e^{-} \rightarrow n + \nu \tag{11.18}$$

The protons and electrons combine to form neutrons; neutron gas is produced whose density reaches about $10^{17}\,\mathrm{kg/m^3}$. The neutrons form a degenerate gas, and a neutron star develops with a diameter of a few 10 km. In the interior there is a neutron liquid, in the outer regions there is a neutron superfluid and a crystalline surface (neutron lattice gas). In the outermost meters there exists an atmosphere of atoms, electrons, and protons, the atoms being mostly iron atoms.

Consider the gravitational red shift of a neutron star of 7 km radius (SI units used throughout):

$$\Delta\lambda/\lambda \approx GM/Rc^2 = \frac{6.67 \times 10^{-11} \times 2 \times 10^{30}}{9 \times 10^{16} \times 7 \times 10^{3}} \approx 0.2 \tag{11.19}$$

The structure of a neutron star is imagined as follows (Fig. 11.9):

- 15−16 km: The top layer consists of degenerate matter as in the white dwarf with an increase in density, from 10^7 to $4 \times 10^{14}\,\mathrm{kg/m^3}$, above, of iron nuclei, and further down also of neutron-rich nuclei (e.g., gold, lead, uranium, . . .).
- 11−15 km: Inner crust, the neutron-rich nuclei dissolve, and free neutrons appear in greater numbers, unable to decay in the relativistically degenerate electron gas surrounding them.
- At a distance of 11 km from the center there is a density of $200 \times 10^{15}\,\mathrm{kg/m^3}$, the state of a strongly incompressible neutron liquid.
- Central region: density up to $400 \times 10^{15}\,\mathrm{kg/m^3}$. Possibly the neutrons dissolve, and subnuclear particles such as free quarks could occur.

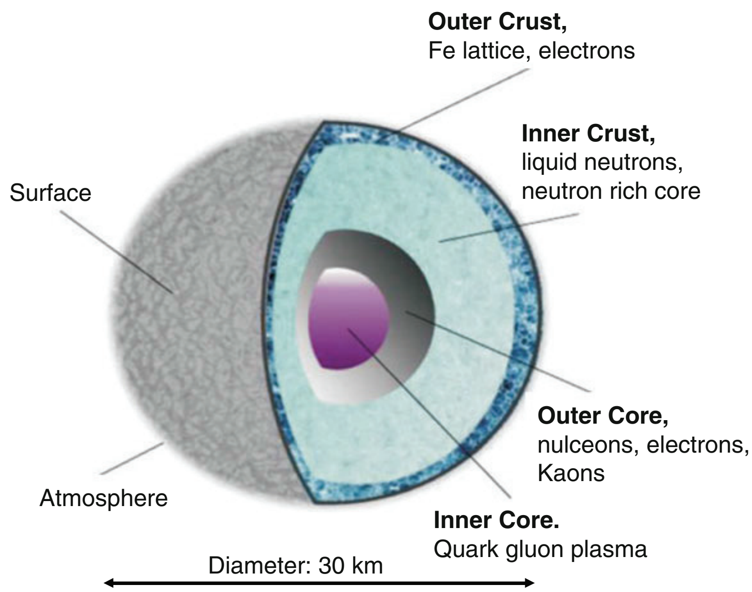


Fig. 11.9 Structure of a neutron star

As with the white dwarf, the diameter decreases with increasing mass, and there is a limiting mass for neutron stars analogous to the Chandrasekhar limiting mass, called the *Oppenheimer-Volkoff limit.*

> Calculations give a limiting mass for neutron stars in the range of three to four solar masses. Thus, even more massive stars evolve into black holes.

11.4.2 Pulsars

In 1967 the Hewish group wanted to study scintillation of radio sources on the sky with the help of a newly constructed radio telescope in Cambridge, England. Scintillation is the flashing of a radio source due to density fluctuations in the interplanetary plasma (caused by the solar wind) and in the interstellar medium. Very big surprise was then when extremely periodic radio signals were found. The Hewish group found a signal with the exact period of 1.33730113 s. This phenomenon was called *Pulsar*, although it will be shown below that the brightness change has nothing to do with pulsations of the star. In the meantime, more than 150 pulsars have been studied.

The regularity of the pulses is better than $1{:}10^8$. The energy of a pulse can vary strongly. Sometimes pulses fail. Typical pulsation durations range from a few 10 s to milliseconds.

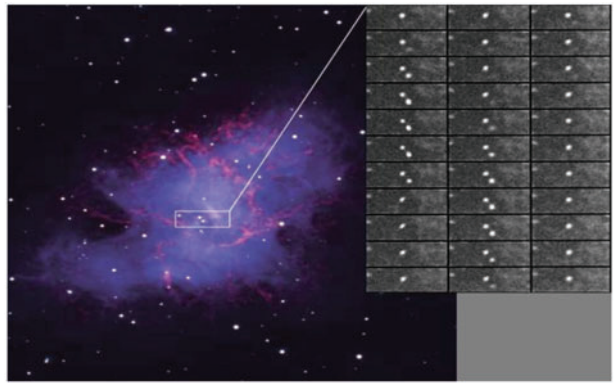


Fig. 11.10 The Pulsar in the *Crab Nebula*

The primary pulses can still be decomposed into sub millisecond pulses. The first pulsars were found at a frequency of 81.5 MHz. The pulsation times change in the long term, one measures an increase of the periods around 10^{-8} s/a (seconds per year). This can only be determined with atomic clocks, which have an accuracy of 10^{-10} s/a. The age of a pulsar follows from its period divided by the rate of change of that period:

$$t_{\text{pulsar}} \approx P \left(\frac{dP}{dt} \right)^{-1} \tag{11.20}$$

A very well known pulsar is the Crab pulsar (Fig. 11.10): $P = 0.03\,\text{s}$, $dP/dt = 1.2 \times 10^{-13}$ s/s, it therefore has age:

$$t_{\text{Pulsar}} \approx 10^{11}\,\text{s} \tag{11.21}$$

indicating the approximate correctness of the formula. The supernova explosion that led to the formation of the Crab pulsar occurred in 1054 AD.

Important in observation is the effect of *Dispersion:* if one examines a given pulsar at lower frequencies, then the photons are slowed down by the electrons that are in the line of sight of the pulsar. Longer wavelengths are slowed down more, and the electron density in the line of sight is estimated from the observations. Conversely, if we know the average electron density, we can determine the distance of the pulsar. Let us assume pulses of two different frequencies f_1, f_2 are emitted at the time t_0 and arrive here at the times t_1, t_2. We then receive:

$$t_1 - t_0 = d/v_1 \qquad t_2 - t_0 = d/v_2$$

Of course, we don't know t_0 but this is omitted in the case of:

$$t_2 - t_1 = (1/v_2 - 1/v_1)d \tag{11.22}$$

The velocities depend on the electron density, and if we know this, we can calculate the distance *d*. Interstellar matter, i.e., the matter between stars, does not have a constant density, a *Dispersion measure DM* has been introduced:

$$DM = \int_0^d n_e dl \tag{11.23}$$

and:

$$t_2 - t_1 = 2\pi e^2/m_e c(1/f_2^2 - 1/f_1^2)DM \tag{11.24}$$

$$D = (t_2 - t_1)/(1/f_2^2 - 1/f_1^2) \tag{11.25}$$

$$DM = 2\pi mcD/e^2 \qquad DM = 2.41 \times 10^{-16} D \tag{11.26}$$

Most pulsars are found at low galactic latitudes.

Furthermore the *Faraday rotation* has to be considered. The plane of polarization of linearly polarized radiation is rotated when it passes through a magnetic plasma. The Faraday rotation depends on:

- average electron density,
- mean magnetic field strength,
- λ^2 of radiation,
- Distance through the medium.

So one can measure the angle through which the plane of polarization is rotated at different wavelengths and then say something about above quantities.

How Do Pulses Occur?
To explain them, one needs:

- a rapidly rotating neutron star, which has a high rotational energy E_{rot} and
- a dipolar magnetic field, which transforms rotational energy into electromagnetic energy. In Fig. 11.11 a model of an oblique rotator is shown.

Pulsars are rapidly rotating neutron stars.

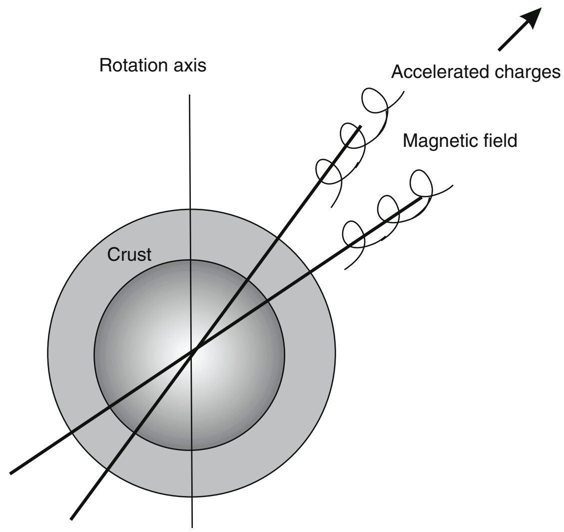


Fig. 11.11 The oblique rotator model of a pulsar; charges are accelerated in the strong magnetic field, and focused synchrotron radiation is produced

Suppose our Sun collapses into a neutron star with radius R. Let us examine the expected magnetic field strengths! Let $R_{\mathrm{ns}} = 73\,\mathrm{km}$, it holds:

$$B_{\mathrm{ns}} = B_{\odot}(R_{\odot}/R_{\mathrm{ns}})^2 \approx 10^6\ \mathrm{T}$$

This is purely hypothetical, since our Sun will evolve into a white dwarf.

The magnetic axis is inclined with respect to the axis of rotation. The rotation creates an electric field by induction, and this accelerates particles of the crust. The particles thus accelerated (mainly electrons) emit *Synchrotron radiation*. The torque of the accelerated particles slows down the rotation, and therefore the slower pulsars rotate, the older they are.

Let us consider rotation for a moment:

$$\frac{v^2}{R} = \frac{GM}{R^2} \tag{11.27}$$

This is the stability condition (centrifugal force must be less than or at most equal to gravity). The rotation period is:

$$P = 2\pi R/v$$

From the above equations we obtain a typical density for neutron stars of. $\rho = 4 \times 10^{16}\,\mathrm{kg/m^3}$.

As we have seen, there are very many binary stars and multiple systems. In 1974 *Hulse* and *Taylor* analyzed a pulsar already known by then and found a period of 7.75 h, which can be explained by the orbital motion of two components. Therefore it is a double pulsar. This object (PSR 1913+16) is 5 kpc from us, and the semi-axis of the double system is only as large as the radius of the Sun. The masses are 1.4 $M_\odot$ and 1.3 $M_\odot$.

Pulsars and neutron stars are formed in supernova outbursts. The pulsar in the Crab Nebula emits pulses with an energy of 10^{28} W from the optical to the X-ray range. Here a deceleration of its rotation by 4×10^{-13} s/s or 10^{-5} s/a has been measured. The rotational energy provides the energy budget for the nebula surrounding the pulsar. There is a conversion of rotational energy into kinetic energy and finally into radiation energy of the nebula. The rotational energy is (I is the moment of inertia):

$$E_{\mathrm{red}} = \frac{1}{2} I \omega^2 \tag{11.28}$$

$$\omega = 2\pi / P \tag{11.29}$$

$$I = \frac{2}{5} M R^2 \tag{11.30}$$

Let us assume that all the rotational energy is converted into radiant energy:

$$\frac{dE_{\mathrm{rad}}}{dt} + \frac{dE_{\mathrm{rot}}}{dt} = 0 \tag{11.31}$$

Now we put in:

$$\begin{aligned}
\frac{dE_{\mathrm{rot}}}{dt} &= \frac{1}{2}\frac{d}{dt}(I\omega^2) \\
&= \frac{1}{2}\frac{d}{dt}\left[\frac{2}{5} M R^2 \left(\frac{2\pi}{P}\right)^2\right] \\
&= \ldots = -\frac{8}{5}\pi^2 M R^2 P^{-3} \frac{dP}{dt}
\end{aligned}$$

and since $L = dE_{\mathrm{rad}}/dt = -dE_{\mathrm{rot}}/dt$, we get:

$$\begin{aligned}
L &= \frac{8}{5}\pi^2 M R^2 P^{-3} \frac{dP}{dt} \\
\frac{dP}{dt} &= \frac{5}{8\pi^2} \frac{L P^3}{M R^2}
\end{aligned}$$

We determine the rate of pulse changes for the Crab pulsar. We use to estimate $M = 1\,M_\odot$, $R = 10$ km and $L = 10^{31}$ W and $P = 1$ s:

$$\frac{dP}{dt} = \frac{5}{8\pi^2} \frac{10^{31}}{2 \times 10^{30} (10^4)^2} = 10^{-8} \mathrm{s/s}$$

For the Crab pulsar: $P \approx 0.03$ s, therefore one has:

$$dP/dt = 10^{-13}\ \mathrm{s/s}$$

This agrees with the observations.

11.5 Supernovae

11.5.1 Classification

In a supernova, the star explodes and the outer shell is ejected.

Supernovae (SN) reach absolute magnitudes of -16^{M} to -20^{M} and can thereby increase the brightness of an entire galaxy. From historical records is known, for example, the SN from the year 1054, which was observed by Chinese astronomers. It was so bright that the star could be seen in the daytime sky. The brightness progression of a SN shows a rapid rise to maximum and then a drop of two to three magnitudes within a month, before a slower decline in brightness. The radiant energy released during an SN explosion is 10^{44} J. The neutrinos carry off much more energy still. The first neutrinos originating from a SN to be recorded were those from SN 1987A. The collapse of a star leads to the release of gravitational energy (be $R = 15$ km):

$$E_{\mathrm{grav}} = \frac{GM^2}{R} \approx \frac{(6.67 \times 10^{-11})(2 \times 10^{30})^2}{1.5 \times 10^4} \approx 2 \times 10^{46}\ \mathrm{J} \tag{11.32}$$

There are two types of supernovae:

- Type I: They occur in elliptical and in spiral galaxies, it is a white dwarf that explodes due to sudden onset of carbon fusion. The white dwarf accretes matter from a companion. Once the Chandrasekhar limit is reached, the star collapses and a type I supernova is formed.

  Another form of a type I supernova is possible in a compact binary star system where both components have evolved into a white dwarf. The masses move around the common center of mass, this is an accelerated motion, and accelerated masses radiate gravitational waves according to general relativity (cf. Electrodynamics: Accelerated charges radiate electromagnetic waves). As a result, the orbital angular momentum decreases, and the two components approach each other until they merge. A type Ia supernova is formed, and the onset of carbon detonation is likely to rupture the star completely, leaving no remnant star (e.g., neutron star). The light curves of SN Ia are very similar.

Type I is divided into type Ia, b, c. In general, type I lacks hydrogen lines. Spectra of SN Ib resemble those of SN Ia near the maximum, but later those of SN II.

- Type II: occur exclusively in spiral galaxies; massive (10 to 100 solar masses) stars at the end of their evolution. The detonation is caused by gravitational collapse. Inside, an iron core is formed in a highly evolved star, which collapses into a neutron star.

At the end of the evolution of a massive star, an inert Fe core remains, which produces no energy; neutrinos escape and dissipate energy. Density increases, protons and electrons form neutrons and neutrinos. Matter around the nucleus impacting the nucleus at 15% of the speed of light causes the nuclear mass to increase. The nucleus collapses as soon as the Chandrasekhar mass is reached. There is no counterforce to gravity at this moment, as the pressure of the degenerate electrons is released ;

The neutron densities become so high that the nucleus becomes incompressible, a repulsive matter wave is formed, which propagates outward as a shock wave . This shock wave causes the actual explosion, the inner region compresses further and forms a neutron star or a black hole (Table 11.1).

The spectra of both types show emission lines that are often accompanied by short-wavelength absorption components, so-called P-Cygni lines → expanding gas shell (absorption lines are produced in the shell that moves toward the observer). One measures ejection velocities of up to 2×10^4 km/s, higher for SN I than for SN II. In SN II one observes similar lines as in novae, Balmer lines, He, metals (Ca II, Fe II) and later forbidden lines like [OI] and [OIII].

Brightnesses:

- SN Ia: $M_{\mathrm{B,max}} = -19^{\mathrm{m}}$; in the first 20 to 30 days after maximum the brightness decreases by two to three magnitudes; light curves are very similar → Standard candles for distance determination!

Table 11.1 Comparison of type Ia and type II supernovae

	Ia	II
Cause	White dwarf in double star	Massive star
Spectrum	No H	H
Max. bright.	brightness $1\overset{\mathrm{m}}{.}5$ > type II	
Light curve	Sharp maximum	Broader maximum
	All have the same brightness	Diverse brightness
Occur in	All galaxies	Only spiral galaxies
Expansion	10,000 km/s	5000 km/s
Radio emission	–	Available

- SN Ib/c: Brightness at maximum $1\overset{\mathrm{m}}{.}5$ lower than SN Ia.
- SN II: stronger dispersion of maximum brightness; $M_{\mathrm{B,max}} = -17 \ldots - 18^{\mathrm{m}}$.

11.5.2 Nuclear Synthesis During a SN

Towards the end of its evolution a star with a mass between 10 and 20 solar masses has a shell-like structure: C, He, and H shells. The Fe core contracts and its temperature increases. At 10^9 K occurs *Photodisintegration of* Fe:

$$^{56}\mathrm{Fe} + \gamma \rightarrow 13\,^{4}\mathrm{He} + 4\mathrm{n} \tag{11.33}$$

This reaction is endothermic and requires about 100 MeV. The nucleus loses energy and contracts more rapidly. The following reactions lead to the formation of a degenerate neutron gas:

$$^{4}\mathrm{He} \rightarrow 2p + 2n \tag{11.34}$$

$$p + e^{-} \rightarrow n + \bar{\nu} \tag{11.35}$$

The upper layers also fall inward, heat up, and nuclear fusion begins. This happens explosively, and the outer layers are repelled. So many energetic neutrons are formed, which can be absorbed by the heavy nuclei. There is an *r*-process *(rapid)* and an *s*-process *(slow),* where the terms "rapid" and "slow" refer to beta decay, respectively:

$$n \rightarrow p + e^{-} + \bar{\nu} \tag{11.36}$$

This takes about 15 min. In the *r* process, neutron capture occurs faster than beta decay. For example, the *r*-process leads to $^{56}\mathrm{Fe} + \mathrm{n} \ldots$ the $^{61}\mathrm{Fe}$. This is only stable for about 6 min, and if during this time the *s process* neutrons are captured, then emerges:

$$^{56}\mathrm{Fe} \rightarrow {}^{56}\mathrm{Co} + \mathrm{e}^{-} + \nu \tag{11.37}$$

For type II supernovae only the *r* process plays a role. In red giants, nucleosynthesis after the *s*-process plays an important role.

For both type I and type II supernovae, the major source of energy in emission is radioactive decay: ^{56}Ni decays with a half-life of 6.1 days to ^{56}Co, and this decays with a half-life of 77.3 days to the stable ^{56}Fe.

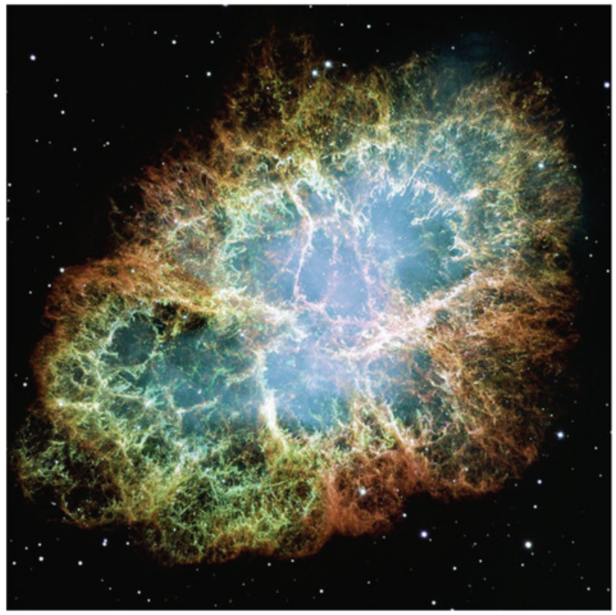


Fig. 11.12 Crab Nebula M1. A supernova remnant, distance 6300 Ly. (Credit: NASA, ESA, S. Beckwith (STScI), and The Hubble Heritage Team STScI/AURA)

11.5.3 Observed Supernovae

Several hundred supernova outbursts have been observed to date, and several dozen per year through surveys. Because of their high luminosity, these outbursts are observable not only in our Galaxy, but even in distant galaxies; at the time of greatest brightness, a supernova even outshines an entire galaxy. In our Milky Way only three supernova outbursts have been registered in the last 900 years:

- 1054 (Remnant = Crab Nebula, see Fig. 11.12) ,
- In 1572 *Tycho Brahe* saw a supernova,
- In 1604, Kepler observed a supernova in the constellation Ophiuchus.

Let us do some calculation. Kepler observed a supernova whose brightness was about the same as Jupiter's during its opposition. How far away was this supernova from us?

Assuming it was a type I SN, then $M = -19$; the apparent brightness of Jupiter is $-2\overset{\mathrm{m}}{.}5$. Therefore, it follows from the distance modulus:

$d \approx 20{,}000$ pc = 65,000 light years.

In 1987 a supernovae was observed in the large Magellanic cloud[5] , Supernova1987A (the "A" stands for the first supernova of 1987). It could be seen with the naked eye in the southern sky (light curve, Fig. 11.13). Before the outburst was detected, neutrinos were

[5] A dwarf galaxy, belongs to our galaxy.

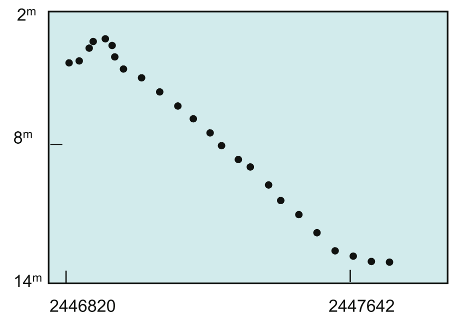


Fig. 11.13 Light curve of supernova 1987 A. The time is given in Julian day count

found with a neutrino detector in Kamioka, Japan, which must have been emitted from SN 1987 A. The neutrinos are produced at the enormous nuclear densities of 10^{14} kg m^{-3} the neutrinos are still scattered in the core region and therefore leave the star with a delay. The neutrino signals were received a few hours before the first optical observation of the outburst. The neutrino burst corresponds to a total energy of 10^{45}–10^{46} J. Only 1% of the gravitational binding energy is released as optical radiation and as kinetic energy of the ejected envelope. Such observations are also essential for the question of the neutrino rest mass. The observations suggest an upper limit for the neutrino rest mass of 10 to 30 eV c^{-2} .

The discovery was made on February 24, 1987, and in July light echoes from small matter rings 140 and 400 pc before the supernova were detected. From July to November the brightness decreased exponentially, which is related to the mean decay time of the ^{56}Co agrees.

On average, one supernova outburst can be expected in 50 years per galaxy.

The evolution of light echoes around object V838 is shown in Fig. 11.14. This originated from a nova outburst (early 2002) of an object in the constellation Monoceros (Unicorn) at a distance of 6.1 kpc.

11.6 Black Holes

The existence of objects whose gravity is so strong that not even light can escape was already suspected by Newton. In principle, all objects could become a black hole if they were suitably compressed. Quantum physics shows that even black holes evaporate over very long periods of time.

Fig. 11.14 Evolution of the light echo around V838 (Source: Hubble Space Telescope)

11.6.1 General

We consider an object whose escape velocity

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} \tag{11.38}$$

equal to the speed of light c; so we get for the radius of a black hole:

$$R = \frac{2GM}{c^2} \tag{11.39}$$

For such an object, nothing can escape, not even light particles, hence the name black hole. The radius of a black hole therefore depends on its mass M and is called *Schwarzschild radius.* Matter falls into a black hole on a spiral path.

From the above formula, the hypothetical Schwarzschild radius for our Sun is 3 km.

A hypothetical trip into a black hole yields that if the black hole has a mass of ten solar masses, at 3000 km from the singularity, you will be torn apart by the enormous tidal forces. 10^{-5} s after passing the Schwarzschild radius, one arrives in the singularity. For an outside observer, the spacecraft suffers an ever-increasing red shift as it flies in, and time passes slower and slower. The light is red shifted until you can no longer detect the signals.

As we have seen, above a mass larger than the Chandrasekhar limit mass, the pressure of the degenerate electrons is not sufficient to stop the gravitational collapse at the end of the star's evolution. As the mass increases, a neutron star is still possible. As the star contracts, the gravitational field at its surface becomes stronger and stronger, and with it relativistic effects, which roughly depend on the ratio of the Schwarzschild radius to the actual radius of an object. Finally when the mass is big enough, light can no longer escape, the escape velocity is equal to the speed of light, and thus everything is trapped by the gravitational field, you have a set of events from which no escape is possible, and this is called a black hole. The limit from which no escape is possible is called the *Event horizon.*

At the center of a black hole, there is a singularity. No known laws of physics exist there anymore, and nothing can be predicted. Such singularities are beyond our knowledge, since they are separated from us by the event horizon. But there are also solutions according to general relativity, which would allow an astronaut to avoid a collision with this singularity, he could instead fall into a *Wormhole* which would mean that he would come out at a completely different location in the universe. Such journeys through space and time have a disadvantage: the solutions are extremely unstable, the slightest disturbance would lead to a fall into the singularity.

The mathematical description using the Schwarzschild metric yields three solutions:

- Black holes;
- White holes: opposite of black holes, only matter, energy flows out; violate 2nd law of thermodynamics;
- Wormholes: also called *Einstein-Rosen-Bridges* connect different parts of the universe.

Rotating black holes are described by a Kerr metric. In addition to the event horizon, there is a so-called ergosphere, which envelops the event horizon; within the ergosphere, matter cannot be kept stationary.

The formation of a black hole results in the emission of *gravitational waves.* These extract energy from the system. Gravitational waves emitted when two black holes collide were directly detected for the first time in 2015.

Gravitational waves are emitted by all accelerating moving masses.

Gravitational waves (ripples in space-time) are also produced by the motion of the Earth around the Sun, energy is thus extracted from the system, but very little, and the effect here is extremely small. This is different, for example, with the pulsar PSR 1913+16, where we have two neutron stars orbiting each other. They lose energy by emitting gravitational waves and thus spiral towards each other.

The size and shape of a black hole depend only on its mass and rotation, but not on other parameters such as chemical composition, etc.

11.6.2 Candidates for Black Holes

There are many candidates for black holes: e.g. the system *Cyg X-1* (Fig. 11.15). This is a powerful X-ray source in the sky (brightness in the range 2–11 keV: 2×10^{30} W, distance: 2.5 kpc). It is a binary star system in which matter is blown away from one component and spirals in an accretion disk towards an unseen companion (which is likely to be a black hole due to its large mass, 16 solar masses), heating it enough to emit X-rays. The mass of the other star is 33 solar masses. The blue supergiant shows periodic Doppler shifts in the absorption lines (period five days), which is interpreted as motion around the system's center of mass. The X-ray intensity varies in the range of 0.001 s, indicating a very compact X-ray source.

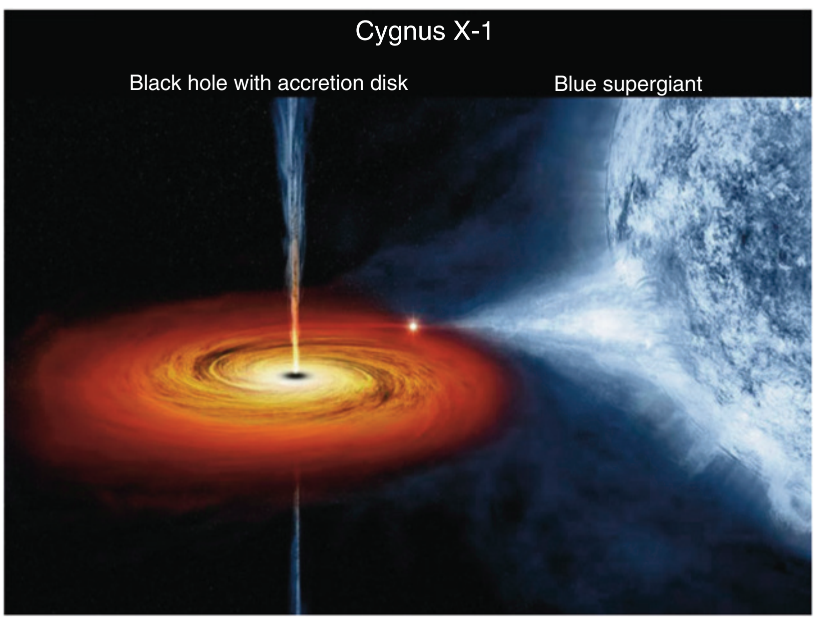


Fig. 11.15 Cygnus-X1: blue supergiant with black hole companion (Adapted from Chandra/Harvard)

11.6.3 Quantum Theory of Black Holes

The Area of a black hole cannot decrease, similar to that Entropy of a closed system.

Classically, black holes cannot emit radiation, yet there is so-called *Hawking radiation*. The emission is smaller, the larger the mass of the black hole is. According to quantum theory, there are quantum fluctuations even in vacuum, and particle/antiparticle pairs are created. One speaks of virtual particles.

Heisenberg's uncertainty principle states:

$$\Delta t \Delta E \geqq \frac{h}{4\pi} \qquad E = hf \tag{11.40}$$

A virtual particle of the energy range ΔE therefore has a lifetime of range Δt. The energy of a photon pair is $2\Delta E$.

Calculate the lifetime of a virtual photon pair of orange light ($f = 5 \times 10^{14}$ Hz).

$$\Delta t = \frac{1}{8\pi f} = 8 \times 10^{-17}\,\mathrm{s}$$

Energy cannot be created out of nothing, one of the partners of a particle/antiparticle pair has positive energy, the other negative energy. The energy of real particles is always positive. A real particle that is close to a mass has less energy than one that is far away from that mass, because energy must be expended to keep it away from the mass. The gravitational field inside a black hole is so strong that even a real particle can get negative energy there. In this way, a virtual particle with negative energy can also fall into a black hole and become a real particle or antiparticle—it does not need to annihilate with its partner. Its partner can also fall into the black hole or escape from the vicinity of the black hole as a real particle or antiparticle. We as observers from outside then have the impression that a particle is emitted from the black hole. The smaller the black hole is, the shorter is the distance for a particle with negative energy to become a real particle. Small black holes therefore radiate more intensely. Therefore, the smaller the mass of a black hole, the higher the temperature. If the mass becomes extremely small, there is a final evaporation and a violent burst of radiation (Table 11.2).

Bekenstein, Hawking et al. showed that a black hole has a non-vanishing temperature which is calculated from:

$$T = \frac{hc^3}{16\pi^2 kGM} \approx 10^{-7}\frac{M_\odot}{M}\,[\mathrm{K}] \tag{11.41}$$

Table 11.2 Various astrophysical objects; escape velocity, v_e, Schwarzschild radius, R_S (hypothetical for Earth, Sun and white dwarf)

Object	$M[M_\odot]$	Radius [km]	v_e	R_S
Earth	$1/3 \times 10^{-5}$	6357	11.3	9 mm
Sun	1	7×10^5	617	2.9 km
White dwarf	0.8	10^4	5000	2.4
Neutron star	2	8	2.5×10^5	5.9
Galact. Core	5×10^6	?	?	15×10^6

and the energy of the radiation is:

$$E = \frac{hc^3}{16\pi GM} \tag{11.42}$$

Hawking temperature: the larger the mass, the slower the black hole evaporates.

Suppose an Earth mass (5.3×10^{26} kg) be a black hole. What would be the energy of its radiation and at what frequency could it be observed?

Solution:

$$E = hc^3/(16\pi GM) = 8.9 \times 10^{-25}\,\mathrm{J}$$

and because of

$$E = hf$$

follows a frequency of $f = 1.35\,\mathrm{GHz}$.

The temperature of a black hole with a solar mass is only 10^{-7} K. This is much less than the temperature of the background radiation (see section Cosmology) which is 2.7 K. At present, such black holes "warm up". However, as the universe continues to expand, at some point the temperature of the background radiation would drop below that of the black holes, and they may cool down. For a black hole with a solar mass, it takes 10^{66} yr for it to evaporate. However, could be very small black holes that were formed during the Big Bang and are now evaporating.

11.6.4 Accretion

As we have seen, there are different phases of stellar evolution in which accretion plays an important role. According to *Zel'dovich* the luminosity of a star due to accretion of matter:

$$L \approx \Phi \frac{dM}{dt} \approx 2 \times 10^{31} \left(\frac{\Phi}{0.1c^2}\right)\left(\frac{M}{M_\odot}\right)^2 \left(\frac{10^4}{T}\right)^{3/2} N \,\mathrm{erg/s} \tag{11.43}$$

Here Φ is the gravitational potential near the surface of the star, T and N are temperature and particle density of the gas, respectively.

Let us consider two cases:

- White dwarf with $M \approx M_\odot$ and $R = 10^9$ cm, $\Phi = 10^{-4}\,c^2$. One finds $L \approx 10^{28} N$ erg/s. For corresponding values of N is then given by L, and from this we can deduce the temperature of T =500–2000 K.
- Neutron star: Also let $M \approx M_\odot$, $R \approx 10^6$ cm; the radiation occurs mainly in the UV (15...900 nm).

11.7 Gamma Ray Bursts

Gamma Ray Bursts (GRB), were first observed around 1970. The discovery was made by chance with the Vela satellites, which were supposed to monitor the ban on nuclear testing. Since then, well over 2000 GRBs have been recorded.

11.7.1 Properties of GRB

The *bursts* come from random celestial directions (Fig. 11.16). There is no concentration to the galactic plane → GRBs could be from the Galactic halo or even from the Oort cloud.

The duration of the bursts ranges from fractions of a second to minutes. In order to reveal the origin of the bursts we need distances but distance determination was not possible until radio or optical sources could be identified.

Energy release: within seconds as strong as some 10^4 Supernova explosions.

With the Burst and Transient Source Experiment Burst (BATSE[6]) aboard the Compton Ray Observatory, it was possible to register this gamma ray radiation and determine its position within seconds. On January 23, 1999, only 22 s after observing a GRB with a robotic telescope in New Mexico, an optical image was obtained of the region of the sky

[6] Removed from Earth orbit by NASA in 2000.

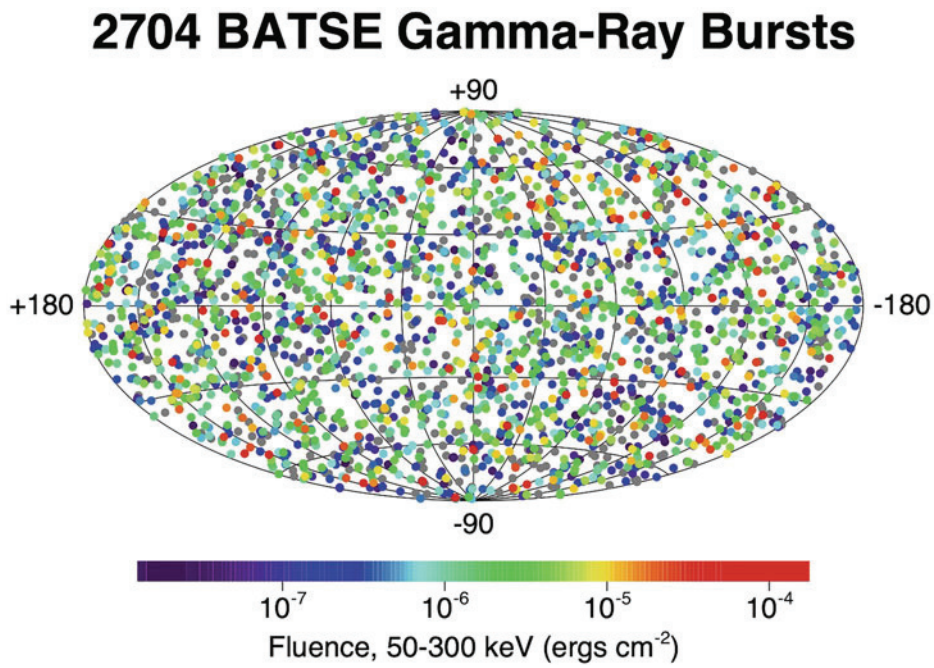


Fig. 11.16 Distribution of GRBs in the sky (galactic coordinates, the galactic equator runs in the middle). (Source: M. Briggs)

where the GRB was observed. Within 25 s a dramatic increase in brightness was observed. Spectra were then taken using the Keck telescope, and the object had a red shift of $z = 1.6$, which corresponds to a distance of 3000 Mpc.

GRBs have also been associated with mass extinctions of animal and plant species in Earth's history *(mass extinction).*

11.7.2 Explanation of GRB

There are several theories:

- Collapse of a massive star → a black hole with an extreme magnetic field arises, one also speaks of a Hypernova.
- Merging of two neutron stars or neutron star + black hole.
- Bursts are directed like in a pulsar. This would lead to an overestimation of the released energy.
- Gravitational lensing effect enhances a less intense burst. Between GRB source and observer is a massive object which acts as a lens .

- Magnetar: Neutron star with extremely strong magnetic field. Starquakes occur from time to time by interaction of the magnetic field with the crust → Gamma rays. During the outburst of object SGR 1900 +14 on August 27, 1998, gamma rays striking the Earth's atmosphere raised the number of ions in the ionosphere at night to daytime levels.

11.8 Variable Stars

Variable stars are interesting for several reasons. On the one hand, they are usually at the end of stellar evolution, and often, in addition to pulsations, envelope expulsion occurs. On the other hand, some groups of variable stars are important standard candles by which distances can be determined. One knows their actual luminosity, and by comparison with the easily measurable apparent brightness follows the distance.

11.8.1 General

The term "variable star" in astrophysics always refers to stars whose brightness changes. As we saw in the section on the Sun, it too is strictly speaking a variable star, but the types of stars discussed here are variable to a much greater extent. If we examine the light curve of a variable, we can derive two parameters:

- Period of the change in brightness P;
- Amplitude of the brightness change A.

The designation is made within a constellation with large Latin letters R, S,. . .Z and then continuing with RR, RS,. . .ZZ and AA, respectively,. . .QZ as well as simply with V and a number. The first star found to be variable is the star Mira (o Ceti) (1596 by *Fabricius* discovered). In general, a distinction is made between:

- Pulsating Variables : giants or supergiants of all spectral classes; the cause of the change in brightness is more or less periodic pulsations of the atmosphere.
- Eruptive Variables: often stars of low luminosity; there are random eruptions of gas.
- Eclipsing Variables: The cause here is the mutual eclipsing; some representatives of these groups are very close binary stars, so that there is also an exchange of matter.

Pulsation Mechanism

Stars pulsate when they are not in hydrostatic equilibrium.

If a star expands due to increased gas pressure, the matter density decreases until the point of hydrostatic equilibrium is reached, gravity dominates again and the star contracts. In both cases, you have overshooting, above the corresponding equilibrium points. In this process, energy dissipation occurs, and normally pulsations therefore come to a rapid halt.

Thus, as in the case of pulsating stars, in order to sustain pulsations over long periods of time, one needs a mechanism that can compensate the dissipation. Opacity plays a significant role in energy transport. If the opacity is large, the radiation cannot escape and the star appears faint. If the star is compressed at the time of greatest opacity, then the excess radiation (cf. virial theorem) is stored and exerts an additional pressure → Mechanism to maintain the pulsations (κ-mechanism).

Pulsation variability: Increase in opacity upon compression as singly ionized helium absorbs UV radiation, becoming doubly ionized. The He^+-Ionization zone is cooler than surrounding regions because energy is consumed to ionize. Thus, one has a region of instability.

There is a simple relation between the density of the star and its pulsation period. Assume that the matter falls freely onto the star after expansion; then Kepler's law applies to this gas:

$$\frac{P^2}{R^3} = \frac{4\pi^2}{GM} \tag{11.44}$$

Thereby P the period of the pulsation, R the radius of the star. Because of

$$P^2 \approx R^3/M \qquad M \approx \bar{\rho} R^3 \tag{11.45}$$

one gets:

$$P^2 \approx R^3/(\bar{\rho}^3 R^3) \approx 1/\bar{\rho} \tag{11.46}$$

If a star radiates like a black body, then:

$$L \propto R^2 T_{\mathrm{eff}}^4 \tag{11.47}$$

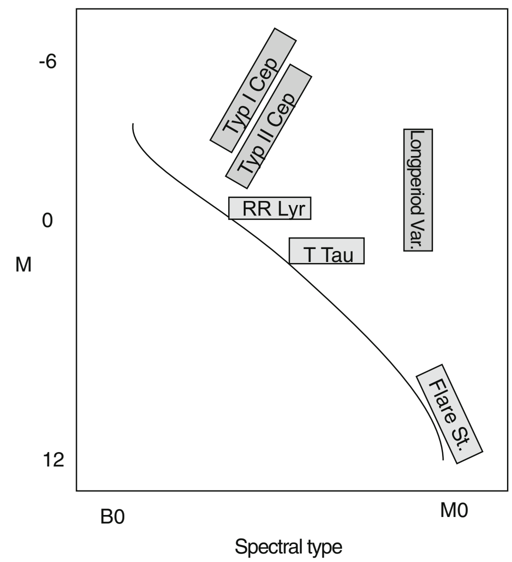


Fig. 11.17 Location of the different types of variable stars in the HRD

If we observe a pulsating star at two different times 1 and 2, then the ratio of the luminosities is given by L_1 and L_2 at these times:

$$\frac{L_1}{L_2} = \left(\frac{R_1}{R_2}\right)^2 \left(\frac{T_1}{T_2}\right)^4 \tag{11.48}$$

Luminosity changes are therefore associated with temperature and radius changes (pulsations).

In addition to regularly variable stars, there are also stars whose variability is not strictly periodic; these irregularly variable stars include ZZ-Ceti stars (hydrogen-rich white dwarfs with periods between 3 and 30 min), BY-Draconis stars (late-type dwarf stars), and L-type variable stars (slowly variable, amplitudes up to 2^{m}, giants and supergiants). The location of different types of variable stars in the HRD is outlined in Fig. 11.17.

11.8.2 Pulsation Variable

Cepheids

The light variation of Cepheids occurs strictly regularly with a period between 1 and 50 days (Fig. 11.18). They are very bright supergiants of types F to K. That is why they can still be observed at great distances (more than 300 have been found, for example, in our neighboring galaxy about 2.5 million light-years away, the *Andromeda galaxy*.

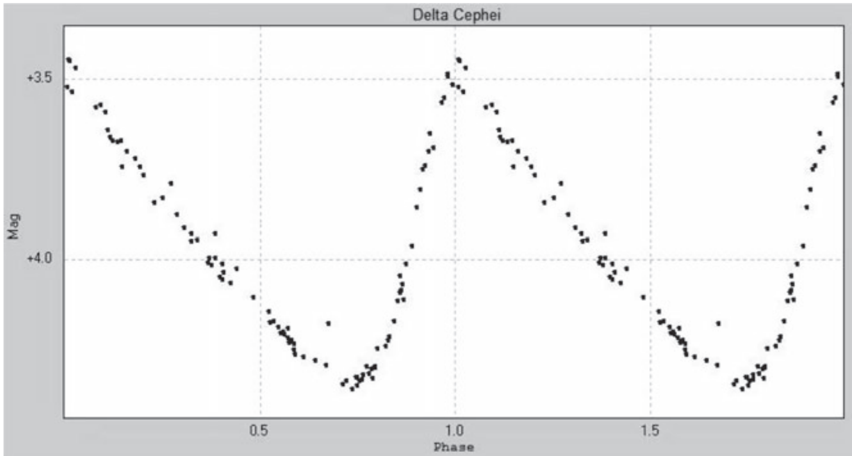


Fig. 11.18 Light curve of the star δ Cephei

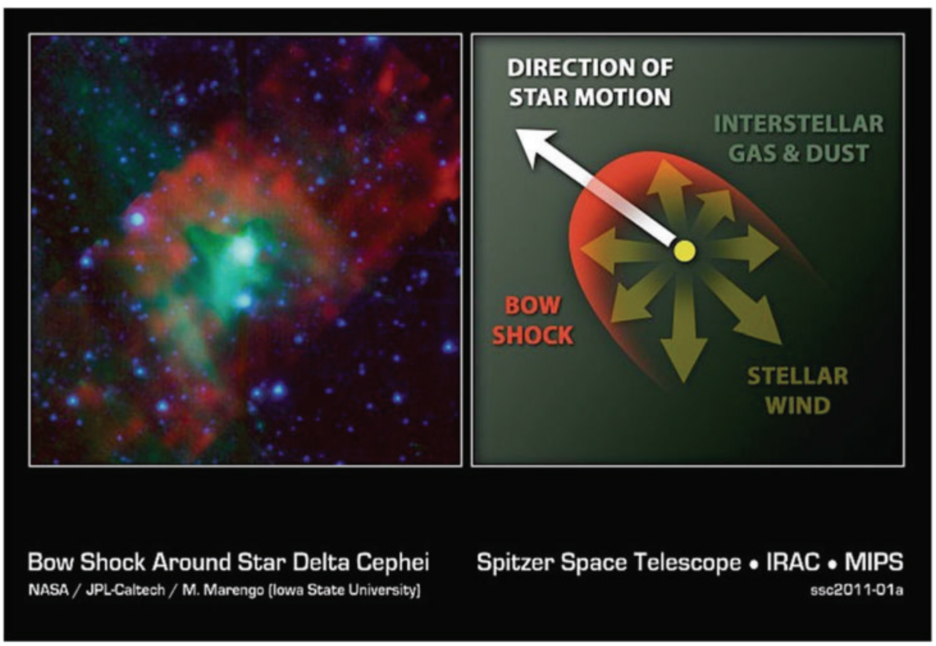


Fig. 11.19 Shock front around δ Cephei (Spitzer Telescope)

The star δ Cephei has a period of 5.37 days and changes its diameter by 2 million km during this time. Using the Spitzer IR telescope, a shock front has been detected resulting from the star's motion through the interstellar medium. Furthermore, strong stellar winds have been observed (Fig. 11.19).

Around 1950, it was found that there were two groups of Cepheids:

- δ-Cephei stars: They occur in the galactic plane, called Cepheids of the population I.
- W-Virginis Cepheids: They occur in the halo of the Galaxy or in the Galactic center, called Cepheids of population II.

The amplitudes are less than 2^m, larger in the blue than in the red. The cause of the change in brightness is pulsation, the radius changes being about 10% for the δ-Cepheids, for the W-Virginis-Cepheids about 50%. The largest diameter occurs during the descent phase of brightness, and the temperature is greatest when the brightness is highest. By measuring the Doppler shift of the spectral lines, one can determine the velocities; from the integration of the radial velocity curve, the change in radius follows.

When this group of variables was detected in the Magellanic Clouds , it was found that there is a relationship between their apparent brightness and period. Since all the stars in this dwarf galaxy neighboring us are about the same distance away, this means that there is a relationship between the periods and luminosities of these objects, the *period-luminosity relation.* The luminosity and absolute magnitude are related and follow from the period P:

$$\delta\text{Ceph Pop I} : M = -1.80 - 1.74 \log P \tag{11.49}$$

$$\text{W Vir Pop II} : M = -0.35 - 1.75 \log P \tag{11.50}$$

So once you have determined the period of these objects, you know their absolute brightness and thus the distance.

This is a very important method of distance determination for nearby Galaxies.

RR Lyrae Stars

They often occur in globular clusters. Globular clusters are spherically arranged collections of some 10^4 Stars distributed in a halo around a galaxy. In them one finds the oldest stars. The periods are $<1^d$, spectral types B8...F2, amplitudes around 1^m. Sometimes several periods overlap. The absolute magnitudes are fairly constant $\rightarrow$ Standard candles

$$M_{vis} = +0.5 \pm 0.4. \tag{11.51}$$

In the HRD they are located at the gap in the horizontal branch where there can be no stable stars.

δ-Scuti Stars
This group is also called dwarf Cepheids. They are pulsating giants (F) with very short periods $0.^{d}5 \ldots 0.^{d}2$ and small amplitudes.

β-Canis-Majoris Stars
Spectral type: B1... B2, III...IV. Periods between 3 and 6 h. Multiple periods always occur. The amplitudes are only a few hundredths of a magnitude.

Mira-Variables
Periods between 80 and 1000 days; striking are the very large amplitudes between two and more than four magnitudes. In terms of light curve shapes, the following types are distinguished:

- α: ascent steeper than descent, minima wider than maxima.
- β: almost symmetrical.
- γ: irregularities in light curves, humps, etc.

Mira stars belong to spectral classes M, S, or C. The enormous changes in brightness result from the strong variation in absorption bands and not from changes in temperature directly.

From the measurements of the radial velocities of emission and absorption lines, it follows that the envelopes are expanding at about 10 km/s.

In addition, these stars lose about 10^{-8}–$10^{-6}\, M_{\odot}$/year in mass (Fig. 11.20). Some Mira stars show maser emission in the radio region at a wavelength of 18 cm from OH.

Let us consider the prototype *Mira (oCeti)* The maximum radius occurs at the minimum luminosity and is about $320\, R_{\odot}$, and the minimum radius occurs at the brightness maximum and is about $220\, R_{\odot}$, the brightness varies between $2.^{m}0$ and $10.^{m}1$. The pulsation period is 331 days. The star itself has a white dwarf as its companion.

11.8.3 Semi-regular Variables

Periods range from 30 to 1000 days, amplitudes are usually less than one to two magnitudes. There is either a good mean period, or the period is disturbed by irregularities. The following subgroups are distinguished:

- SRa: Red giants, M, C, S; smaller amplitude than Mira stars, otherwise the same.
- SRb: Also M, C or S. The periodicity is interrupted by completely irregular phases; example: AF Cyg.
- SRc: Supergiants of intermediate type (G8...M6); example: μ Cep.
- SRd: yellow giants and supergiants (F...K).

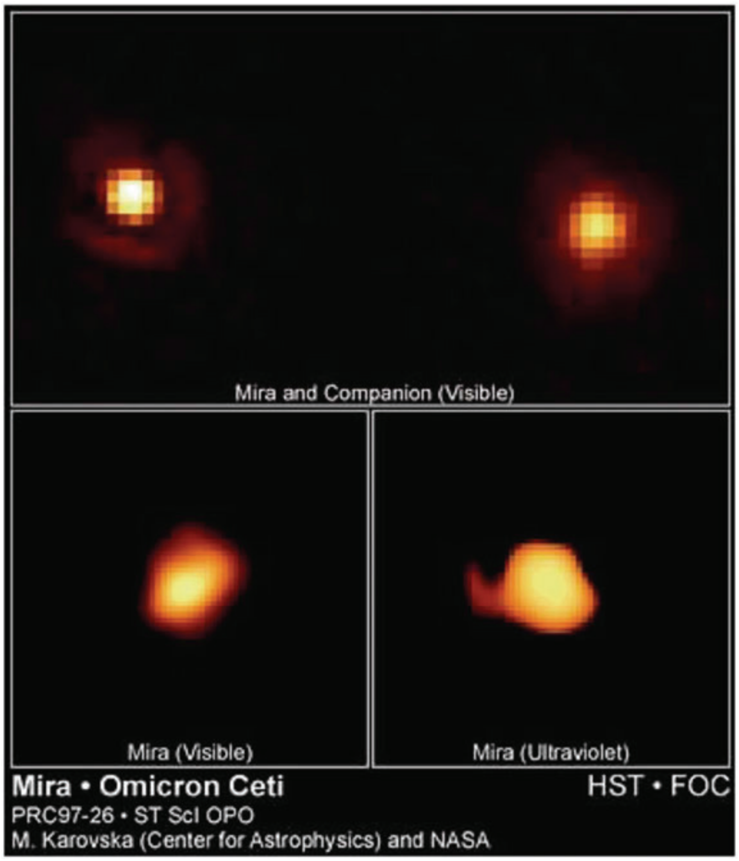


Fig. 11.20 HST image of the variable star Mira. Mira is seen to be losing mass in UV light (GALEX (Galaxy Evolution Explorer))

RV Tauri Stars

Type F. . .K, I, II. regular alternation of shallow and deep minima. Periods between 30 and 150 days. The amplitudes are up to three magnitudes.

α^2-Canum Venaticorum Stars, Ap Stars

These stars have the following features: (a) strong magnetic fields (measured by Zeeman effect, 0.1 to 1 T), (b) abnormally strong lines of rare elements like Si, Mn, Cr, Sr, Eu. Individual line groups change their intensity, which leads to variations in brightness of about $0.^{m}1$. Periods range from 1 to 25 days, magnetic fields are variable, which is explained by:

- Oblique rotator: Magnetic axis does not coincide with axis of rotation of star.
- Activity cycle similar to Sun, but orders of magnitude stronger.
- Huge magnetic spots on surface, change by rotation.

About 10–15% of late B and A stars belong to this group. The elemental anomalies (e.g. Os I, Pt II, Pm) can be explained by neutron irradiation or by complex diffusion processes.

Metal Line Stars, Am Stars
Cooler than the Ap stars; mostly members of binary systems, rotational velocity <100 km/s; strong lines of the Fe group; about 10% of the A stars belong to this group.

11.8.4 Eruptive Variables

Novae and Nova-Like Variable Stars
These objects are sometimes called *cataclysmic variables*. They are hot dwarf stars whose brightness increases by seven to 20 magnitudes within a short time (hours to months). After a short maximum the brightness returns to the original value in the course of years to decades. About the Praenovae is only known that they are A-subdwarfs. Sometimes there is a slight increase in brightness up to 1.5 magnitudes before the outburst. One divides the novae into:

- Na: rapid nova, very steep rise. Decline by 3^{m} in less than 100 days.
- Nb: slow nova, descent around 3^{m} in more than 100 days.
- Nc: extremely slow nova, many years at maximum.
- Nr (Nd): recurrent nova, recurrent bursts of brightness. Example: TCrB; outburst 1866, 1946 at $\Delta m = 8.6$.
- Nl (Ne): Nova-like variable.

The spectra for Na and Nb are also divided into Q0 to Q9. The typical progression of these two types looks like this:

- steep slope: 7–10^{m} within one day;
- short standstill, possibly even decline before the maximum;
- steep final rise to maximum, Q0;
- main maximum, Q1;
- first descent around $3.^{\mathrm{m}}0$; first spectrum similar to an F supergiant (Q2), then broad, vigorous emission of H and ionized metals (Q3). Later H, OII, NII, NIII (Q4, Q5);
- Transitional stage, nebular spectrum, NII, forbidden line [OIII] (Q7);
- Postnovae.

Knowing the time of brightness decline by three magnitude classes in days (t_3), then one can determine the absolute maximum brightness from the following empirical formula:

$$M_{\max} = -11^{\mathrm{M}} + 2.5 \log t_3 \tag{11.52}$$

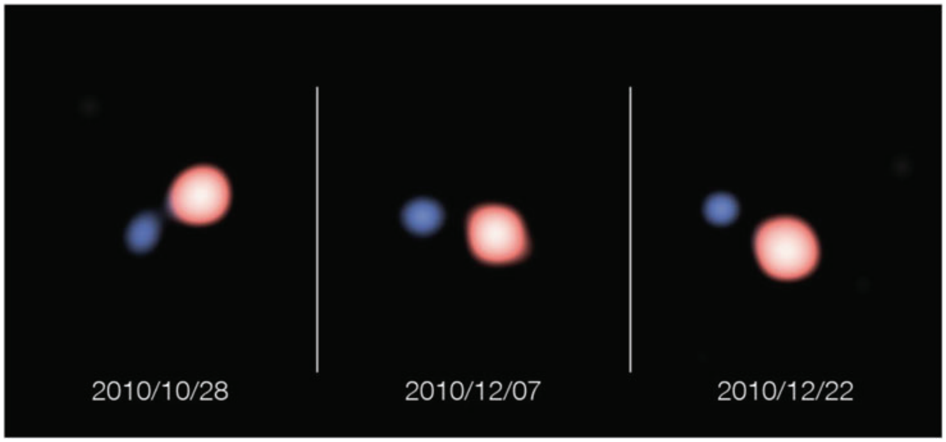


Fig. 11.21 The symbiotic star SS Leporis (17Lep). The image was taken with the VLT interferometer. A red giant orbits a hotter companion. The stellar images were coloured according to the temperatures (VLTI/ESO)

Novae are close binary stars with a hot, blue component (white dwarf). The cooler red component, which has less mass, gives off matter to the white dwarf. For the white dwarf its temperature and density at the bottom of the atmosphere increases, a hot spot forms, and thermonuclear reactions can begin. During the eruption, about 10^{38} J is released.

In Fig. 11.21 the *symbiotic double star* SS Lep is shown. There is an exchange of matter between the components. The object is about 270 pc from us, and the orbital period is 260 days. The size of the orbit is about 0.005". Such observations are only possible with the VLT-PIONIER (Precision Integrated Optics Near-infrared Imaging ExpeRiment). The light from the four VLT telescopes is made to interfere with each other; this results in a greater baselength, which in turn increases the telescope resolution.

About 200 novae have been observed in our Galaxy. It is estimated that there are about 50 novae per year in our Galaxy. Postnovae are often surrounded by expanding nebulae.

In August 1975 a nova was observed in the constellation Cygnus (maximum brightness $1\overset{\mathrm{m}}{.}8$).

Among the Nl stars are the S Doradus stars, the γ-Cassiopeiae stars, Z-Andromedae stars, P-Cygni stars and others. Among the dwarf novae (DN) one counts the U-Geminorum stars. They all have weak fluctuations in brightness in common, and then suddenly there are outbursts between two and six magnitudes within a few days. The longer the pause between the eruptions, the more violent they are (pent-up energy).

R Coronae Borealis Stars

These are supergiants; the brightness remains constant for months or even years. Then within days the brightness decreases by several magnitudes, amplitudes up to 7^{m}. Almost all of them are located in regions of interstellar matter, nebular patches. Based on the blue shift of the spectral lines, we know that matter is ejected from the Star, carbon condenses, and soot clouds eclipse the star. The expansion velocity is about 60 km/s and the mass loss rate $\dot{M} \approx 10^{-5} M_{\odot}/\mathrm{year}$.

T Tauri Stars

These are pre-main sequence stars with masses between 0.2 and two solar masses. They have an extended convection zone. Some are thought to have large spots on their surfaces (based on brightness changes when the spots move due to stellar rotation). Emission lines of H and ionized Ca indicate an active chromosphere. Forbidden lines similar to nebulae are also found in some T Tauri stars. This indicates circumstellar material. Observations in X-rays show very strong variations (up to factor 10) within one day. These are probably huge flare outbursts in their photospheres. The observed excess of IR radiation can be explained simply: The dust/gas cloud surrounding it absorbs the star's shortwave radiation and re-emits in the IR. One observes strong stellar winds (10^{-7}–10^{-8} $M_{\odot}/\mathrm{year}$).

Our Sun also went through a T Tauri stage, which had a significant impact on the formation of the planets' primordial atmospheres. During this stage, it emitted an amount of X-ray radiation equal to a factor of 10^3 stronger than today.

Flare Stars

In dwarf M stars, flares are observed with released energies between 10^{21} and 10^{27} J. The increase in brightness by up to six magnitudes occurs within a few seconds to minutes; they are also called UV Ceti Stars. Such stars are probably very common, but the probability of detection is low because of their low luminosity.

RS Canum Venaticorum Stars

Are found in binary star systems. Orbital periods are around seven days. The stars exert strong tidal interactions on each other. The radio flares observed in these types are 10^5–10^6times stronger than for the Sun. The radio spectra are polarized and non-thermal, suggesting synchrotron radiation. One also detects a brightness modulation with the rotation period, suggesting giant stars pots.

11.8.5 Peculiar Stars

The suffix *p* in the spectral type indicates a special feature, e.g. anomalous metal abundance.

Wolf-Rayet Stars
Stars of very high luminosity, expanding atmosphere, extremely broad emission lines. Temperature about 30,000 K, radii between 3 and 25 $R_\odot$, masses 10–20 $M_\odot$. The emissions come from the expanding envelope. One distinguishes the following types:

- WC: strong C lines,
- WN: strong N lines.

Be and Shell Stars
Shells can form around Be stars due to the high rotational velocities that characterize these stars.

11.8.6 Planetary Nebulae

Due to their nebular appearance, which resembles that of a planet in the telescope, this misleading name was introduced for white dwarfs that have ejected a luminous gas envelope. Again, there is an expanding atmosphere, and H-lines as well as He-lines are observed. The strongest lines are the forbidden lines of the elements O and Ne. Such forbidden lines arise as a result of the low gas density. Atoms can be excited at metastable levels, since collision, which would contribute to depopulation of the level, is very unlikely. The forbidden lines of [O III] are at 500.7 and 495.5 nm, respectively. The forbidden nitrogen line [NII] is at 658.4 nm.

The hot star at the center heats the gases to about 10,000 K. The gas temperature increases as one moves away from the center. High-energy photons are absorbed less often than low-energy photons. In the outer nebular regions, the low-energy photons have already been absorbed, and the remaining high-energy photons cause the temperature increase.

In general, Mira stars are thought to be the precursors of planetary nebulae.

A prominent example of a planetary nebula is the Cat's Eye Nebula, NGC 6543 (Fig. 11.22). It is located at a distance of 1500 light-years in the constellation Dragon. The inner part of the nebula has only 20″ extension, the outer part extends over 6.4 arcminutes and was formerly ejected by the very old central stare. The inner part has a temperature of 8000 K and a particle density of 5000 particles per cm^3, the outer part is much thinner and has a temperature of about 15,000 K. The 80,000 K hot central star loses about 20 trillion tons per second due to stellar winds, which is about $3 \times 10^{-7}\,M_\odot$/year.

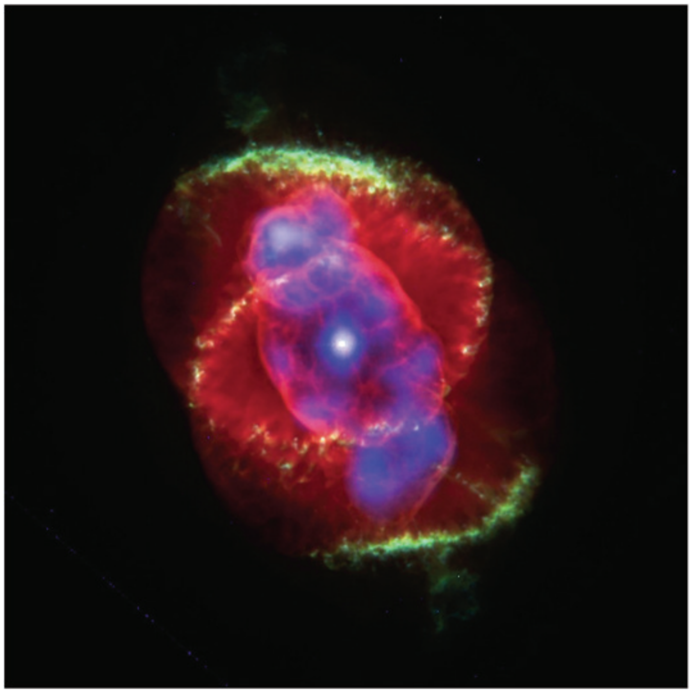


Fig. 11.22 The Cat's Eye Nebula, NGC 6543, an example of a planetary nebula. The image was taken with three filters: Red (Hα, 6563 nm), blue (neutral oxygen, 630 nm), and green (ionized nitrogen, 658 nm). (HST image)

Other well-known examples of planetary nebulae are the Dumbbell Nebula (M27, distance about 1400 light-years, diameter 3 light-years) and the Ring Nebula (M57, distance about 2300 light-years, diameter about 0.9 light-years, age about 20,000 years). In total, more than 1500 planetary nebulae are known in the Milky Way. Compared to the several 100 billion stars in the Milky Way, this is not much, but these nebulae only shine for a few 10,000 years. .

11.9 Stellar Activity

In addition to the stars with very strong variabilities discussed above, it is now possible to determine activity cycles, spots, etc. for "normal" stars.

11.9.1 Stellar Activity and Convection

The activity of the Sun can be explained by a dynamo process, where

- rotation,
- magnetic fields,
- convection zone

interacts. Magnetoacoustic waves are generated in the region of the convection zone, which heat up the chromosphere and the corona, among others.

From the theory of stellar structure we know: Stars of later spectral type than F0 ($T \approx 6500$ K) have a convection zone that extends to the surface. The convection zone extends deeper into the stellar interior the later the spectral type. This can be explained simply. Convection occurs when the adiabatic temperature gradient of an element moving upwards due to a random perturbation is smaller than the radiation gradient of the surroundings. If there is ionization of hydrogen H^+ or helium (in this case He^+, is singly ionized He, and He^{++}, i.e., doubly ionized He), then the radiative gradient increases and the adiabatic gradient decreases, respectively, which favors convection. For cooler stars the surface temperature is lower, the zone above which hydrogen is ionized extends deeper into the stellar interior than for hotter stars.

The chromospheric activity of a star can be determined by measuring the Ca-II H and K emission lines. Long series of measurements then allow the stellar activity cycle to be determined. In 1957 *Wilson* and *Bappu* found that the width of the Ca-II emission lines is a function of the absolute luminosity, *Wilson-Bappu effect* → Method of distance determination. The width of a line can have various causes (rotation of the star, magnetic fields, turbulence)—turbulence is the most important here.

Skumanich found that rotational velocity and chromospheric activity of the Stars decrease with age.

The decrease in stellar rotation can be given as follows:

$$\Omega_{\mathrm{eq}} \approx t^{-1/2} \tag{11.53}$$

Ω_{eq} is the angular velocity at the star's equator, t is the age of the star. One can only determine the equatorial velocity if the inclination of the rotation axis is also known (Fig. 11.23). *Gyrochronology* is the method of determining the age of a star from its rotation rate. This is calibrated at the sun.

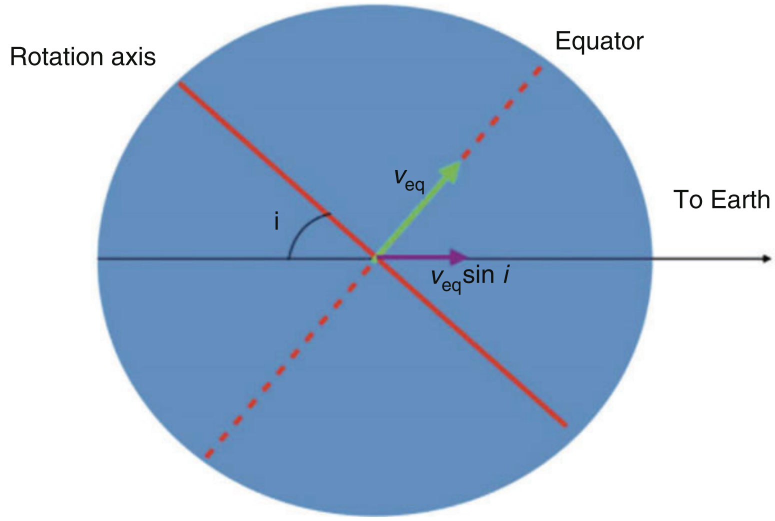


Fig. 11.23 When determining the rotation rate of a star, the mostly unknown inclination of the rotation axis must be taken into account. Thus, apparently slowly rotating stars may nevertheless rotate rapidly if their axis is only slightly inclined

Table 11.3 Summary young stars/old stars

	Young stars	Old stars
Activity	Large amplitudes	Low amplitudes
Activity cycle	Irregular	Regular
Rotation	Rapid	Slow
Chrome. activity	High	Low

From the rotation rate of stars one can infer their age.

In young, rapidly rotating stars, for example, huge flare outbursts are observed in the X-ray region. A summary is given in Table 11.3

Out flowing stellar winds with magnetic fields dissipate angular momentum to the interstellar medium, leading to the slower rotation rates observed in old stars.

Mass losses can be explained by blue-shifted star absorption components of strong lines (Ca II H and K lines, Fig. 11.24). Mass-loss rates can be extracted from (after Reimers, 1975):

$$\dot{M} = 4 \times 10^{-13} \frac{L}{gR} \tag{11.54}$$

Where $\dot{M}$ in solar masses/year, L, g and R in units of the sun.

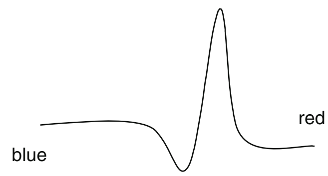


Fig. 11.24 P-Cygni profile. The component shifted to blue is from an envelope moving towards the observer (stellar wind)

Example
cool super giants: $\dot{M} \approx 10^{-7}$–10^{-5}, Sun $\dot{M} \approx 2 \times 10^{-14}$.

The X-ray luminosity also depends on the age and rotation of the star.

Another interesting observation is that of coronae in O and B stars. In their spectra one observes highly excited atoms (N V, O VI) as well as strong X-ray emissions. To explain the existence of these coronae, a mechanism other than the hydrogen convection zone is needed, since these stars show no convection near the surface. Here, compressions in the stellar wind are assumed to be heated by shock waves.

11.9.2 Mass Loss of Stars

It is very easy to show that outer layers of stars (coronae) are not stable. We give here the derivation already developed by Parker in 1958. Let us replace in the hydrostatic equation the density ρ by

$$\rho = \frac{\mu m_H}{k} \frac{P}{T}$$

then The hydrostatic equation:

$$\frac{dP}{dr} = -\frac{GM}{r^2} \frac{\mu m_H}{k} \frac{P}{T} \tag{11.55}$$

At the lower boundary of the corona let $r = r_0$ and $T = T_0$. The heat flux through a surface $4\pi r^2$ be

$$4\pi r^2 K \frac{dT}{dr} \tag{11.56}$$

The thermal conductivity K of a Plasma is

$$K \propto T^{5/2} \tag{11.57}$$

and one obtains

$$r^2 T^{5/2} \frac{dT}{dr} = \text{const} \tag{11.58}$$

or the solution:

$$T = T_0 \left(\frac{r_0}{r}\right)^{2/7} \tag{11.59}$$

Thus

$$\frac{dP}{P} = -\frac{GM\mu m_H}{kT_0^{2/7}} \frac{dr}{r^{12/7}} \tag{11.60}$$

and with $P = P_0$ at the position $r = r_0$

$$P = P_0 \exp\left[\frac{7GM\mu m_H}{5kT_0 r_0}\left[\left(\frac{r_0}{r}\right)^{5/7} - 1\right]\right] \tag{11.61}$$

$\rightarrow$ If $r \rightarrow \infty$, the pressure remains finite and does not vanish. The asymptotic value of P is larger than the typical pressure of the interstellar medium.

How are stellar winds driven? A distinction is made between:

- Thermally driven stellar winds: high temperature drives the wind; example: corona of the Sun.
- Radiation pressure: Its magnitude can become as large as gravity. This mechanism works for massive stars. A star remains stable as long as:

$$\frac{GM}{R^2}\rho > \frac{L}{4\pi R^2}\frac{\rho\kappa}{c} \tag{11.62}$$

It follows the Eddington Limit:

$$L < \frac{4\pi cGM}{\kappa} \tag{11.63}$$

- Stars that are rotating rapidly loose mass as well.

Currently, the Sun's mass loss is: $10^{-14}\, M_\odot$/year.
Red Giants: Low surface gravity, therefore stronger stellar winds.
Mass loss at late stages of stellar evolution leads to the formation of planetary nebulae.

11.10 Further Literature

We give a small selection of recommended further reading.
The Life of Stars, G. Shaviv, Springer, 2010
Theory of Stellar Structure and Evolution, D. Prialnik, Cambridge Univ. Press, 2009
Stars and Stellar Evolution, K. de Boer, W. Seggewiss, EDP, 2008
Stellar Evolution Physics, I. Iben, Cambridge Univ. Press, 2013
Solar and Stellar Magnetic Activity, C. Schrijver, C. Zwaan, Cambridge Univ. Press, 2000
Solar and Stellar Activity Cycles, P. Wilson, Cambridge Univ. Press, 2005

Tasks

11.1 Calculate the free fall time for the above cloud with $R = 10^{15}$ m to a radius with $R = 10^{11}$ m.

Solution
20,000 years.

11.2 Estimate the Jeans mass for an interstellar cloud 100 light-years in diameter, its temperature 30 K. $R = 100 \times 10^{16}$ m

Solution
$M > \frac{3}{2}\frac{1.38\times10^{-23}\times100\times10^{16}\times30}{6.67\times10^{-11}\times1.6\times10^{-27}} \approx 10^{34}\,\text{kg} \approx 10^4 M_\odot$

11.3 Calculate the relativistic Doppler effect (a) for the Sun, (b) for a white dwarf with 0,8 $M_\odot$, $R = 0.01\, R_\odot$.

Solution
Substituting the values for the Sun gives $c\frac{\Delta}{\nu} = 2.117 \times 10^{-6}$...and this corresponds to a Doppler effect of... $c\frac{\Delta\nu}{\nu} = 635\,\text{m/s}$.
The values for the white dwarf are: $\frac{\Delta\nu}{\nu} = 1.7 \times 10^{-4}$...and this corresponds to a Doppler effect... $c\frac{\Delta\nu}{\nu} = 58.8\,\text{km/s}$.

11.4 Calculate the magnetic field strength of a white dwarf with R = 7000 km.

Solution
Assume the values for the Sun: $B_\odot = 10^{-4}$ T, $R_\odot = 7\times10^5$ km. Then you get $B_{\text{WD}} = 1$ T.

11.5 How can brown dwarfs be found? Why are they so difficult to observe?

Solution
If companion in a binary system, due to motion of the primary; low luminosity.

11.6 At what wavelength do you observe light emitted at 600 nm from a neutron star of about 1 solar mass and radius 7 km?

Solution
One observes the radiation at 720 nm.

11.7 What is the apparent brightness of a type Ia supernova in the Andromeda Galaxy?

Solution
The Andromeda Galaxy is 2.5 million light-years away from us, this corresponds to $2.5 \times 10^6/3.26$ pc. If one substitutes into the formula for the distance modulus $M = -19$: $m = -19 + 5\log d - 5 \rightarrow 5.^{\mathrm{m}}3$. So you could just see a supernova exploding in the Andromeda Galaxy with the naked eye under very good conditions.

11.8 At what distance would a SN Ia have to explode so that it surpasses the full moon in brightness?

Solution
A little less than 250 pc = 815 Ly.

11.9 Show that the energy released in a nova outburst is equivalent to the thermal energy content of a thin shell of 5×10^6 K the mass $10^{-3}\ M_\odot$ is equal to.

Solution
$E = 3/2[kTM/m_u] = 3/2[1.38 \times 10^{-16} \times 2 \times 10^{33} \times 10^{-3} \times 5 \times 10^6/1.66 \times 10^{-24}]$

11.10 Show that the mass of a black hole is approx. 2×10^{19} kg must amount to in order to glow deep red. Could you actually see this radiation?

Solution
No, you calculate the energy!

11.11 Show the mass at which black holes could annihilate today.

Solution
Solution: If a black hole had a mass of $10^{-9}\ M_\odot$, then it would explode today ($T > 2.7$ K).

11.12 Visually, the luminosity variation of a Mira stars is, say, 1:100 (how many magnitudes would that correspond to?). Bolometrically, the variation is only 1:2, which corresponds to a $\Delta T \sim 500\,°\mathrm{C}$ corresponds. Verify that.

Solution
Approach: compare the Planck curves.